Thai Journal of Mathematics
Volume 13 (2015) Number 3 : 627–640
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence of the S-Iteration Process for a Pair of Single-valued and Multi-valued Generalized Nonexpansive Mappings in CAT(κ) Spaces

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Abstract : In this paper, we study Δ and strong convergence of modified S-iteration process for a pair of a total asymptotically nonexpansive single-valued mapping and a multi-valued mapping satisfying the condition (C_λ) and (E) in the setting of complete CAT(κ) spaces.

Keywords : total asymptotically nonexpansive mappings; quasi-nonexpansive mappings; CAT(κ) spaces.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

A CAT(κ) space (κ is a real number) is a geodesic metric space whose geodesic triangle is thinner than the corresponding comparison triangle in a model space

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with curvature κ. Fixed point theory in CAT(κ) spaces was first studied by Kirk [1, 2]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0) spaces. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, *e.g.*, [3, 4, 5, 6, 7, 8, 9]). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(ω) space for every $\omega \geq \kappa$ (see, *e.g.*, [10]). However, there are only a few research papers that contain fixed point results in the setting of CAT(κ) spaces with $\kappa > 0$.

In 2011, Sokhuma and Kaewkhao [11] introduced a modified Ishikawa iterative process for finding a common fixed point of a pair of single-valued and multi-valued nonexpansive mappings in Banach spaces. They also proved a strong convergence theorem for the proposed iterative process in uniformly convex Banach spaces. Recently, Uddin *et al.* [12] generalized and improved several results contained in [11]. They proved convergence theorems of modified Ishikawa iteration process involving a pair of mappings satisfying the condition (C_λ) on Banach spaces.

In 2007, Agarwal *et al.* [13] introduced the S-iteration process for finding a fixed point of a nearly asymptotically nonexpansive single-valued mapping in a Banach space. They also showed, theoretically as well as numerically, that the S-iteration process is faster than the Mann and Ishikawa iteration processes for contraction operators. Later in 2011, Khan and Abbas [3] have modified S-iteration process in CAT(0) spaces for finding a fixed point of a nonexpansive single-valued mapping. Recently, Akkasriworn and Sokhuma [14] defined the modified S-iteration process for a pair of single-valued and multi-valued nonexpansive mappings in Banach spaces. However, there is not any result in CAT(κ) spaces concerning the convergence of S-iteration process for a pair of single-valued and multi-valued mappings.

The purpose of this paper is to study the modified S-iteration process for a pair of a total asymptotically nonexpansive single-valued mapping and a multi-valued mapping satisfying the condition (C_λ) in complete CAT(κ) spaces.

2 Preliminaries and some useful lemmas

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map φ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\varphi(0) = x$, $\varphi(l) = y$, and $d(\varphi(t_1), \varphi(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, φ is an isometry and $d(x, y) = l$. The image of φ is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. For each $x, y \in X$ and $\alpha \in (0, 1)$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic metric space* (D-geodesic metric space) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (D-*uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for

$x, y \in X$ with $d(x,y) < D$). A nonempty subset K of X is said to be *convex* if K includes every geodesic segment joining any two of its points. The set K is said to be *bounded* if $\operatorname{diam}(K) = \sup\{d(x,y) : x, y \in K\} < \infty$.

We now introduce the model spaces M_κ^n, for more details on these spaces the reader is referred to [10]. Let $n \in \mathbb{N}$, we denote the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance by $\mathbb{E}^n$. The Euclidean scalar product in $\mathbb{R}^n$ is denote by $(\cdot|\cdot)$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n \text{ where } x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n).$$

Let $\mathbb{S}^n$ denote the *n-dimensional sphere* defined by $\mathbb{S}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\}$ with metric $d_{\mathbb{S}^n}(x,y) = \arccos(x|y)$ for $x, y \in \mathbb{S}^n$. Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$ the real number $\langle x|y\rangle$ defined by

$$\langle x|y\rangle = -x_{n+1}y_{n+1} + \sum_{i=1}^{n} x_iy_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic n-space* defined by $\mathbb{H}^n = \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n,1} : (x|x) = -1, x_{n+1} > 0\}$ with metric $d_{\mathbb{H}^n}$ such that $\cosh d_{\mathbb{H}^n}(x,y) = -\langle x|y\rangle$ for $x, y \in \mathbb{H}^n$.

Given a real number κ, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$, then M_κ^n is the Euclidean space $\mathbb{E}^n$;

(ii) if $\kappa > 0$, then M_κ^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{\kappa}}$;

(iii) if $\kappa < 0$, then M_κ^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $\frac{1}{\sqrt{-\kappa}}$.

A *geodesic triangle* $\Delta(x,y,z)$ in a geodesic metric space (X,d) consists of three points $x, y, z \in X$ (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the *edges* of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x,y,z)$ in (X,d) is a triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ in M_κ^2 such that $d(x,y) = d_{M_\kappa^2}(\overline{x},\overline{y})$, $d(y,z) = d_{M_\kappa^2}(\overline{y},\overline{z})$, and $d(z,x) = d_{M_\kappa^2}(\overline{z},\overline{x})$. If $\kappa \leq 0$, then such a comparison triangle always exists M_κ^2. If $\kappa > 0$, then such a triangle exists whenever $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$, where $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$. A point $\overline{w} \in [\overline{x},\overline{y}]$ is called a *comparison point* for $w \in [x,y]$ if $d(x,w) = d_{M_\kappa^2}(\overline{x},\overline{w})$.

A geodesic triangle $\Delta(x,y,z)$ in X is said to satisfy the CAT(κ) *inequality* if for any $p, q \in \Delta(x,y,z)$ and for their comparison points $\overline{p}, \overline{q} \in \Delta(\overline{x},\overline{y},\overline{z})$, one has $d(p,q) \leq d_{M_\kappa^2}(\overline{p},\overline{q})$.

Definition 2.1.

(i) If $\kappa \leq 0$, then X is called a CAT(κ) space *if and only if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(κ) inequality.*

(ii) If $\kappa > 0$, then X is called a CAT(κ) space *if and only if X is D_κ-geodesic and any geodesic triangle $\Delta(x,y,z)$ in X with $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa$ satisfies the CAT(κ) inequality.*

Let $R \in (0,2]$. Recall that a geodesic metric space (X,d) is said to be *R-convex* (see [15]) if for any three points $x,y,z \in X$ and $\alpha \in [0,1]$, we have

$$d^2(x,(1-\alpha)y \oplus \alpha z) \leq (1-\alpha)d^2(x,y) + \alpha d^2(x,z) - \frac{R}{2}\alpha(1-\alpha)d^2(y,z). \quad (2.1)$$

It is known that a geodesic metric space (X,d) is a CAT(0) space if and only if (X,d) is R-convex for $R=2$. The following lemma is a consequence of Proposition 3.1 in [15].

Lemma 2.2. *Let $\kappa > 0$ and (X,d) be a CAT(κ) space with diam$(X) \leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$. Then (X,d) is R-convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

We now collect some elementary facts about CAT(κ) spaces; see [16].

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space X with $\kappa > 0$. For $x \in X$, we define a mapping $r(\cdot,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

The asymptotic center $A(\{x_n\})$ is singleton for a CAT(κ) space with diameter smaller than $\frac{\pi}{2\sqrt{\kappa}}$; see [17].

Definition 2.3. A sequence $\{x_n\}$ in a CAT(κ) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 2.4 ([17])**.** *Let $\kappa > 0$ and X be a complete CAT(κ) space with diam$(X) \leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$. Then every bounded sequence in X has a Δ-convergent subsequence.*

Lemma 2.5 ([10])**.** *Let $\kappa > 0$ and X be a complete CAT(κ) space with diam$(X) \leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\frac{\pi}{2})$. Then, we have*

$$d((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d(x,z) + \alpha d(y,z)$$

for all $x,y,z \in X$ and $\alpha \in [0,1]$.

Lemma 2.6 ([4])**.** *Let $\kappa > 0$ and X be a complete CAT(κ) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then x=u.*

The following lemma is a characterization of CAT(0) spaces. It can be applied to a CAT(κ) space with $\kappa > 0$ as well.

Lemma 2.7 ([5])**.** *Let X be a CAT(0) space, and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and that $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and*

$$\lim_{n\to\infty} d((1 - t_n)x_n \oplus t_n y_n, x) = r \text{ for some } r \geq 0.$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Let K be a nonempty subset of a CAT(κ) space X, and $T : K \to K$ be a single-valued mapping. The set of all fixed points of T will be denoted by $F(T) = \{x \in K : x = Tx\}$.

Definition 2.8. A single-valued mapping $T : K \to K$ is said to be

(i) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in K$ and $n \in \mathbb{N}$;

(iii) *generalized asymptotically nonexpansive* if there exist two sequences $\{k_n\}$, $\{s_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} s_n = 0$ and $d(T^n x, T^n y) \leq k_n d(x, y) + s_n$ for all $x, y \in K$ and $n \in \mathbb{N}$;

(iv) *total asymptotically nonexpansive* if there exist two sequences $\{k_n\}, \{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} s_n = 0$ and a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that $d(T^n x, T^n y) \leq d(x, y) + k_n \phi(d(x, y)) + s_n$ for all $x, y \in K$ and $n \in \mathbb{N}$.

Remark 2.9.

(i) The concept of total asymptotically nonexpansive single-valued mappings was first introduced in Banach spaces by Alber et al. *[18].*

(ii) If $\phi(\lambda) = \lambda$, then a total asymptotically nonexpansive mapping reduces to a generalized asymptotically nonexpansive mapping. If $\phi(\lambda) = \lambda$ and $k_n = 0$ for all $n \in \mathbb{N}$, then a total asymptotically nonexpansive mapping reduces to an asymptotically nonexpansive mapping. If $\phi(\lambda) = \lambda$ and $k_n = 0$ and $s_n = 0$ for all $n \in \mathbb{N}$, then a total asymptotically nonexpansive mapping reduces to a nonexpansive mapping.

The following two lemmas can be found in [16].

Lemma 2.10. *Let $\kappa > 0$ and X be a complete CAT(κ) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a continuous and total asymptotically nonexpansive mapping. Then T has a fixed point in K.*

Lemma 2.11. *Let $\kappa > 0$ and X be a complete CAT(κ) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$. Let K be a nonempty closed convex subset of X, and let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive mapping. If $\{x_n\}$ is a sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Δ-$\lim_{n\to\infty} x_n = p$, then $p \in K$ and $p = Tp$.*

We shall denote the family of nonempty closed bounded subsets of K by $CB(K)$, and the family of nonempty compact convex subsets of K by $CC(K)$. The *Pompeiu-Hausdorff distance* [19] on $CB(K)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} \text{dist}(x, B), \sup_{y\in B} \text{dist}(y, A)\right\} \text{ for } A, B \in CB(K),$$

where $\text{dist}(x, K) = \inf\{d(x, y) : y \in K\}$ is the distance from a point x to a subset K. Let S be a multi-valued mapping of K into $CB(K)$. The set of all fixed points of S will be denoted by $F(S) = \{x \in K : x \in Sx\}$.

Definition 2.12. A multi-valued mapping $S : K \to CB(K)$ is said to

(i) be *nonexpansive* if $H(Sx, Sy) \leq d(x, y)$ for all $x, y \in K$;

(ii) be *quasi-nonexpansive* if $F(S) \neq \emptyset$ and $H(Sx, Sz) \leq d(x, z)$ for all $x \in D$ and $z \in F(S)$;

(iii) satisfy *condition* (E_μ) if there exists $\mu \geq 1$ such that for each $x, y \in K$, $\text{dist}(x, Sy) \leq \mu\text{dist}(x, Sx) + d(x, y)$. We say that S satisfies *condition* (E) whenever S satisfies (E_μ) for some $\mu \geq 1$.

(iv) satisfy *condition* (C_λ) if there exists $\lambda \in (0, 1)$ such that for each $x, y \in K$, $\lambda\text{dist}(x, Sx) \leq d(x, y)$ implies $H(Sx, Sy) \leq d(x, y)$.

Remark 2.13.

(i) If $S : K \to CB(K)$ is nonexpansive, then S satisfies the condition (E_1).

(ii) As in the single-valued case, if $0 < \lambda_1 < \lambda_2 < 1$ then the condition (C_{λ_1}) implies the condition (C_{λ_2}).

The following lemma is also needed.

Lemma 2.14 ([20])**.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq a_n + b_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3 Main results

In this section, we first introduce the modified S-iteration process for a pair of single-valued and multi-valued mappings in CAT(κ) spaces.

Definition 3.1. For K a nonempty convex subset of a CAT(κ) space X, $T : K \to K$ a single-valued mapping and $S : K \to CB(K)$ a multi-valued mapping, the iterative sequence $\{x_n\}$ is generated from $x_1 \in K$, and is defined by

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n z_n, \ \ z_n \in Sx_n, \\ x_{n+1} = (1-\beta_n)z_n \oplus \beta_n T^n y_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. We will call it *the modified S-iteration process.*

Before proving the Δ and strong convergence theorems, we need the following two lemmas.

Lemma 3.2. *Let $\kappa > 0$, X be a complete CAT(κ) space with diam(X) $\leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$, and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive single-valued mapping with sequences $\{k_n\}, \{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, and $S : K \to CB(K)$ be a multi-valued mapping satisfying the condition (C_λ). Assume that $\mathfrak{F} := F(T) \cap F(S)$ is nonempty and $Sp = \{p\}$ for all $p \in \mathfrak{F}$. For $x_1 \in K$, the sequence $\{x_n\}$ generated by* (3.1)*. Then,* $\lim_{n\to\infty} d(x_n, p)$ *exists for all $p \in \mathfrak{F}$.*

Proof. Let $p \in \mathfrak{F}$ and $M = \text{diam}(K)$. As, $\lambda \in (0,1)$, implies $\lambda \text{dist}(p, Sp) = 0 \leq d(x_n, p)$, owing to the condition (C_λ), we have $H(Sx_n, Sp) \leq d(x_n, p)$. Since T is total asymptotically nonexpansive, it follows by Lemma 2.5 that

$$\begin{aligned} d(x_n, p) &\leq (1-\beta_n)d(z_n, p) + \beta_n d(T^n y_n, T^n p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n(d(y_n, p) + k_n\phi(d(y_n, p)) + s_n) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n(d(y_n, p) + k_n\phi(M) + s_n) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(y_n, p) + \beta_n(k_n\phi(M) + s_n) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n((1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p)) + \beta_n(k_n\phi(M) + s_n) \\ &= (1-\beta_n + \beta_n\alpha_n)d(z_n, p) + \beta_n(1-\alpha_n)d(x_n, p) + \beta_n(k_n\phi(M) + s_n) \\ &= (1-\beta_n + \beta_n\alpha_n)\text{dist}(z_n, Sp) + \beta_n(1-\alpha_n)d(x_n, p) + \beta_n(k_n\phi(M) + s_n) \\ &\leq (1-\beta_n + \beta_n\alpha_n)H(Sx_n, Sp) + \beta_n(1-\alpha_n)d(x_n, p) + \beta_n(k_n\phi(M) + s_n) \\ &\leq d(x_n, p) + \beta_n(k_n\phi(M) + s_n). \end{aligned}$$

Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it implies by Lemma 2.14 that $\lim_{n\to\infty} d(x_n, p)$ exists. □

Lemma 3.3. *Let $\kappa > 0$, X be a complete CAT(κ) space with diam$(X) \leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$, and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive single-valued mapping with sequences $\{k_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, and $S : K \to CB(K)$ be a multi-valued mapping satisfying the condition (C_λ). Assume that $\mathfrak{F} := F(T) \cap F(S)$ is nonempty and $Sp = \{p\}$ for all $p \in \mathfrak{F}$. For $x_1 \in K$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then, we have $\lim_{n\to\infty}$ dist$(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Let $p \in \mathfrak{F}$ and $M = \text{diam}(K)$. In view of Lemma 3.2, we assume that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.2}$$

If $c = 0$, then all the conclusions are trivial. Therefore we will assume that $c > 0$. As, $\lambda \in (0,1)$, implies $\lambda\text{dist}(p, Sp) = 0 \leq d(x_n, p)$, owing to the condition (C_λ), we have $H(Sx_n, Sp) \leq d(x_n, p)$. Thus, $d(z_n, p) = \text{dist}(z_n, Sp) \leq H(Sx_n, Sp) \leq d(x_n, p)$. This implies that

$$\limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{3.3}$$

Consider,

$$\begin{aligned}
d(T^n y_n, p) &\leq d(y_n, p) + k_n\phi(d(y_n, p)) + s_n \\
&\leq d(y_n, p) + k_n\phi(M) + s_n \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(z_n, p) + k_n\phi(M) + s_n \\
&= (1-\alpha_n)d(x_n, p) + \alpha_n \text{dist}(z_n, Sp) + k_n\phi(M) + s_n \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n H(Sx_n, Sp) + k_n\phi(M) + s_n \\
&\leq d(x_n, p) + k_n\phi(M) + s_n.
\end{aligned}$$

This implies by $\lim_{n\to\infty} k_n = 0$ and $\lim_{n\to\infty} s_n = 0$ that

$$\limsup_{n\to\infty} d(T^n y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = c. \tag{3.4}$$

Since $c = \limsup_{n\to\infty} d(x_{n+1}, p) = \limsup_{n\to\infty} d((1-\beta_n)z_n \oplus \beta_n T^n y_n, p)$, it follows by Lemma 2.7 that

$$\lim_{n\to\infty} d(z_n, T^n y_n) = 0. \tag{3.5}$$

By the definition of the sequence $\{x_n\}$, we have

$$\begin{aligned}
d(x_{n+1}, p) &\leq (1-\beta_n)d(z_n, p) + \beta_n d(T^n y_n, p) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n(d(y_n, p) + k_n\phi(d(y_n, p)) + s_n) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n(d(y_n, p) + k_n\phi(M) + s_n).
\end{aligned}$$

This implies that

$$d(x_{n+1},p) - d(x_n,p) \le \beta_n(d(y_n,p) - d(x_n,p) + k_n\phi(M) + s_n).$$

Therefore,

$$\begin{aligned}\frac{d(x_{n+1},p) - d(x_n,p)}{b} + d(x_n,p) &\le \frac{d(x_{n+1},p) - d(x_n,p)}{\beta_n} + d(x_n,p)\\ &\le d(y_n,p) + k_n\phi(M) + s_n.\end{aligned}$$

It implies by (3.2) and (3.4) that

$$\begin{aligned}c &= \liminf_{n\to\infty}\left(\frac{d(x_{n+1},p) - d(x_n,p)}{b} + d(x_n,p)\right)\\ &\le \liminf_{n\to\infty}(d(y_n,p) + k_n\phi(M) + s_n)\\ &= \liminf_{n\to\infty} d(y_n,p)\\ &\le \limsup_{n\to\infty} d(y_n,p) \le c.\end{aligned}$$

Then we have

$$c = \lim_{n\to\infty} d(y_n,p) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n z_n, p). \tag{3.6}$$

Since $d(z_n,p) \le d(x_n,p)$,

$$\limsup_{n\to\infty} d(z_n,p) \le \limsup_{n\to\infty} d(x_n,p) = c. \tag{3.7}$$

By (3.6), (3.7), and Lemma 2.7 that

$$\lim_{n\to\infty} d(x_n,z_n) = 0. \tag{3.8}$$

Since $z_n \in Sx_n$, we have $\operatorname{dist}(x_n,Sx_n) \le d(x_n,z_n)$. This implies by (3.8) that

$$\lim_{n\to\infty} \operatorname{dist}(x_n,Sx_n) = 0.$$

By T is total asymptotically nonexpansive, we have

$$\begin{aligned}d(T^nx_n,x_n) &\le d(T^nx_n,T^ny_n) + d(T^ny_n,x_n)\\ &\le d(x_n,y_n) + k_n\phi(d(x_n,y_n)) + s_n + d(T^ny_n,x_n)\\ &\le d(x_n,y_n) + k_n\phi(M) + s_n + d(T^ny_n,x_n)\\ &\le \alpha_n d(x_n,z_n) + k_n\phi(M) + s_n + d(T^ny_n,z_n) + d(z_n,x_n)\\ &= (1+\alpha_n)d(x_n,z_n) + d(T^ny_n,z_n) + k_n\phi(M) + s_n.\end{aligned}$$

Then, by (3.5) and (3.8), we get

$$\lim_{n\to\infty} d(T^nx_n,x_n) = 0. \tag{3.9}$$

By the uniform continuity of T, we have

$$\lim_{n\to\infty} d(T^{n+1}x_n, Tx_n) = 0. \tag{3.10}$$

Consider,

$$\begin{aligned} d(x_n, x_{n+1}) &\le (1-\beta_n)d(x_n, z_n) + \beta_n d(x_n, T^n y_n) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n(d(x_n, T^n x_n) + d(T^n x_n, T^n y_n)) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n(d(x_n, T^n x_n) + d(x_n, y_n) + k_n\phi(d(x_n, y_n)) + s_n) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n(d(x_n, T^n x_n) + d(x_n, y_n) + k_n\phi(M) + s_n) \\ &\le (1-\beta_n)d(x_n, z_n) + \beta_n(d(x_n, T^n x_n) + \alpha_n d(x_n, z_n) + k_n\phi(M) + s_n) \\ &= (1-\beta_n+\beta_n\alpha_n)d(x_n, z_n) + \beta_n(d(x_n, T^n x_n) + k_n\phi(M) + s_n). \end{aligned}$$

This implies by (3.8) and (3.9) that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.11}$$

By (3.9), (3.10), and (3.11), we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) \\ &\quad + d(T^{n+1}x_n, Tx_n) \\ &\le 2d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + k_{n+1}\phi(M) + s_{n+1} \\ &\quad + d(T^{n+1}x_n, Tx_n) \to 0 \text{ as } n\to\infty. \end{aligned}$$

This completes the proof. □

Now, we prove a Δ-convergence theorem for a pair of a total asymptotically nonexpansive single-valued mapping and a multi-valued mapping satisfying the condition (C_λ) and (E) in complete CAT(κ) spaces.

Theorem 3.4. *Let $\kappa > 0$, X be a complete CAT(κ) space with diam$(X) \le \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$, and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive single-valued mapping with sequences $\{k_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, and $S : K \to CC(K)$ be a multi-valued mapping satisfying the condition (C_λ) and (E). Assume that $\mathfrak{F} := F(T)\cap F(S)$ is nonempty and $Sp = \{p\}$ for all $p \in \mathfrak{F}$. For $x_1 \in K$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. Then the sequence $\{x_n\}$ Δ-converges to a point in $\mathfrak{F}$.*

Proof. By Lemma 3.2, it implies that $\{x_n\}$ is bounded. Let $\omega_\Delta(x_n) = \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. First, we show that $\omega_\Delta(x_n) \subseteq \mathfrak{F}$. To show this, we let $u \in \omega_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4, there exists a subsequence

$\{u_{n_j}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{j\to\infty} u_{n_j} = v \in K$. From Lemma 3.3, we have $\lim_{j\to\infty} \text{dist}(u_{n_j}, Su_{n_j}) = 0$ and $\lim_{j\to\infty} d(u_{n_j}, Tu_{n_j}) = 0$. It implies by Lemma 3.3 and Lemma 2.11 that $v \in F(T)$. Since Sv is compact, for all $j \in \mathbb{N}$, we can choose $q_{n_j} \in Sv$ such that $d(u_{n_j}, q_{n_j}) = \text{dist}(u_{n_j}, Sv)$ and the sequence $\{q_{n_j}\}$ has a convergent subsequence $\{q_{n_k}\}$ with $\lim_{k\to\infty} q_{n_k} = q \in Sv$. By condition (E), there exists $\mu \geq 1$ such that

$$\text{dist}(u_{n_k}, Sv) \leq \mu\text{dist}(u_{n_k}, Su_{n_k}) + d(u_{n_k}, v).$$

Then we have

$$\begin{aligned} d(u_{n_k}, q) &\leq d(u_{n_k}, q_{n_k}) + d(q_{n_k}, q) \\ &= \text{dist}(u_{n_k}, Sv) + d(q_{n_k}, q) \\ &\leq \mu\text{dist}(u_{n_k}, Su_{n_k}) + d(u_{n_k}, v) + d(q_{n_k}, q). \end{aligned}$$

This implies that

$$\limsup_{k\to\infty} d(u_{n_k}, q) \leq \limsup_{k\to\infty} d(u_{n_k}, v).$$

By the uniqueness of asymptotic centers, we have $v = q \in Sv$.

Hence, we obtain $v \in \mathfrak{F}$ and so $\lim_{n\to\infty} d(x_n, v)$ exists. Suppose that $u \neq v$. By the uniqueness of asymptotic centers, we get

$$\begin{aligned} \limsup_{k\to\infty} d(u_{n_k}, v) &< \limsup_{k\to\infty} d(u_{n_k}, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u) \\ &< \limsup_{n\to\infty} d(u_n, v) \\ &= \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{k\to\infty} d(u_{n_k}, v). \end{aligned}$$

This is a contradiction. Then we have $u = v \in \mathfrak{F}$. This shows that $\omega_\Delta(x_n) \subseteq \mathfrak{F}$.

To show that $\{x_n\}$ Δ-converges to a point in $\mathfrak{F}$, it is sufficient to show that $\omega_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{z\}$. Since $u \in \omega_\Delta(x_n) \subseteq \mathfrak{F}$, it follows that $\lim_{n\to\infty} d(x_n, u)$ exists. By Lemma 2.6, $u = z$. This completes the proof. □

We now get a strong convergence theorem of modified S-iteration for a pair of a total asymptotically nonexpansive single-valued mapping and a multi-valued mapping satisfying the condition (C_λ) and (E) in complete CAT(κ) spaces.

Theorem 3.5. *Let $\kappa > 0$, X be a complete CAT(κ) space with diam$(X) \leq \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$, and K be a nonempty compact convex subset of X. Let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive* *single-valued mapping with sequences* $\{k_n\}, \{s_n\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} k_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$*, and* $S : K \to CB(K)$ *be a multi-valued mapping satisfying the condition* (C_λ) *and* (E)*. Assume that* $\mathfrak{F} := F(T) \cap F(S)$ *is nonempty and* $Sp = \{p\}$ *for all* $p \in \mathfrak{F}$*. For* $x_1 \in K$*, the sequence* $\{x_n\}$ *generated by* (3.1)*, where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *such that* $0 < a \leq \alpha_n, \beta_n \leq b < 1$*. Then the sequence* $\{x_n\}$ *converges strongly to a point in* $\mathfrak{F}$*.*

Proof. By Lemma 3.2, we have $\{x_n\}$ is bounded. Since K is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to p in K. By condition (E), there exists $\mu \geq 1$ such that

$$\begin{aligned} \text{dist}(p, Sp) &\leq d(p, x_{n_i}) + \text{dist}(x_{n_i}, Sp) \\ &\leq d(x_{n_i}, p) + \mu \text{dist}(x_{n_i}, Sx_{n_i}) + d(x_{n_i}, p) \\ &= 2d(x_{n_i}, p) + \mu \text{dist}(x_{n_i}, Sx_{n_i}). \end{aligned}$$

Then, by Lemma 3.3, we have $p \in F(S)$. Again, by Lemma 3.3 and the uniform continuity of T, we have

$$d(Tp, p) \leq d(Tp, Tx_{n_i}) + d(Tx_{n_i}, x_{n_i}) + d(x_{n_i}, p) \to 0 \text{ as } n \to \infty.$$

That is, $p \in F(T)$. Therefore, $p \in \mathfrak{F}$. By Lemma 3.2, $\lim_{n\to\infty} d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$ itself. □

Recall that a single-valued mapping $T : K \to K$ is said to be *semi-compact* if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in K$. The following theorem, we show that the compactness of K in Theorem 3.5 can be dropped if a single-valued mapping T^m is semi-compact for some $m \in \mathbb{N}$.

Theorem 3.6. *Let* $\kappa > 0$*,* X *be a complete CAT(*κ*) space with diam*$(X) \leq \frac{\frac{\pi}{2} - \varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \frac{\pi}{2})$*, and* K *be a nonempty closed convex subset of* X*. Let* $T : K \to K$ *be a uniformly continuous and total asymptotically nonexpansive single-valued mapping with sequences* $\{k_n\}, \{s_n\} \subset [0, \infty)$ *such that* $\sum_{n=1}^{\infty} k_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$*, and* $S : K \to CB(K)$ *be a multi-valued mapping satisfying the condition* (C_λ) *and* (E)*. Assume that* $\mathfrak{F} := F(T) \cap F(S)$ *is nonempty and* $Sp = \{p\}$ *for all* $p \in \mathfrak{F}$*. For* $x_1 \in K$*, the sequence* $\{x_n\}$ *generated by* (3.1)*, where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *such that* $0 < a \leq \alpha_n, \beta_n \leq b < 1$*. If* T^m *is semi-compact for some* $m \in \mathbb{N}$*, then the sequence* $\{x_n\}$ *converges strongly to a point in* $\mathfrak{F}$*.*

Proof. By Lemma 3.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. This follows by the uniform continuity of T that

$$d(x_n, T^m x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{m-1} x_n, T^m x_n) \to 0,$$

as $n \to \infty$. By the semi-compactness of T^m, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to p in K for some $p \in K$. As in the proof of Theorem 3.5, we obtain that the sequence $\{x_n\}$ converges strongly to a point in $\mathfrak{F}$. □

Recall that a multi-valued mapping $S : K \to CB(K)$ is said to be *hemi-compact* if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \mathrm{dist}(x_n, Sx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p \in K$. The following theorem, we show that the compactness of K in Theorem 3.5 can be dropped if a multi-valued mapping S is hemi-compact.

Theorem 3.7. *Let $\kappa > 0$, X be a complete CAT(κ) space with diam$(X) \le \frac{\frac{\pi}{2}-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \frac{\pi}{2})$, and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a uniformly continuous and total asymptotically nonexpansive single-valued mapping with sequences $\{k_n\}, \{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, and $S : K \to CB(K)$ be a multi-valued mapping satisfying the condition (C_λ) and (E). Assume that $\mathfrak{F} := F(T) \cap F(S)$ is nonempty and $Sp = \{p\}$ for all $p \in \mathfrak{F}$. For $x_1 \in K$, the sequence $\{x_n\}$ generated by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $0 < a \le \alpha_n, \beta_n \le b < 1$. If S is hemi-compact, then the sequence $\{x_n\}$ converges strongly to a point in $\mathfrak{F}$.*

Proof. Since S is hemi-compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges strongly to p in K for some $p \in K$. As in the proof of Theorem 3.5, we obtain that the sequence $\{x_n\}$ converges strongly to a point in $\mathfrak{F}$. □

Acknowledgement : The authors would like to thank the referees for valuable suggestions on the research paper.

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(Received 4 July 2015)
(Accepted 10 December 2015)