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On Connected Cayley Graphs of Semigroups

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Abstract : It is well known that for a finite group G with a nonempty subset A, the Cayley graph of G with respect to A is connected if and only if A is a generating set of G. In this paper we generalize this result to the case of semigroups and give some necessary and sufficient conditions for Cayley graphs of a semigroup to be connected. We also characterize weakly connected Cayley graphs of a semigroup and indicate a relationship between a Cayley graph of a semigroup and the Hasse diagram of a partially ordered set.

Keywords : Cayley graph of semigroup; right simple semigroup; connected digraph; strongly connected graph; weakly connected graph.
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1 Introduction

Cayley graphs of a semigroup have been extensively studied because they reflect the structure of the semigroup. Further, we can visualize a semigroup by constructing its Cayley graphs. Numerous interesting results have been found. For instance, the monograph [1] includes one section devoted to the study of Cayley graphs of a semigroup, and [2] includes fundamental properties and recent research on Cayley graphs of a semigroup. Many properties of Cayley graphs of particular types of semigroups have been investigated, see for instance [3, 4, 5].

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Recall the well-known property of Cayley graphs of a group: *for a finite group G with a nonempty subset A, the Cayley graph of G with respect to A is (strongly) connected if and only if A is a generating set of G.* This is a necessary and sufficient condition for Cayley graphs of a group to be connected. The main purpose of this article is to find out such condition for semigroups. We also characterize weakly connected Cayley graphs of a semigroup, using Green's relations taken relative to a subsemigroup. Finally, we connect a Cayley graph of a semigroup with the Hasse diagram of a partially ordered set.

In the next section we give the definition of a Cayley graph of a semigroup and summarize basic knowledge of graphs and semigroups. In Section 3 we give a condition to determine whether or not a Cayley graph of a semigroup is strongly connected. In Section 4 we characterize weakly connected Cayley graphs of a semigroup.

2 Preliminaries

For a finite semigroup S and a nonempty subset A of S, we define the *(right) Cayley graph* of S with respect to A, denoted by $\mathrm{Cay}\,(S, A)$, to be the directed graph with vertex set S and edge set $\{(s, sa) \colon s \in S \text{ and } a \in A\}$. To shorten the notation, we write $\mathrm{Cay}\,(S, a)$ instead of $\mathrm{Cay}\,(S, \{a\})$.

We introduce the basic theory of graphs and semigroups, following [6, 7] and [8, 9].

Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. A digraph H is a *subgraph* of D provided that $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. If a subgraph H of D has the additional property that for any pair of vertices u and v in H, $(u, v) \in E(H)$ if and only if $(u, v) \in E(D)$, then H is called an *induced subgraph* of D. Let A be a set of vertices of D. The subgraph of D induced by A, denoted by $\mathrm{ind}\, A$, is the induced subgraph of D with vertex set A.

An alternating sequence $u = u_0, e_1, u_1, \dots, e_n, u_n = v$ of vertices and edges in a digraph is a u-v *(directed) walk* of length n if $e_i = (u_{i-1}, u_i)$ for $i = 1, 2, \dots, n$. A u-v walk is a *(directed) path* if the vertices are all distinct and is *closed* if $u = v$. A closed walk of length at least 2 containing no repeated vertex except for the beginning and the end is called a *(directed) cycle*. A *u-v semiwalk* is an alternating sequence $u = u_0, e_1, u_1, \dots, e_n, u_n = v$ of vertices and edges in which either $e_i = (u_{i-1}, u_i)$ or $e_i = (u_i, u_{i-1})$ for all i. A semipath and semicycle are defined in a similar way. Note that every path is a semipath and that every cycle is a semicycle.

We will use the result that if there are semipaths both from u to v and from v to w, then there is a u-w semipath. More precisely, we have

Proposition 2.1. *Let D be a digraph. For any two distinct vertices u and v, define*

$$u \sim v \quad \Leftrightarrow \quad \text{there is a semipath from } u \text{ to } v.$$

Then $\sim$ is symmetric and transitive.

When the word "semipath" is replaced by "path", the relation defined above is only transitive.

The notion of paths and semipaths is used to describe connectedness of a digraph as follows. Let D be a digraph and let u and v be distinct vertices of D. The digraph D is *strongly connected (or strong)* if a u-v path and v-u path exist. It is *unilaterally connected (or unilateral)* if a u-v path or v-u path exists. It is *weakly connected (or weak)* if a u-v semipath or v-u semipath exists. A digraph is said to be *disconnected* if it is not even weak. Note that the trivial digraph is vacuously strong for it cannot contain two different vertices.

Let A be a nonempty subset of a semigroup S. The subsemigroup of S generated by A is denoted by $\langle A\rangle$ and consists of the elements of S that can be expressed as finite products of elements in A. In particular, we have $\langle a\rangle = \{a^n : n \in \mathbb{N}\}$. If A is such that $\langle A\rangle = S$, then A is called a *generating set* of S or, equivalently, S is generated by A. Clearly, S is a generating set of itself.

A semigroup S is said to be *right simple* provided that it contains no proper right ideals. It is straightforward to check that S is right simple if and only if $aS = S$ for all a in S if and only if the linear equation $ax = b$ in the variable x possesses a solution in S for all $a, b \in S$.

Recall that an equivalence relation is a relation that is reflexive, symmetric, and transitive. Let ρ be an equivalence relation on a set S and let $s \in S$. Denote by $s\rho$ the equivalence class containing s and by S/ρ the collection $\{s\rho \colon s \in S\}$ of equivalence classes of S. An element t of S is a *representative* for $s\rho$ if $t \in s\rho$.

Let S be a semigroup with a subsemigroup T and let aT^1 denote the set $aT \cup \{a\}$. Following [10], Green's relation taken relative to T, written $\mathcal{R}^T$, is defined by

$$a\,\mathcal{R}^T\,b \quad\Leftrightarrow\quad aT^1 = bT^1 \tag{2.1}$$

for $a, b \in S$. This relation will play an important role in Sections 3 and 4.

A partial order is a relation that is reflexive, antisymmetric, and transitive. An ordered set $(S, \leq)$ consists of a set S together with a partial order $\leq$ on S. Two ordered sets $(S, \leq)$ and $(T, \leq)$ are *order-isomorphic* if there exists a surjective map φ from S onto T that preserves the order, that is, $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$ for all $a, b \in S$.

As in the case of Cayley graphs of a semigroup, one may visualize the structure of a finite ordered set by the so-called *Hasse diagram*, see [11] for more details.

Throughout the article by a graph we mean a directed graph in which loops are permitted but multiple edges are not; S stands for an arbitrary finite semigroup and A stands for a nonempty subset of S. There is no loss of generality when we work with Cayley graphs of *nontrivial* semigroups.

3 Strongly Connected Graphs

The condition given in the introduction determines whether a Cayley graph of a *group* is connected. Such condition does not guarantee connectedness in the case of semigroups, however.

We present two simple examples, which show that the condition that A is a generating set of S is neither necessary nor sufficient for Cayley graphs of a semigroup to be connected. Let $S_4 = \{s_1, s_2, s_3, s_4\}$ and $T_4 = \{t_1, t_2, t_3, t_4\}$ be semigroups with multiplication defined by Table 1 and Table 2. It is easy to see that $A = \{s_3, s_4\}$ is a generating set of S_4, but the Cayley graph $\mathrm{Cay}\,(S_4, A)$ is disconnected (see Figure 1). Moreover, the Cayley graph $\mathrm{Cay}\,(T_4, t_3)$ is weakly connected, even though $\{t_3\}$ does not generate T_4.

$\cdot$	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_4	s_4
s_2	s_2	s_2	s_3	s_3
s_3	s_3	s_3	s_2	s_2
s_4	s_4	s_4	s_1	s_1

Table 1. Multiplication table for S_4

$\cdot$	t_1	t_2	t_3	t_4
t_1	t_1	t_2	t_3	t_4
t_2	t_1	t_2	t_3	t_4
t_3	t_3	t_4	t_1	t_2
t_4	t_3	t_4	t_1	t_2

Table 2. Multiplication table for T_4

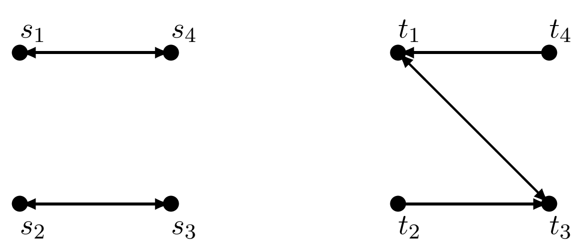

Figure 1. $\mathrm{Cay}\,(S, \{s_3, s_4\})$ and $\mathrm{Cay}\,(T, t_3)$

The next theorem gives a criterion for the existence of a path from one vertex to another. Consequently, necessary and sufficient conditions for connectedness of Cayley graphs of a semigroup are obtained. This result is consistent with Lemma 5.1 of [3]. In light of the proof of Theorem 3.1, we also obtain Proposition 3.4.

Theorem 3.1. *Let u and v be distinct elements of S. A path from u to v in* $\mathrm{Cay}\,(S, A)$ *exists if and only if the linear equation $ux = v$ in the variable x has a solution in $\langle A \rangle$. Furthermore,* $\mathrm{Cay}\,(S, A)$ *has a cycle containing u or a loop at u if and only if $ux = u$ has a solution in $\langle A \rangle$.*

Proof. Let $u = s_0, s_1, \ldots, s_n = v$ be a path. By definition, there are elements $a_1, a_2, \ldots, a_n$ of A such that $s_i = s_{i-1}a_i$ for $i = 1, 2, \ldots, n$. This proves $v = u(a_1a_2 \cdots a_n)$, and $a_1a_2 \cdots a_n \in \langle A \rangle$ is a solution to the equation.

Conversely, there is an $x_0 \in \langle A \rangle$ for which $ux_0 = v$. Since $x_0 \in \langle A \rangle$, we can write x_0 as $a_1a_2 \cdots a_n$, where $a_i \in A$. Define $x_1 = ua_1$ and $x_i = x_{i-1}a_i$ for $i = 2, 3, \ldots, n$. Hence, $x_n = v$ and (u, x_1), (x_1, x_2), $\ldots$, (x_{n-1}, v) are edges in $\mathrm{Cay}\,(S, A)$. Note that loops and cycles may occur in these edges. However, we can exclude them to obtain a u-v path. The remaining part can be proved in a similar way. □

Corollary 3.2. *A Cayley graph* $\mathrm{Cay}\,(S, A)$ *is strong if and only if the linear equation $ux = v$ in the variable x has a solution in $\langle A \rangle$ for all $u, v \in S$.*

Proof. To complete the proof, we need only show that a solution for $ux = u$ exists for $u \in S$. Since S is a nontrivial semigroup, there is an element $w \neq u$ of S such that a u-w path and w-u path exist. Hence, $ux_0 = w$ and $wy_0 = u$ for some $x_0, y_0 \in \langle A \rangle$ and so $ux_0y_0 = u$. □

Corollary 3.3. *A Cayley graph* $\mathrm{Cay}\,(S, A)$ *is unilateral if and only if for all $u, v \in S$ with $u \neq v$, the linear equation $ux = v$ or $vx = u$ in the variable x has a solution in $\langle A \rangle$.*

Proposition 3.4. *If the subsemigroup $\langle A \rangle$ of S has order m, then the Cayley graph* $\mathrm{Cay}\,(S, A)$ *contains a path of length at most m.*

Proof. The statement is clear if there is no path in $\mathrm{Cay}\,(S, A)$. We may therefore assume that $u = s_0, s_1, \ldots, s_n = v$ is a path in $\mathrm{Cay}\,(S, A)$. Repeating the argument given in the proof of Theorem 3.1, we have $s_i = s_0a_1 \cdots a_i$ for $i = 1, 2, \ldots, n$. Furthermore, the elements a_1, a_1a_2, $\ldots$, $a_1a_2 \cdots a_n$ must be distinct for if there were indices $j \neq k$ such that $a_1a_2 \cdots a_j = a_1a_2 \cdots a_k$, then we would have $s_j = s_k$, contrary to the definition of a path. Since $a_1, a_1a_2, \ldots, a_1a_2 \cdots a_n$ belong to $\langle A \rangle$, we conclude that $n \leq m$. □

In order to obtain a result as in the group case, we have to restrict classes of semigroups to the class of right simple semigroups. In fact, if S is not a right simple semigroup, then a Cayley graph $\mathrm{Cay}\,(S, A)$ fails to be strongly connected for any subset A of S.

The following theorem is a generalization of the statement given in the introduction as every group is a right simple semigroup. Recall that every (right) linear equation in a right simple semigroup always has a solution.

Theorem 3.5. *Let S be a right simple semigroup and let $A \subseteq S$. A Cayley graph* $\mathrm{Cay}\,(S, A)$ *is strong if and only if $\langle A \rangle = S$.*

Proof. Suppose that $y \in S$ and $a \in A$. By Corollary 3.2, the equation $y = ax$ has a solution in $\langle A \rangle$. Hence, y belongs to $\langle A \rangle$. This proves $S \subseteq \langle A \rangle$ and so equality holds. The converse holds trivially by the same corollary. □

Corollary 3.6. *There is a subset A of S such that the Cayley graph* $\mathrm{Cay}\,(S, A)$ *is strong if and only if S is right simple.*

Proof. For each $a \in S$, we have $aS \subseteq S$. For each $s \in S$, we can pick, by Corollary 3.2, an element x_0 of $\langle A \rangle$ such that $s = ax_0 \in aS$. Hence, $S \subseteq aS$ and so equality holds. This proves that S is right simple. Verification of the converse is straightforward. Since $\langle S \rangle = S$, the Cayley graph $\mathrm{Cay}\,(S, S)$ is strong. □

We close this section by introducing the equivalence relation that will be useful in the next section. According to (2.1), if $T = \langle A \rangle$, then for all $u, v \in S$,

$$u \,\mathcal{R}^{\langle A \rangle}\, v \quad \Leftrightarrow \quad u\langle A \rangle^1 = v\langle A \rangle^1. \tag{3.1}$$

For simplicity, we write π instead of $\mathcal{R}^{\langle A \rangle}$ and $A(u, v)$ instead of the cumbersome expression $u\langle A \rangle \cap v\langle A \rangle$. The following theorem gives a characterization of π.

Theorem 3.7. *Let u and v be distinct elements of S. Then $u \,\pi\, v$ if and only if there exist elements a and b of $\langle A \rangle$ such that $v = ua$ and $u = vb$.*

Proof. Clearly, $u \in v\langle A \rangle^1$ and $v \in u\langle A \rangle^1$ because $u\langle A \rangle^1 = v\langle A \rangle^1$. Since $u \neq v$, there is an element $a \in \langle A \rangle$ for which $v = ua$. Repeating this argument shows that $u = vb$ for some b in $\langle A \rangle$.

Let $x \in u\langle A \rangle^1$. If $x = u$, then $x = vb \in v\langle A \rangle^1$. Otherwise, $x = uc$ for some $c \in \langle A \rangle$. It follows that $x = vbc \in v\langle A \rangle^1$ and hence $u\langle A \rangle^1 \subseteq v\langle A \rangle^1$. One obtains similarly that $v\langle A \rangle^1 \subseteq u\langle A \rangle^1$. This proves $u \,\pi\, v$. □

From Theorem 3.1, we conclude that $u \,\pi\, v$ if and only if the Cayley graph contains a u-v path and v-u path. This leads to a remarkable consequence: for an equivalence class $s\pi$ in S/π, the induced subgraph $\mathrm{ind}\, s\pi$ of a graph $\mathrm{Cay}\,(S, A)$ is always strongly connected. We state this result as Theorem 3.8. Note that this theorem enables us to study weakly connected graphs through the collection S/π of equivalence classes instead of the entire semigroup S.

Theorem 3.8. *The induced subgraph* $\operatorname{ind} s\pi$ *of* $\operatorname{Cay}(S, A)$ *is strongly connected.*

Proof. If $s\pi$ consists exactly of one vertex, then the graph $\operatorname{ind} s\pi$ is vacuously strong. We may therefore assume that there are at least two distinct vertices in $s\pi$, say u and v. Hence $u \,\pi\, v$, and a u-v path and v-u path exist, namely

$$u = u_0, u_1, \ldots, u_m = v \quad \text{and} \quad v = v_0, v_1, \ldots, v_n = u.$$

It remains to prove that both paths lie in the graph $\operatorname{ind} s\pi$ or, equivalently, u_i and v_j belong to $s\pi$ for all i, j. In fact, the existence of u-u_i, u_i-v, and v-u paths ensures that $u_i \,\pi\, u$, that is, $u_i \in s\pi$. Similarly, we have $v_j \in s\pi$ for all j. □

4 Weakly Connected Graphs

We begin with a condition that guarantees the existence of a semipath, and then use this result to characterize weakly connected Cayley graphs of a semigroup, as shown in Theorem 4.3. Recall that the notation $A(u, v)$ stands for the intersection $u\langle A\rangle \cap v\langle A\rangle$.

Lemma 4.1. *Let u and v be distinct vertices of* $\operatorname{Cay}(S, A)$. *If $A(u, v) \neq \emptyset$, then there is a semipath between u and v.*

Proof. Let z be an element of $A(u, v)$. Then $ua = z = vb$ for some $a, b \in \langle A\rangle$. If $z = u$, then a v-u path exists. If $z = v$, then a u-v path exists. In the case $z \notin \{u, v\}$, there are a u-z path and v-z path. Hence, a u-v semipath exists. □

The converse of Lemma 4.1 is not in general true, as the following example indicates. Let $S_6 = \{s_1, s_2, \ldots, s_6\}$ be a semigroup with multiplication defined by Table 3. In the case $A = \{s_4, s_6\}$, the Cayley graph $\operatorname{Cay}(S, A)$ contains two s_2-s_4 semipaths and yet $A(s_2, s_4) = s_2S \cap s_4S = \emptyset$ (see Figure 2).

$\cdot$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_2	s_3	s_4	s_5	s_6
s_2	s_2	s_2	s_2	s_5	s_5	s_5
s_3	s_3	s_3	s_3	s_4	s_4	s_4
s_4	s_4	s_4	s_4	s_3	s_3	s_3
s_5	s_5	s_5	s_5	s_2	s_2	s_2
s_6	s_6	s_4	s_5	s_2	s_3	s_1

Table 3. Multiplication table for S_6

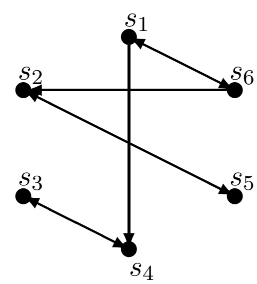

Figure 2. Cay $(S, \{s_4, s_6\})$

For simplicity, let π denote the relation defined by (3.1). Because S is a finite semigroup, we can let

$$S/\pi = \{s_i\pi \colon s_i \in S, i = 1, 2, \ldots, m\} \quad \text{and} \quad P = \{s_1, s_2, \ldots, s_m\} \tag{4.1}$$

be the collection of all distinct π-classes and the set of all distinct representatives, respectively.

Lemma 4.2. *Suppose that s and t are representatives for $u\pi$ and $v\pi$. If there is an s-t path or a t-s path, then $A(u, v)$ is nonempty.*

Proof. If an s-t path exists, then $t = sa$ for some $a \in \langle A\rangle$. By assumption and Theorem 3.7, $s = u$ or $s = ub$, and $t = v$ or $t = vc$ for some $b, c \in \langle A\rangle$. It follows that $sab = tb$ belongs to $A(u, v)$. The same reasoning applies to the case of a t-s path. □

For convenience, let us introduce terminology that will be used in the proof of Theorem 4.3. Note that we can divide any semipath in a graph into partial paths. To be more precise, given a semipath $u_1, u_2, \ldots, u_n$, a semipath $u_p, u_{p+1}, \ldots, u_q$ with $1 \le p < q \le n$ is called a *subpath* of the u_1-u_n semipath if it is either a u_p-u_q path or u_q-u_p path. A subpath $u_p, u_{p+1}, \ldots, u_q$, which is denoted by $[u_p, u_q]$, is said to have the most length if one of the following conditions holds:

(1) $p = 1$ and $q = n$

(2) $p = 1, q \neq n$ and $u_p, \ldots, u_q, u_{q+1}$ is not a path

(3) $p \neq 1, q = n$ and $u_{p-1}, u_p, \ldots, u_q$ is not a path

(4) $p \neq 1, q \neq n$, $u_p, \ldots, u_q, u_{q+1}$ and $u_{p-1}, u_p, \ldots, u_q$ are not paths.

Theorem 4.3. *Let P be the set defined by (4.1). A Cayley graph* Cay (S, A) *is weak if and only if for all $u, v \in P$ with $u \neq v$, there exists a sequence $u = t_0, t_1, \ldots, t_n = v$ of distinct elements in P such that*

$$A(t_{i-1}, t_i) \neq \emptyset \quad \textit{for } i = 1, 2, \ldots, n. \tag{4.2}$$

Proof. Assume that $u, v \in P$ with $u \neq v$. It is clear that u and v are not related by π and that a u-v semipath exists, say $u = u_0, u_1, \ldots, u_p = v$. Divide the semipath into k subpaths such that each of them has the most length, namely that it consists of subpaths $[u_{r_0}, u_{r_1}], [u_{r_1}, u_{r_2}], \ldots, [u_{r_{k-1}}, u_{r_k}]$, where $0 = r_0 < r_1 < \cdots < r_k = p$. Set $v_i = u_{r_i}$ for $i = 0, 1, \ldots, k$. For each i, either a v_{i-1}-v_i path or v_i-v_{i-1} path exists since $[v_{i-1}, v_i]$ is a subpath. Next, define $V := \{v_0, v_1, \ldots, v_k\}$ and $V_1 := \{s \in V : s\,\pi\, u\}$. Obviously, $V_1 \neq \emptyset$ and so $n_1 := \max\{0 \leq i \leq k : v_i \in V_1\}$ is meaningful and $v_{n_1+1} \notin V_1$. The inequality $n_1 < k$ is true since $v_k \notin V_1$, so we may define $V_2 := \{s \in V : s\,\pi\, v_{n_1+1}\}$ and $n_2 := \max\{0 \leq i \leq k : v_i \in V_2\}$. Observe that $n_1 + 1 \leq n_2 \leq k$. If $n_2 = k$, then $v_{n_1+1} \in v\pi$. This combined with the existence of a v_{n_1}-v_{n_1+1} path or v_{n_1+1}-v_{n_1} path implies $A(u, v) \neq \emptyset$ by Lemma 4.2, and the sequence u, v is what we wanted. Otherwise, $n_2 < k$ and we define $V_3 := \{s \in V : s\,\pi\, v_{n_2+1}\}$ and $n_3 := \max\{0 \leq i \leq k : v_i \in V_3\}$. For $n_3 = k$, we have $v \in V_3$. Because $v_{n_1+1}\,\pi\, v_{n_2}$, two possibilities arise: (1) $n_1 + 1 = n_2$ or (2) a v_{n_1+1}-v_{n_2} path and v_{n_2}-v_{n_1+1} path exist. In either case, there is a path from v_{n_1} to v_{n_2} or from v_{n_2} to v_{n_1}. Furthermore, since S/π forms a disjoint partition of S, there is an index j_2 in $\{1, 2, \ldots, m\}$ for which $v_{n_2} \in s_{j_2}\pi$. By Lemma 4.2, $A(u, s_{j_2}) \neq \emptyset$. Also, $A(s_{j_2}, v)$ is nonempty. Since $s_{j_2} \notin \{u, v\}$, the sequence u, s_{j_2}, v works. If $n_3 < k$, we will continue this process to define V_i and n_i. The finiteness of V together with $n_1 < n_2 < \cdots$ ensures that the process must terminate and the required sequence arises.

To prove the converse, let u and v be distinct elements of S. Of course, a u-v path exists if $u\,\pi\, v$. Otherwise, $u \in s_j\pi$ and $v \in s_k\pi$, where $s_j, s_k \in P$ and $j \neq k$. By assumption, there is a sequence $s_j = t_0, t_1, \ldots, t_n = s_k$ having the property described in (4.2). Since $A(t_{i-1}, t_i) \neq \emptyset$, there is a semipath from t_{i-1} to t_i for each i by Lemma 4.1. An application of Proposition 2.1 shows that an s_j-s_k semipath exists and so a u-v semipath exists because $u \in s_j\pi$ and $v \in s_k\pi$. □

Corollary 4.4. *A graph* Cay (S, A) *is weak if* $A(u, v) \neq \emptyset$ *for all* $u, v \in P$.

Proof. The sequence u, v meets the criterion given in Theorem 4.3. □

Two immediate consequences of the preceding corollary are as follows. Let S be a semigroup with a right zero element z. If $\langle A \rangle$ contains z, then the graph Cay (S, A) is weak since $z \in A(u, v)$ for all u, v in P. According to the example below Lemma 4.1, the converse does not hold in general.

A semigroup S is *left reversible* provided that $aS \cap bS$ is not empty for all $a, b \in S$, see [8]. If a left reversible semigroup S is generated by a set A, then the graph Cay (S, A) is weak because $A(u, v) = uS \cap vS \neq \emptyset$. The converse is not true, as shown in Figure 1 (right). The semigroup T_4 is left reversible and the graph Cay (T_4, t_3) is weak even though $\{t_3\}$ does not generate T_4.

Corollary 4.5. *If A is a subset of S with $|P| > 1$ and $s\pi = s\langle A \rangle^1$ for all $s \in P$, then the Cayley graph* Cay (S, A) *is disconnected.*

Proof. For $u, v \in P$ with $u \neq v$, we have

$$A(u,v) \subseteq u\langle A\rangle^1 \cap v\langle A\rangle^1 = u\pi \cap v\pi = \emptyset,$$

which implies $A(u,v) = \emptyset$. Hence, no sequences in P can meet the criterion given in Theorem 4.3, and the corollary follows. □

The remaining of this section is devoted to the study of the set P in terms of an ordered set. A connection between a Cayley graph of a semigroup and the Hasse diagram is presented here. We first define the relation on P by the condition

$$u \leq v \qquad \Leftrightarrow \qquad u\langle A\rangle^1 \subseteq v\langle A\rangle^1. \tag{4.3}$$

The relation $\leq$ is easily seen to be a partial order on P. A characterization of $\leq$ in terms of paths is formulated in the next theorem. As usual, the expression $u < v$ means $u \leq v$ and $u \neq v$.

Theorem 4.6. *For all u, v in P, $u < v$ if and only if* $\mathrm{Cay}\,(S, A)$ *contains a path from v to u.*

Proof. If $u < v$, then $u \neq v$ and $u = va$ for some $a \in \langle A\rangle$. Hence, a v-u path exists. Conversely, the hypothesis implies $u = va$ with $a \in \langle A\rangle$, which implies $u\langle A\rangle^1 \subseteq v\langle A\rangle^1$. This proves $u < v$. □

The important point to note here is that the ordered set $(P, \leq)$ does not depend on the choice of a representative for each π-class. More mathematically, suppose that t_i is an arbitrary representative of $s_i\pi$ and $Q = \{t_1, t_2, \ldots, t_m\}$. Then the ordered sets $(P, \leq)$ and $(Q, \leq)$ have the same structure. In fact, the map $\varphi\colon P \to Q$ defined by $\varphi(s_i) = t_i$ for $i = 1, 2, \ldots, m$ is an order-isomorphism between them.

Given an ordered set $(P, \leq)$, the Hasse diagram of P is *connected* if for each pair of distinct elements u and v in P, there exists a sequence $u = u_0, u_1, \ldots, u_n = v$ of distinct elements in P such that u_{i-1} and u_i are comparable for all i.

Theorem 4.7. *A Cayley graph* $\mathrm{Cay}\,(S, A)$ *is weak if and only if the Hasse diagram of the ordered set $(P, \leq)$ given by* (4.1) *and* (4.3) *is connected.*

Proof. Let u and v be distinct elements of P. There is, by Theorem 4.3, a sequence $u = t_0, t_1, \ldots, t_n = v$ in P such that $A(t_{i-1}, t_i) \neq \emptyset$ for $i = 1, 2, \ldots, n$. As in the proof of Lemma 4.1, one of a u_{i-1}-u_i path, u_i-u_{i-1} path, or u_{i-1}-z_i and u_i-z_i paths must exist, that is, one of $u_i < u_{i-1}$, $u_{i-1} < u_i$, or $z_i' < u_{i-1}$ and $z_i' < u_i$, where $z_i \in z_i'\pi$ and $z_i' \in P$ must be the case. Hence, in the Hasse diagram, there is a line joining u_{i-1} and u_i for all i. This proves the connectedness of the Hasse diagram.

Conversely, let u and v be distinct elements of P. By assumption, there is a sequence $u = u_0, u_1, \ldots, u_n = v$ of distinct elements in P such that u_{i-1} and u_i are comparable. Hence, a u_i-u_{i-1} path or u_{i-1}-u_i path exists. By Lemma 4.2, $A(u_{i-1}, u_i) \neq \emptyset$ and so the graph $\mathrm{Cay}\,(S, A)$ is weak by Theorem 4.3. □

Corollary 4.8. *If the ordered set* $(P, \leq)$ *has the maximum or minimum element, then the Cayley graph* Cay (S, A) *is weak.*

Proof. Whenever P has the maximum or minimum element, any two elements of P are necessarily comparable, which implies the Hasse diagram is connected. □

For instance, consider the semigroup $\mathbb{Z}_{10}$ with multiplication modulo 10 and $A = \{3\}$. A direct computation gives $0\pi = \{0\}$, $1\pi = \{1, 3, 7, 9\}$, $2\pi = \{2, 4, 6, 8\}$, and $5\pi = \{5\}$. The Cayley graph Cay $(\mathbb{Z}_{10}, 3)$ and the Hasse diagram of $P = \{0, 1, 2, 5\}$ are pictured in Figure 3.

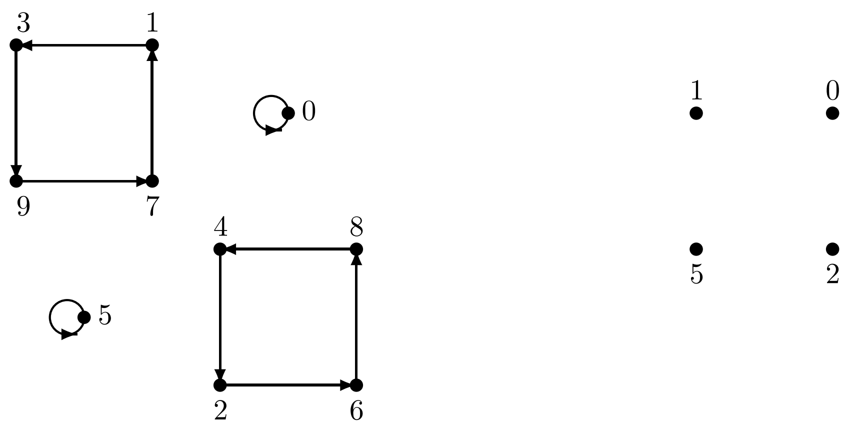

Figure 3. Cay $(\mathbb{Z}_{10}, 3)$ and the Hasse diagram of $P = \{0, 1, 2, 5\}$

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