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Approximation Method for Fixed Points of Nonlinear Mapping and Variational Inequalities with Application[1]

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Abstract : In this paper, we introduce the new method of iterative scheme $\{x_n\}$ for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without demiclose condition and $T_\omega := (1-\omega)I + \omega T$, when T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$ in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.

Keywords : quasi-nonexpansive mapping; variational inequality; fixed point; nonspreading mapping.
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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote $F(T)$ by the set of all fixed points of T. Recall that the mapping $T : C \to C$ is

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said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$. Fixed point problems have been investigated in the following literature; see [1–3].

A mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

Let $B : C \to H$. The *variational inequality* is to find a point $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, B)$.

The variational inequalities were initially studied and introduced by Stampacchia [4, 5]. This problem is widely used in economics, social sciences and other fields, see for example [6–8].

Let $D_1, D_2 : C \to H$ be two mappings. In 2008, Ceng et al. [9] introduced a problem for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 D_1 z^* + x^* - z^*, x - x^*\rangle \geq 0, \forall x \in C, \\ \langle \lambda_2 D_2 x^* + z^* - x^*, x - z^*\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.2}$$

which is called a system of variational inequalities where $\lambda_1, \lambda_2 > 0$.

In 2013, Kangtunyakarn [10] modified (1.2) for finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1 - a) z^*), x - x^*\rangle \geq 0, \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2) x^*, x - z^*\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.3}$$

which is called a modification of system of variational inequalities, for every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$. If $a = 0$, (1.3) reduces to (1.2). He introduced the relation between solutions of (1.3) and fixed point of the mapping G as follows:

Lemma 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0, 1]$, the following statements are equivalent:*

1. *$(x^*, z^*) \in C \times C$ is a solution of problem (1.3),*
2. *x^* is a fixed point of the mapping $G : C \to C$, i.e., $x^* \in F(G)$, defined by*

$$G(x) = P_C(I - \lambda_1 D_1)(ax + (1 - a)P_C(I - \lambda_2 D_2)x),$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$.

Moreover, he introduced a new iterative algorithm $\{x_n\}$ for finding a common element of the set of fixed points of a finite family of κ_i-strictly pseudo-contractive mappings and the set of solutions of problem (1.3) in Hilbert space. The sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(I - \lambda_2 D_2)x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C\left(a x_n + (1-a)y_n - \lambda_1 D_1\left(a x_n + (1-a)y_n\right)\right), \forall n \geq 1, \end{cases}$$

where $D_1, D_2 : C \to H$ are d_1, d_2-inverse strongly monotone mappings, respectively, and $S : C \to C$ is S-mapping generated by a finite family of strictly pseudo-contractive mapping and finite real numbers. Under suitable conditions of the parameters $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \lambda_1, \lambda_2, a$, he proved a strong convergence theorem of iterative scheme $\{x_n\}$.

In 2012, Tian and Jin [11] proved the following strong convergence theorem of iterative scheme $\{x_n\}$ generated by (1.4).

Theorem 1.2. *Starting with an arbitrary chosen $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A) T_\omega x_n, \tag{1.4}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\omega \in (0, \frac{1}{2}), T_\omega := (1-\omega)I + \omega T$ with two conditions on T:

1. *$\|Tx - q\| \leq \|x - q\|$ for any $x \in H$, and $q \in F(T)$; this means that T is a quasi-nonexpansive mapping;*
2. *T is demiclosed on H; that is: if $\{y_k\} \subset H, y_k \rightharpoonup z$, and $(I-T)y_k \to 0$, then $z \in F(T)$.*

Then $\{x_n\}$ converges strongly to the $x^* \in F(T)$ which is the unique solution of the VIP:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \forall x \in F(T).$$

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by assuming the following conditions:

(1) $T_\omega := (1-\omega)I + \omega T$,

(2) T is demiclosed on H.

see for example [12] and [13].

Motivated by [10], we introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without the conditions (1) and (2) in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Throughout this paper, we denote weak and strong convergence by notations "$\rightharpoonup$" and "$\rightarrow$", respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C.

Remark 2.1. *It is well-known that metric projection P_C has the following properties:*

1. *P_C is firmly nonexpansive, i.e.,*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \ \forall x, y \in H.$$

2. *For each $x \in H$,*

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \ \forall y \in C.$$

Recall that H satisfies *Opial's condition* [14], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.2. *Let H be a real Hilbert space. Then there holds the following well-known results:*

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2,$
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$

for all $x, y \in H$.

Lemma 2.3 ([15])**.** *Let $(E, \langle .,. \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \ & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ & - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.\end{aligned}$$

Lemma 2.4 ([16])**.** *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.5 ([17])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 1$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6 ([10])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively, which $VI(C, D_1) \cap VI(C, D_2) \neq \emptyset$. Define a mapping $G : C \to C$ by*

$$G(x) = P_C (I - \lambda_1 D_1)(ax + (1-a) P_C (I - \lambda_2 D_2) x),$$

for every $\lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2)$ and $a \in (0,1)$. Then $F(G) = VI(C, D_1) \cap VI(C, D_2)$.

Lemma 2.7 ([18])**.** *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C (I - \lambda A) u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

The next result is very important for our main result.

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Then $VI(C, I-T) = F(T)$.*

Proof. It is easy to see that $F(T) \subseteq VI(C, I-T)$.
Let $u \in VI(C, I-T)$, then we have

$$\langle v - u, (I-T)u \rangle \ge 0, \ \forall v \in C. \tag{2.1}$$

Let $v^* \in F(T)$, then we have

$$\|Tu - v^*\|^2 \le \|u - v^*\|^2. \tag{2.2}$$

On the other hand

$$\begin{aligned} \|Tu - v^*\|^2 &= \|(u - v^*) - (I-T)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I-T)u \rangle + \|(I-T)u\|^2. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have

$$\|u - v^*\|^2 - 2\langle u - v^*, (I-T)u \rangle + \|(I-T)u\|^2 \le \|u - v^*\|^2.$$

From (2.1), we have

$$\|(I-T)u\|^2 \le 2\langle u - v^*, (I-T)u \rangle.$$

It follows that $u \in F(T)$. Hence $VI(C, I-T) \subseteq F(T)$. □

Remark 2.9. *From Lemma 2.7 and 2.8, we have*

$$F(T) = VI(C, I-T) = F(P_C (I - \lambda (I-T))),$$

for all $\lambda > 0$.

3 Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Gx_n, \ \forall n \geq 1, \tag{3.1}$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Suppose the following conditions holds:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. We divide the proof into five steps.
Step 1. We show that $\{x_n\}$ is bounded.
Let $x, y \in C$. Since A is α-inverse strongly monotone and $\lambda_1 \in (0, 2\alpha)$, we have

$$\begin{aligned}
\|(I - \lambda_1 A)x - (I - \lambda_1 A)y\|^2 &= \|x - y\|^2 - 2\lambda_1 \langle x - y, Ax - Ay\rangle + \lambda_1^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha\lambda_1 \|Ax - Ay\|^2 + \lambda_1^2 \|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda_1(\lambda_1 - 2\alpha) \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 .
\end{aligned}$$

Therefore $(I - \lambda_1 A)$ is a nonexpansive mapping. Similarly, $(I - \lambda_2 B)$ is a nonexpansive mapping. Hence $P_C(I - \lambda_1 A)$ and $P_C(I - \lambda_2 B)$ are nonexpansive mappings. From definition of the mapping G, we have G is a nonexpansive mapping.
Let $x^* \in \mathcal{F}$. From Remark 2.9, we have

$$x^* \in F\left(P_C(I - \lambda_n(I - T))\right).$$

By Lemma 2.6, we have

$$x^* = G(x^*) = P_C(I - \lambda_1 A)(ax^* + (1-a)P_C(I - \lambda_2 B)x^*).$$

Observe that

$$\begin{aligned}\|Tx_n - Tx^*\|^2 &= \|(x_n - x^*) - (I-T)x_n\|^2 \\ &= \|x_n - x^*\|^2 - 2\langle x_n - x^*, (I-T)x_n\rangle + \|(I-T)x_n\|^2 .\end{aligned}$$

Since T is a quasi-nonexpansive mapping, we have

$$\|(I-T)x_n\|^2 \leq 2\langle x_n - x^*, (I-T)x_n\rangle. \tag{3.2}$$

From the nonexpansiveness of P_C and (3.2), we have

$$\begin{aligned}\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 &= \|P_C(I-\lambda_n(I-T))x_n - P_C(I-\lambda_n(I-T))x^*\|^2 \\ &\leq \|(I-\lambda_n(I-T))x_n - (I-\lambda_n(I-T))x^*\|^2 \\ &= \|(x_n - x^*) - \lambda_n((I-T)x_n - (I-T)x^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, (I-T)x_n\rangle \\ &\quad + \lambda_n^2 \|(I-T)x_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \lambda_n \|(I-T)x_n\|^2 + \lambda_n^2 \|(I-T)x_n\|^2 \\ &\leq \|x_n - x^*\|^2 . \end{aligned} \tag{3.3}$$

Put $M_n = ax_n + (1-a)P_C(I-\lambda_2 B)x_n$ and $W_n = P_C(I-\lambda_1 A)M_n$. From (3.1), we have

$$x_{n+1} = \alpha_n u + \beta_n P_C(I-\lambda_n(I-T))x_n + \gamma_n W_n.$$

From the definition of x_n, (3.3) and nonexpansiveness of G, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(u-x^*) + \beta_n(P_C(I-\lambda_n(I-T))x_n - x^*) + \gamma_n(W_n - x^*)\| \\ &\leq \alpha_n \|u-x^*\| + \beta_n \|P_C(I-\lambda_n(I-T))x_n - x^*\| + \gamma_n \|W_n - x^*\| \\ &\leq \alpha_n \|u-x^*\| + \beta_n \|x_n - x^*\| \\ &\quad + \gamma_n\|P_C(I-\lambda_1 A)(ax_n + (1-a)P_C(I-\lambda_2 B)x_n) \\ &\quad - P_C(I-\lambda_1 A)(ax^* + (1-a)P_C(I-\lambda_2 B)x^*)\| \\ &= \alpha_n \|u-x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|G(x_n) - G(x^*)\| \\ &\leq \alpha_n \|u-x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= \alpha_n \|u-x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u-x^*\|, \|x_n - x^*\|\}.\end{aligned}$$

By induction, we can conclude that

$$\|x_n - x^*\| \leq \max\{\|u-x^*\|, \|x_1 - x^*\|\},$$

for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is bounded and so is $\{(I-T)x_n\}$.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
From the definition of x_n and nonexpansiveness of G, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| =& \|(\alpha_n - \alpha_{n-1})u + (\beta_n - \beta_{n-1})P_C(I - \lambda_{n-1}(I-T))x_{n-1} \\
&+ \beta_n(P_C(I - \lambda_n(I-T))x_n - P_C(I - \lambda_{n-1}(I-T))x_{n-1}) \\
&+ \gamma_n(W_n - W_{n-1}) + (\gamma_n - \gamma_{n-1})W_{n-1}\| \\
\le& |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ \beta_n\|P_C(I - \lambda_n(I-T))x_n - P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ \gamma_n\|W_n - W_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|W_{n-1}\| \\
\le& |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ \beta_n\|(x_n - x_{n-1}) - \lambda_n(I-T)x_n + \lambda_{n-1}(I-T)x_{n-1}\| \\
&+ \gamma_n\|P_C(I - \lambda_1 A)(ax_n + (1-a)P_C(I - \lambda_2 B)x_n) \\
&- P_C(I - \lambda_1 A)(ax_{n-1} + (1-a)P_C(I - \lambda_2 B)x_{n-1})\| \\
&+ |\gamma_n - \gamma_{n-1}|\|W_{n-1}\| \\
=& |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ \beta_n\|(x_n - x_{n-1}) - \lambda_n\left((I-T)x_n - (I-T)x_{n-1}\right) \\
&- (\lambda_n - \lambda_{n-1})(I-T)x_{n-1}\| + \gamma_n\|G(x_n) - G(x_{n-1})\| \\
&+ |\gamma_n - \gamma_{n-1}|\|W_{n-1}\| \\
\le& |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ \beta_n\|x_n - x_{n-1}\| + \lambda_n\|(I-T)x_n - (I-T)x_{n-1}\| \\
&+ |\lambda_n - \lambda_{n-1}|\,\|(I-T)x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|W_{n-1}\| \\
=& (1-\alpha_n)\|x_n - x_{n-1}\| + \lambda_n\|(I-T)x_n - (I-T)x_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|P_C(I - \lambda_{n-1}(I-T))x_{n-1}\| \\
&+ |\gamma_n - \gamma_{n-1}|\|W_{n-1}\| + |\lambda_n - \lambda_{n-1}|\,\|(I-T)x_{n-1}\| \\
\le& (1-\alpha_n)\|x_n - x_{n-1}\| + \lambda_n M + |\alpha_n - \alpha_{n-1}|M + |\beta_n - \beta_{n-1}|M \\
&+ |\gamma_n - \gamma_{n-1}|M + |\lambda_n - \lambda_{n-1}|M,
\end{aligned}$$

where $M := \max_{n\in\mathbb{N}}\{\|(I-T)x_{n+1} - (I-T)x_n\|,\ \|u\|,\ \|P_C(I - \lambda_n(I-T))x_n\|,\ \|W_n\|,$
$\|(I-T)x_n\|\}$.

From the condition $(ii), (iv), (v)$ and Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Step 3. We show that $\lim_{n\to\infty} \|P_C(I - \lambda_n(I-T))x_n - x_n\| = 0$.
Since $x^* = P_C(I - \lambda_1 A)(ax^* + (1-a)P_C(I - \lambda_2 B)x^*)$ and
$M^* = ax^* + (1-a)P_C(I - \lambda_2 B)x^*$, we have $x^* = P_C(I - \lambda_1 A)M^*$.

Since $x^* \in VI(C,B)$, we obtain

$$\begin{aligned} M^* - x^* &= (1-a)\left(P_C(I-\lambda_2 B)x^* - x^*\right) \\ &= (1-a)\left(P_C(I-\lambda_2 B)x^* - P_C(I-\lambda_2 B)x^*\right) \\ &= 0. \end{aligned} \tag{3.5}$$

From the definition of M_n and M^*, we have

$$\begin{aligned} \|M_n - M^*\| &= \|a(x_n - x^*) + (1-a)\left(P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\right)\| \\ &\le a\|x_n - x^*\| + (1-a)\|P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\| \\ &\le a\|x_n - x^*\| + (1-a)\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{3.6}$$

From the definition of W_n, we have

$$\begin{aligned} \|W_n - x^*\|^2 &= \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)M^*\|^2 \\ &\le \langle (I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*, W_n - x^*\rangle \\ &= \frac{1}{2}\left(\|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*\|^2 + \|W_n - x^*\|^2\right. \\ &\quad \left. - \|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^* - W_n + x^*\|^2\right) \\ &\le \frac{1}{2}\left(\|M_n - M^*\|^2 + \|W_n - x^*\|^2\right. \\ &\quad \left. - \|(M_n - W_n) - \lambda_1(AM_n - AM^*)\|^2\right), \end{aligned}$$

which implies that

$$\begin{aligned} \|W_n - x^*\|^2 &\le \|M_n - M^*\|^2 - \|(M_n - W_n) - \lambda_1(AM_n - AM^*)\|^2 \\ &= \|M_n - M^*\|^2 - \|M_n - W_n\|^2 + 2\lambda_1\langle M_n - W_n, AM_n - AM^*\rangle \\ &\quad - \lambda_1^2\|AM_n - AM^*\|^2. \end{aligned} \tag{3.7}$$

From the definition of W_n, we have

$$\begin{aligned} &\|W_n - x^*\|^2 \\ &= \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)M^*\|^2 \\ &\le \|(I-\lambda_1 A)M_n - (I-\lambda_1 A)M^*\|^2 \\ &= \|(M_n - M^*) - \lambda_1(AM_n - AM^*)\|^2 \\ &= \|M_n - M^*\|^2 - 2\lambda_1\langle M_n - M^*, AM_n - AM^*\rangle + \lambda_1^2\|AM_n - AM^*\|^2 \\ &\le \|M_n - M^*\|^2 - 2\lambda_1\alpha\|AM_n - AM^*\|^2 + \lambda_1^2\|AM_n - AM^*\|^2 \\ &= \|M_n - M^*\|^2 - \lambda_1(2\alpha - \lambda_1)\|AM_n - AM^*\|^2. \end{aligned} \tag{3.8}$$

From the definition of x_n, (3.3), (3.6) and (3.8), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq\ & \alpha_n\|u-x^*\|^2+\beta_n\|P_C(I-\lambda_n(I-T))x_n-x^*\|^2\\ & +\gamma_n\|W_n-x^*\|^2\\ \leq\ & \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2\\ & +\gamma_n\left(\|M_n-M^*\|^2-\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2\right)\\ \leq\ & \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\ & -\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2\\ =\ & \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2\\ & -\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2.\end{aligned}$$

It implies that

$$\begin{aligned}&\gamma_n\lambda_1(2\alpha-\lambda_1)\|AM_n-AM^*\|^2\\ &\quad\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad\leq \alpha_n\|u-x^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|). \qquad (3.9)\end{aligned}$$

From the condition (ii) and (3.4), we derive

$$\lim_{n\to\infty}\|AM_n-AM^*\|=0. \qquad (3.10)$$

From the definition of x_n, (3.3), (3.6) and (3.7), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|u-x^*\|^2+\beta_n\|P_C(I-\lambda_n(I-T))x_n-x^*\|^2+\gamma_n\|W_n-x^*\|^2\\ &\leq \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2\\ &\quad+\gamma_n(\|M_n-M^*\|^2-\|M_n-W_n\|^2+2\lambda_1\langle M_n-W_n,AM_n-AM^*\rangle\\ &\quad-\lambda_1^2\|AM_n-AM^*\|^2)\\ &\leq \alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2-\gamma_n\|M_n-W_n\|^2\\ &\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|\\ &= \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\gamma_n\|M_n-W_n\|^2\\ &\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|M_n-W_n\|^2 &\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|\\ &\leq \alpha_n\|u-x^*\|^2+\|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\ &\quad+2\lambda_1\|M_n-W_n\|\|AM_n-AM^*\|. \qquad (3.11)\end{aligned}$$

From the condition (ii), (3.4) and (3.10), we derive

$$\lim_{n\to\infty} \|M_n - W_n\| = 0. \tag{3.12}$$

From the property of P_C, we have

$$\begin{aligned}
\|P_C(I-\lambda_2 B)x_n - x^*\|^2 =& \|P_C(I-\lambda_2 B)x_n - P_C(I-\lambda_2 B)x^*\|^2 \\
\leq& \langle (I-\lambda_2 B)x_n - (I-\lambda_2 B)x^*, P_C(I-\lambda_2 B)x_n - x^*\rangle \\
=& \frac{1}{2}\ (\|(I-\lambda_2 B)x_n - (I-\lambda_2 B)x^*\|^2 \\
&+ \|P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
&- \|(I-\lambda_2 B)x_n - (I-\lambda_2 B)x^* - P_C(I-\lambda_2 B)x_n + x^*\|^2) \\
\leq& \frac{1}{2}\ (\|x_n - x^*\|^2 + \|P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
&- \|(x_n - P_C(I-\lambda_2 B)x_n) - \lambda_2(Bx_n - Bx^*)\|^2).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|P_C(I-\lambda_2 B)x_n - x^*\|^2 \leq& \|x_n - x^*\|^2 \\
&- \|(x_n - P_C(I-\lambda_2 B)x_n) - \lambda_2(Bx_n - Bx^*)\|^2 \\
=& \|x_n - x^*\|^2 - \|x_n - P_C(I-\lambda_2 B)x_n\|^2 \\
&+ 2\lambda_2\langle x_n - P_C(I-\lambda_2 B)x_n, Bx_n - Bx^*\rangle \\
&- \lambda_2^2 \|Bx_n - Bx^*\|^2 .
\end{aligned} \tag{3.13}$$

By using the same method as (3.8), we have

$$\|P_C(I-\lambda_2 B)x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_2(2\beta - \lambda_2)\|Bx_n - Bx^*\|^2 . \tag{3.14}$$

Since $x^* \in VI(C, A)$, we have

$$\begin{aligned}
\|W_n - x^*\|^2 =& \|P_C(I-\lambda_1 A)M_n - P_C(I-\lambda_1 A)x^*\|^2 \\
\leq& \|ax_n + (1-a)P_C(I-\lambda_2 B)x_n - x^*\|^2 \\
=& \|a(x_n - x^*) + (1-a)(P_C(I-\lambda_2 B)x_n - x^*)\|^2 \\
\leq& a\|x_n - x^*\|^2 + (1-a)\|P_C(I-\lambda_2 B)x_n - x^*\|^2 .
\end{aligned} \tag{3.15}$$

From the definition of x_n, (3.3), (3.14) and (3.15), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 + \gamma_n\|W_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \beta_n\|P_C(I-\lambda_n(I-T))x_n - x^*\|^2 \\
&\quad + \gamma_n\left(a\|x_n - x^*\|^2 + (1-a)\|P_C(I-\lambda_2 B)x_n - x^*\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\|^2 \\
&\quad + \gamma_n \left(a \|x_n - x^*\|^2 + (1 - a) \left(\|x_n - x^*\|^2 - \lambda_2(2\beta - \lambda_2) \|Bx_n - Bx^*\|^2\right)\right) \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n(a \|x_n - x^*\|^2 + (1 - a) \|x_n - x^*\|^2 \\
&\quad - (1 - a)\lambda_2(2\beta - \lambda_2) \|Bx_n - Bx^*\|^2) \\
&= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - (1 - a)\lambda_2\gamma_n(2\beta - \lambda_2) \|Bx_n - Bx^*\|^2 .
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1 - a)\lambda_2\gamma_n(2\beta - \lambda_2) \|Bx_n - Bx^*\|^2 \\
&\qquad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|) . \qquad (3.16)
\end{aligned}$$

From the condition (ii) and (3.4), we have

$$\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0. \qquad (3.17)$$

From the definition of x_n, (3.3) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\|^2 \\
&\quad + \gamma_n \left(a \|x_n - x^*\|^2 + (1 - a) \|P_C(I - \lambda_2 B)x_n - x^*\|^2\right) \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n(a \|x_n - x^*\|^2 \\
&\quad + (1 - a)(\|x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\langle x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\rangle - \lambda_2^2 \|Bx_n - Bx^*\|^2)) \\
&\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 \\
&\quad - \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\gamma_n(1 - a)\langle x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\rangle \\
&\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\quad + 2\lambda_2\gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\| .
\end{aligned}$$

This implies that

$$\begin{aligned}
&\gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2 \\
&\qquad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad\quad + 2\lambda_2\gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\| \\
&\qquad \leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\qquad\quad + 2\lambda_2\gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\| . \qquad (3.18)
\end{aligned}$$

From the condition (ii), (3.4) and (3.17), we derive

$$\lim_{n\to\infty} \|x_n - P_C(I - \lambda_2 B)x_n\| = 0.$$

Since

$$\begin{aligned}\|M_n - x_n\| &= \|ax_n + (1-a)P_C(I-\lambda_2 B)x_n - x_n\| \\ &= (1-a)\|P_C(I-\lambda_2 B)x_n - x_n\|\end{aligned}$$

and $\|P_C(I-\lambda_2 B)x_n - x_n\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|M_n - x_n\| = 0. \tag{3.19}$$

From (3.12) and (3.19), we have

$$\lim_{n\to\infty} \|W_n - x_n\| = 0. \tag{3.20}$$

Since

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \beta_n(P_C(I-\lambda_n(I-T))x_n - x_n) + \gamma_n(W_n - x_n),$$

it implies by the condition (ii), the condition (iii), (3.4) and (3.20) that

$$\lim_{n\to\infty} \|P_C(I-\lambda_n(I-T))x_n - x_n\| = 0. \tag{3.21}$$

Step 4. We show that $\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0$, where $z_0 = P_{\mathcal{F}}u$. To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle = \lim_{j\to\infty}\langle u - z_0, x_{n_j} - z_0\rangle.$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, where $\omega \in C$. First, we show that $\omega \in F(T)$. From Remark 2.9, we have $F(T) = VI(C, I-T) = F(P_C(I-\lambda_{n_j}(I-T)))$. Assume that $\omega \notin F(T)$, that $\omega \ne P_C(I-\lambda_{n_j}(I-T))\omega$. By $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, (3.21) and Opial's property, we have

$$\begin{aligned}\liminf_{j\to\infty}\left\|x_{n_j} - \omega\right\| &< \liminf_{j\to\infty}\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))\omega\right\| \\ &\le \liminf_{j\to\infty}(\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))x_{n_j}\right\| \\ &\quad + \left\|P_C(I-\lambda_{n_j}(I-T))x_{n_j} - P_C(I-\lambda_{n_j}(I-T))\omega\right\|) \\ &\le \liminf_{j\to\infty}(\left\|x_{n_j} - P_C(I-\lambda_{n_j}(I-T))x_{n_j}\right\| \\ &\quad + \left\|x_{n_j} - \omega\right\| + \lambda_{n_j}\left\|(I-T)x_{n_j} - (I-T)\omega\right\|) \\ &\le \liminf_{j\to\infty}\left\|x_{n_j} - \omega\right\|.\end{aligned}$$

This is a contradiction, we have

$$\omega \in F(T). \tag{3.22}$$

Next, we show that $\omega \in VI(C,A) \cap VI(C,B)$. From Lemma 2.6, we have $VI(C,A) \cap VI(C,B) = F(G)$. From (3.20), we have $W_{n_j} \rightharpoonup \omega$ as $j \to \infty$.

$$\begin{aligned}\|W_n - G(W_n)\| &= \|P_C(I-\lambda_1 A)\left(ax_n + (1-a)P_C(I-\lambda_2 B)x_n\right) - G(W_n)\| \\ &= \|G(x_n) - G(W_n)\| \\ &\le \|x_n - W_n\|.\end{aligned}$$

From (3.20), we have

$$\lim_{n\to\infty} \|W_n - G(W_n)\| = 0.$$

From $W_{n_j} \rightharpoonup \omega$ as $j \to \infty$ and Lemma 2.4, we have

$$\omega \in F(G) = VI(C,A) \cap VI(C,B). \tag{3.23}$$

From (3.22) and (3.23), we have $\omega \in \mathcal{F}$. Since $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle u - z_0, x_n - z_0\rangle &= \lim_{j\to\infty} \langle u - z_0, x_{n_j} - z_0\rangle \\ &= \langle u - z_0, \omega - z_0\rangle \le 0. \end{aligned} \tag{3.24}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$. From the definition of x_n and $z_0 = P_{\mathcal{F}}u$, we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + \beta_n(P_C(I - \lambda_n(I-T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2 \\ &\le \|\beta_n(P_C(I - \lambda_n(I-T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2 \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le \beta_n \|P_C(I - \lambda_n(I-T))x_n - z_0\|^2 + \gamma_n \|W_n - z_0\|^2 \\ &\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le \beta_n \|x_n - z_0\|^2 + \gamma_n \|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &= (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.\end{aligned}$$

From the condition (ii), (3.24) and Lemma 2.5, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}u$. This completes the proof. □

From our main result, Lemma 1.1 and Lemma 2.6, we have the following corollary:

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = F(G) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I-T))x_n + \gamma_n G x_n, \ \forall n \ge 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Suppose the following conditions holds:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \le \beta_n \le c < 1$ *for all* $n \ge 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$ *and* (z_0, y_0) *is a solution of (1.3), where* $y_0 = P_C (I - \lambda_2 B) z_0$.

4 Application

In this section, we prove strong convergence theorems involving the set of fixed points of nonspreading mapping.

A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$

The such mapping is defined by Kohsaka and Takahashi [19].

The following lemma is needed to prove in application.

Lemma 4.1 ([19]). *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H*, and let* S *be a nonspreading mapping of* C *into itself. Then* $F(S)$ *is closed and convex.*

In 2009, Kangtunyakarn and Suantai [20] introduced the S-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as following. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

For every $i = 1, 2, ..., N$. Put $\alpha_3^i = 0$ in Definition 4.1, then the S-mapping is reduced to the K-mapping defined by Kangtunyakarn and Suantai [21] as following. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself, and let $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, ..., N$. We define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\
U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\
U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\
&\ \ \vdots \\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\
K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}.
\end{aligned}$$

Such a mapping K is called the *K-mapping* generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$.

Lemma 4.2 ([22])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0, 1)$ for all $j = 1, 2, ..., N - 1$ and $\alpha_1^N \in (0, 1], \alpha_3^N \in [0, 1)$ $\alpha_2^j \in [0, 1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, T_2, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.*

Lemma 4.3 ([23])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, ..., \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, ..., N - 1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$ and K is quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, ..., N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, α_1^j, $\alpha_3^j \in (0, 1)$ for all $j = 1, 2, ..., N - 1$ and $\alpha_1^N \in$*

$(0,1], \alpha_3^N \in [0,1)$ $\alpha_2^j \in [0,1)$ *for all* $j = 1, 2, ..., N$. *Let* S *be the mapping generated by* $T_1, T_2, ..., T_N$ *and* $\alpha_1, \alpha_2, ..., \alpha_N$. *Let* $A, B : C \to H$ *be* α, β*-inverse strongly monotone mappings, respectively. Define the mapping* $G : C \to C$ *by* $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ *for all* $x \in C$. *Assume* $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Suppose that* $x_1, u \in C$ *and let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - S))x_n + \gamma_n G x_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $[0,1]$. *Suppose the following conditions hold:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$.

Proof. By using Theorem 3.1 and Lemma 4.2, we obtain the conclusion. □

Theorem 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonspreading mappings of* C *into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, ..., \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, ..., N-1$ *and* $0 < \lambda_N \leq 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, ..., T_N$ *and* $\lambda_1, \lambda_2, ..., \lambda_N$. *Let* $A, B : C \to H$ *be* α, β*-inverse strongly monotone mappings, respectively. Define the mapping* $G : C \to C$ *by* $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ *for all* $x \in C$. *Assume* $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Suppose that* $x_1, u \in C$ *and let* $\{x_n\}$ *be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - K))x_n + \gamma_n G x_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are sequences in* $[0,1]$. *Suppose the following conditions hold:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^\infty |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$.

Proof. By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion. □

The following result is direct proved from Theorem 4.4. Therefore, we omit the prove.

Corollary 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space. Let T be a nonspreading mappings of C into itself with $F(T) \neq \emptyset$. Let $A, B : C \to H$ be α, β-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1-a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $\mathcal{F} = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Gx_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ *for all* $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

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