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# Some Proximally Coincidence Points for Non-Self Mappings and Self Mappings in Partially Ordered Metric Spaces

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**Abstract :** In this paper, we first introduce the concept of proximally coincidence points for non-self mapping and self mappings. We prove the existence of a proximally coincidence point for non-self mappings and self mapping satisfying the $(\psi, \alpha, \beta)$-contractive conditions in the setting of partially ordered set which is endowed with a metric. Further, our result provides an extension of a result due to Choudhury and Kundu [B.S. Choudhury, A. Kundu, $(\psi, \alpha, \alpha)$-weak contractions in partially ordered metric spaces, Appl. Math. Lett. 25 (2012) 6–10.] to the case of non-self-mapping. The examples illustrating our results are given.

**Keywords :** proximally coincidence points; partially ordered set; proximally increasing mapping; best proximity point.
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## 1 Introduction

The weak contraction in Hilbert spaces, a generalization of the Banach contraction, was first introduced and proved by Alber and Guerre-Delabriere [1]. Rhoades

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[2] had shown that the result which Alber et is also valid in complete metric spaces. Weakly contractive mappings satisfying other weak contractive inequalities have been discussed in several works, please refer to [3–10]. Khan *et al.* [11] introduced the use of a control function in metric fixed point problems. This function and its extensions have been used in several problems of fixed point theory, please refer to [12–16]. In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, in metric spaces endowed with a partial ordering [17–20]. Using the control functions the weak contraction principle has been generalized in metric spaces [4] and in partially ordered metric spaces in [21]. In [22], the weak contraction principle has been generalized by using three functions.

In [23], Choudhury and Kundu gave necessary and sufficient to claim that the weak contraction principle to the case of two functions in partially ordered complete metric spaces as follows:

**Theorem 1.1** ([23, Theorem 2.1])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $f, g : X \longrightarrow X$ be such that $f(X) \subseteq g(X)$, $f$ is $g$-non-decreasing, $g(X)$ is closed and*

$$\psi(d(fx, fy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)) \text{ for all } x, y \in X \text{ such that } gx \preceq gy, \tag{1.1}$$

*where $\psi, \alpha, \beta : [0, \infty) \longrightarrow [0, \infty)$ are such that, $\psi$ is continuous and monotone non-decreasing, $\alpha$ is continuous, $\beta$ is lower semi-continuous,*

$$\psi(t) = 0 \quad \text{if and only if} \quad t = 0, \alpha(0) = \beta(0) = 0 \tag{1.2}$$

*and*

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all} \quad t > 0. \tag{1.3}$$

*Also, if any nondecreasing sequence $\{x_n\}$ in $X$ converges to $z$, then we assume*

$$x_n \preceq z \quad \text{for all} \quad n \geq 0. \tag{1.4}$$

*If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then $f$ and $g$ have a coincidence point.*

In this article, we attempt to give a generalization of Theorem 1.1 by considering a non-self-map $f$. Before getting into the details of our main theorem, let us give a brief discussion of best proximity point results.

Let $A$ be nonempty subset of a metric space $(X, d)$ and $f : A \longrightarrow X$ has a fixed point in $A$ if the fixed point equation $fx = x$ has at least one solution. That is, $x \in A$ is a fixed point of $f$ if $d(x, fx) = 0$. If the fixed point equation $fx = x$ does not possess a solution, then $d(x, fx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, fx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Here we sate the following well-known best approximation theorem due to Ky Fan [24].

**Theorem 1.2** ([24]). *Let $A$ be a nonempty compact convex subset of a normed linear space $X$ and $f : A \longrightarrow X$ be a continuous function. Then there exists $x \in A$ such that*

$$\| x - fx \| = d(fx, A) := \inf\{\| fx - a \| : a \in A\}.$$

Such an element $x \in A$ in Theorem 1.2 is called a *best approximant* of $f$ in $A$. Note that if $x \in A$ is a best approximant, then $\| x - fx \|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x\in A} \| x - fx \|$ has at least one solution. To have a concrete lower bound.

Let us consider two nonempty subsets $A$, $B$ of a metric space $X$ and a mapping $f : A \longrightarrow B$. The natural question is whether one can find an element $x_0 \in A$ such that

$$d(x_0, fx_0) = \min\{d(x, fx) : x \in A\}.$$

Since $d(x, fx) > d(A, B)$,the optimal solution to the problem of minimizing the real valued function $x \mapsto d(x, fx)$ over the domain $A$ of the mapping $f$ will be the one for which the value $d(A, B)$ is attained.

A point $x_0 \in A$ is called a *best proximity point* of $f$ if $d(x_0, fx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of $f$. The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [7–14]. Also one can find the existence of best proximity point in the setting of partially order metric space in [15–17].

The purpose of this article is to give the concept of proximally coincidence points for non-self mapping $f$ and a self mapping $g$. We prove the existence of a proximally coincidence point for non-self mappings and self mapping satisfying the $(\psi, \alpha, \beta)$-contractive conditions in the setting of partially ordered set which is endowed with a metric. When the map $f$ is considered to be a self-map, then our result reduces to the fixed point theorem of Choudhury and Kundu [23].

## 2 Preliminaries

Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$, we recall the following notations and notions that will be used in what follows.

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then $A_0$ and $B_0$ are nonempty. Further, it is interesting to notice that $A_0$ and $B_0$ are contained in the boundaries of $A$ and $B$ respectively, provided $A$ and $B$ are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [25]).

**Definition 2.1** (Best proximity point)**.** A point $x \in A$ is said to be a *best proximity point* of the mapping $f : A \to B$ if it satisfies the following condition

$$d(x, fx) = d(A, B).$$

**Definition 2.2** ([26, $\mathcal{P}$-property])**.** Let $(A, B)$ be a pair of non-empty subsets of a metric space $X$ with $A_0 \neq \emptyset$. Then the pair $(A, B)$ is said to have the *$\mathcal{P}$-property* if and only if

$$\left.\begin{array}{r} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

**Example 2.3.** Let $A, B$ be two non-empty closed convex subsets of a Hilbert space $X$. Then $(A, B)$ satisfies the $\mathcal{P}$-property.

**Example 2.4.** Let $A, B$ be two non-empty subsets of a metric space $(X, d)$ such that $A_0 \neq \emptyset$ and $d(A, B) = 0$. Thus $(A, B)$ has the $\mathcal{P}$-property.

**Definition 2.5** ([27, $g$-non decreasing mapping])**.** Suppose that $(X, \preceq)$ is a partially ordered set and $f, g : X \longrightarrow X$ are mappings of $X$ into itself. $f$ is said to be *$g$-non-decreasing* if for $x, y \in X$,

$$gx \preceq gy \quad \text{implies} \quad fx \preceq fy.$$

# 3 Proximity coincidence points and proximally increasing mappings

We introduce the concept of proximally coincidence points for non-self mapping and self mappings as follows:

**Definition 3.1** (Proximity coincidence point)**.** A point $x \in A$ is said to be a *proximity coincidence point* of the non-self mapping $f : A \to B$ and a self mapping $g : A \to A$, if it satisfies the following condition

$$d(gx, fx) = d(A, B).$$

**Definition 3.2** ($g$-proximally increasing)**.** Suppose $(X, \preceq)$ is a partially ordered set. Let $f : A \longrightarrow B$ and $g : A \longrightarrow A$. A mapping $f$ is said to be *$g$-proximally increasing* if it satisfies the condition that

$$\left.\begin{array}{r} gy_1 \preceq gy_2 \\ d(x_1, fy_1) = d(A, B) \\ d(x_2, fy_2) = d(A, B) \end{array}\right\} \Longrightarrow x_1 \preceq x_2$$

where $x_1, x_2, y_1, y_2 \in A$.

**Remark 3.3.**

(i) An example of $g$-proximally increasing mapping can be found in Example 3.5.

(ii) One can see that, for a self-mapping $f, g : A \to A$, the concept of $g$-proximally increasing mapping reduces to that of $g$-non decreasing mapping.

Now, let us state our main result.

**Theorem 3.4.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $f : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions.*
*(i) $f$ is a $g$-proximally increasing and $(A, B)$ satisfies the $\mathcal{P}$-property;*
*(ii) $g(A_0)$ is closed and $f(A_0) \subseteq B_0, A_0 \subseteq g(A_0)$;*
*(iii)*

$$\psi(d(fx, fy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)) \text{ for all } x, y \in A \text{ such that } gx \preceq gy, \tag{3.1}$$

*where $\psi, \alpha, \beta : [0, \infty) \longrightarrow [0, \infty)$ are such that, $\psi$ is continuous and monotone non-decreasing, $\alpha$ is continuous, $\beta$ is lower semi-continuous,*

$$\psi(t) = 0 \quad \text{if and only if} \quad t = 0,\ \alpha(0) = \beta(0) = 0 \tag{3.2}$$

*and*

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all} \quad t > 0; \tag{3.3}$$

*(iv) There exist elements $x_0$ and $x_1$ in $A_0$ such that*

$$d(gx_1, fx_0) = d(A, B) \quad \text{and} \quad gx_0 \preceq gx_1.$$

*Also, if any nondecreasing sequence $\{x_n\}$ in $g(A_0)$ converges to $z$, then we assume*

$$x_n \preceq z \quad \text{for all} \quad n \geq 0. \tag{3.4}$$

*Then, there exists an element $x^*$ in $A$ such that*

$$d(gx^*, fx^*) = d(A, B).$$

**Proof.** By hypothesis (iv) there exist elements $x_0$ and $x_1$ in $A_0$ such that

$$d(gx_1, fx_0) = d(A, B) \quad \text{and} \quad gx_0 \preceq gx_1.$$

Because of the fact that $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2$ in $A_0$ such that

$$d(gx_2, fx_1) = d(A, B).$$

Since $f$ is $g$-proximally increasing, we get $gx_1 \preceq gx_2$. Continuing this process, we can construct a sequence $\{gx_n\}$ in $g(A_0)$ such that

$$d(gx_{n+1}, fx_n) = d(A,B) \quad \text{for all} \quad n \geq 0. \tag{3.5}$$
$$\text{with} \quad gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \cdots$$

Since $(A, B)$ satisfies the $\mathcal{P}$-property, we conclude that

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n), \quad \text{for all} \quad n \geq 1. \tag{3.6}$$

If $gx_{n_0} = gx_{n_0+1}$ for some $n_0 \geq 0$, then $d(gx_{n_0}, fx_{n_0}) = d(gx_{n_0+1}, fx_{n_0}) = d(A, B)$. The conclusion of the theorem follows.

Suppose that

$$d(gx_n, gx_{n+1}) \neq 0 \quad \text{for all} \quad n \geq 0. \tag{3.7}$$

We claim that $d(gx_n, gx_{n+1}) \leq d(gx_{n-1}, gx_n)$ for all $n \geq 1$. Suppose that

$$d(gx_{n-1}, gx_n) < d(gx_n, gx_{n+1}) \quad \text{for some} \quad n \geq 1.$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (3.1), using (3.5), (3.6) and the monotone property of $\psi$, we have

$$\begin{aligned} \psi(d(gx_{n-1}, gx_n)) \leq \psi(d(gx_n, gx_{n+1})) &= \psi(d(fx_{n-1}, fx_n)) \\ &\leq \alpha(d(gx_{n-1}, gx_n)) - \beta(d(gx_{n-1}, gx_n)). \end{aligned} \tag{3.8}$$

By (3.3), we have that $d(gx_{n-1}, gx_n) = 0$, which contradicts (3.7). Therefore,

$$d(gx_n, gx_{n+1}) \leq d(gx_{n-1}, gx_n) \quad \text{for all} \quad n \geq 1.$$

It follows that the sequence $\{d(gx_n, gx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(gx_n, gx_{n+1}) = r.$$

Taking $n \longrightarrow \infty$ in (3.8) and using the lower semi continuity of $\beta$ and the continuities of $\psi$ and $\alpha$, we obtain $\psi(r) \leq \alpha(r) - \beta(r)$, which, by (3.2), (3.3) implies that $r = 0$. Hence

$$\lim_{n \longrightarrow \infty} d(gx_n, gx_{n+1}) = 0. \tag{3.9}$$

Next, we will prove that $\{gx_n\}$ is a Cauchy sequence. We distinguish two cases.
**Case I.** Suppose that there exists $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, we observe that

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \alpha(d(gx_n, gx_{n+1})) - \beta(d(gx_n, gx_{n+1})) \\ &= \alpha(d(gx_n, gx_n)) - \beta(d(gx_n, gx_n)) \\ &= 0, \end{aligned}$$

which implies that $gx_{n+1} = gx_{n+2}$. So, for every $m > n$, we conclude that $gx_m = gx_n$. Hence $\{gx_n\}$ is a Cauchy sequence in $A$.

**Case II.** The successive terms of $\{gx_n\}$ are different. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequence $\{gx_{m(k)}\}, \{gx_{n(k)}\}$ of $\{gx_n\}$ such that $n(k)$ is the smallest index for which with

$$n(k) > m(k) > k, \qquad d(gx_{m(k)}, gx_{n(k)}) \geq \varepsilon. \tag{3.10}$$

This means that

$$d(gx_{m(k)}, gx_{n(k)-1}) < \varepsilon. \tag{3.11}$$

Now we have for all $k \geq 0$, by (3.11)

$$\begin{aligned}\varepsilon &\leq d(gx_{m(k)}, gx_{n(k)})\\ &\leq d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)})\\ &< \varepsilon + d(gx_{n(k)-1}, gx_{n(k)}).\end{aligned}$$

Taking $k \longrightarrow \infty$ in the above inequality and using (3.9) we obtain

$$\lim_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)}) = \varepsilon. \tag{3.12}$$

Also, by triangular inequality, for all $k \geq 0$, we have

$$d(gx_{m(k)+1}, gx_{n(k)+1}) \leq d(gx_{m(k)+1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gx_{n(k)}, gx_{n(k)+1})$$

and

$$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{n(k)}).$$

Taking limit as $k \longrightarrow \infty$ in the above two inequalities and using (3.9) and (3.12) we have

$$\lim_{k\longrightarrow\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = \varepsilon. \tag{3.13}$$

Again, by (3.5), we have that the elements $gx_{m(k)}$ and $gx_{n(k)}$ are comparable. Putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (3.1), for all $k \geq 0$, we have

$$\begin{aligned}\psi(d(gx_{m(k)+1}, gx_{n(k)+1})) &= \psi(d(fx_{m(k)}, fx_{n(k)}))\\ &\leq \alpha(d(gx_{m(k)}, gx_{n(k)})) - \beta(d(gx_{m(k)}, gx_{n(k)})).\end{aligned}$$

Taking $k \longrightarrow \infty$ in the above inequality, using (3.13), the continuities of $\psi$ and $\alpha$ and the lower semi-continuity of $\beta$, we obtain $\psi(\varepsilon) \leq \alpha(\varepsilon) - \beta(\varepsilon)$. Then, by (3.2), we have $\varepsilon = 0$, which is a contradiction. It then follows that $\{gx_n\}$ is a Cauchy sequence in $g(A_0)$. Since $(X, d)$ is complete and $g(A_0) \subseteq X$ is closed, we have

$g(A_0)$ is also complete. By using the completeness of the space $g(A_0)$ and the fact that $g(A_0)$ is closed, there exists $x^* \in A_0$ such that

$$gx_n \longrightarrow gx^* \in g(A_0).$$

By (3.4), we have $gx_n \preceq gx^*$ for all $n \in \mathbb{N}$.

Since $x^* \in A_0$, we have that $fx^* \in f(A_0) \subseteq B_0$ and therefore there exists a point $z \in A_0$ such that

$$d(z, fx^*) = d(A, B).$$

By (3.5) and applying $(A, B)$ satisfies the $\mathcal{P}$-property, we get

$$d(gx_{n+1}, z) = d(fx_n, fx^*).$$

Again, by (3.1)

$$\psi(d(gx_{n+1}, z)) = \psi(d(fx_n, fx^*)) \leq \alpha(d(gx_n, gx^*)) - \beta(d(gx_n, gx^*)).$$

Taking $n \longrightarrow \infty$ in the above inequality, using the continuities of $\psi$ and $\alpha$ and the lower semi-continuity of $\beta$, we have that $d(gx_{n+1}, z) \longrightarrow 0$ and consequently $z = gx^*$. Thus, we conclude that $d(gx^*, fx^*) = d(A, B)$. This completes the proof. $\square$

Now we give an example to support our result.

**Example 3.5.** Consider the complete metric space $X = \mathbb{R}^2$ with Euclidean metric. We define a partial order $\preceq$ on $X$ as $(x, y) \preceq (u, v)$ if and only if $x \leq u$ for all $(x, y), (u, v) \in X$. Let

$$A = \{(x, 1) : 0 \leq x \leq 1\}$$

and

$$B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $f : A \longrightarrow B,\ g : A \longrightarrow A$ as follows:

$$f(x, 1) = (\frac{3x}{4x + 8}, -1)$$

and

$$g(x, 1) = (\frac{x}{2 - x}, 1).$$

Let $\psi, \alpha, \beta : [0, \infty) \longrightarrow [0, \infty)$ be defined as $\psi(t) = \alpha(t) = t$ and $\beta(t) = \frac{t}{4}$ for all $t \in [0, 1]$.

It is easy to see that $d(A, B) = 2$, $A_0 = A$ and $B_0 = B$. Further $g(A_0)$ is closed and $f(A_0) \subseteq B_0, A_0 \subseteq g(A_0)$.

First, we will show that the pair $(A, B)$ have the $\mathcal{P}$-property.

Let $(x, 1), (u, 1) \in A_0$ and $(y, -1), (v, -1) \in B_0$ be such that

$$d((x, 1), (y, -1)) = d(A, B) \quad \text{and} \quad d((u, 1), (v, -1)) = d(A, B).$$

So that

$$\begin{aligned} 2 = d((x, 1), (y, -1)) &= \sqrt{(x - y)^2 + 2^2} \\ 2 = d((u, 1), (v, -1)) &= \sqrt{(u - v)^2 + 2^2}. \end{aligned}$$

It follows that

$$x = y \quad \text{and} \quad u = v,$$

and hence

$$d((x, 1), (u, 1)) = d((y, 1), (v, 1)) = d((y, -1), (v, -1)).$$

Next, we will show that $f$ is a $g$-proximally increasing. Let $(x, 1), (y, 1), (u, 1), (v, 1) \in A$ be such that

$$g(y, 1) \preceq g(v, 1) \quad \text{and} \quad d((x, 1), f(y, 1)) = d((u, 1), f(v, 1)) = d(A, B).$$

It follows that

$$\begin{aligned} g(y, 1) \preceq g(v, 1) &\Longleftrightarrow \frac{y}{2 - y} \leq \frac{v}{2 - v} \\ &\Longleftrightarrow 2y - vy \leq 2v - vy \\ &\Longleftrightarrow y \leq v \\ &\Longleftrightarrow 12vy + 24y \leq 12vy + 24v \\ &\Longleftrightarrow \frac{3y}{4y + 8} \leq \frac{3v}{4v + 8} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} 2 = d((x, 1), f(y, 1)) = d((x, 1), (\frac{3y}{4y + 8}, -1)) &\Longleftrightarrow x = \frac{3y}{4y + 8} \\ 2 = d((u, 1), f(v, 1)) = d((u, 1), (\frac{3v}{4v + 8}, -1)) &\Longleftrightarrow u = \frac{3v}{4v + 8}. \end{aligned}$$

That is $x \leq u \Longleftrightarrow (x, 1) \preceq (u, 1)$.

Finally, we will show that $f$ and $g$ are satisfy the condition (iii) of Theorem 3.4. Let $(x, 1), (y, 1) \in A$ be such that $g(x, 1) \preceq g(y, 1)$. By (3.14), we get that $x \leq y$.

Then

$$\begin{aligned}\psi(d(f(x,1),f(y,1))) &= \psi\Big(d\Big(\Big(\frac{3x}{4x+8},1\Big),\Big(\frac{3y}{4y+8},1\Big)\Big)\Big)\\ &= \psi\Big(\frac{3}{4}\Big|\frac{2(y-x)}{(x+2)(y+2)}\Big|\Big)\\ &= \frac{3}{2}.\frac{(y-x)}{(x+2)(y+2)}\\ &\leq \frac{3}{2}.\frac{(y-x)}{(2-x)(2-y)}\\ &= 2.\frac{(y-x)}{(2-x)(2-y)}-\frac{2}{4}.\frac{(y-x)}{(2-x)(2-y)}\\ &= \alpha(d(g(x,1),g(y,1)))-\beta(d(g(x,1),g(y,1))).\end{aligned}$$

Therefore, all hypothesis of Theorem 3.4 are satisfied. Furthermore, $x^* = (0,1) \in A$, because

$$d(g(0,1),f(0,1)) = d((0,1),(0,-1)) = 2 = d(A,B).$$

□

Taking $g = I_A$ (identity function) in Theorem 3.4, we have best proximity point theorem the following.

**Corollary 3.6.** *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Let $f : A \longrightarrow B$ satisfy the following conditions.*

*(i) $f$ is a proximally increasing and $(A,B)$ satisfies the $\mathcal{P}$-property;*

*(ii) $(A_0)$ is closed and $f(A_0) \subseteq B_0$;*

*(iii) $\psi(d(fx,fy)) \leq \alpha(d(x,y)) - \beta(d(x,y))$ for all $x,y \in A$ such that $x \preceq y$ where $\psi,\alpha,\beta : [0,\infty) \longrightarrow [0,\infty)$ are such that, $\psi$ is continuous and monotone non-decreasing, $\alpha$ is continuous, $\beta$ is lower semi-continuous,*

$$\psi(t) = 0 \quad \textit{if and only if} \quad t = 0,\ \alpha(0) = \beta(0) = 0$$

*and*

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all} \quad t > 0;$$

*(iv) There exist elements $x_0$ and $x_1$ in $A_0$ such that*

$$d(x_1, fx_0) = d(A,B) \quad \textit{and} \quad x_0 \preceq x_1.$$

*Also, if any nondecreasing sequence $\{x_n\}$ in $A_0$ converges to $z$, then we assume*

$$x_n \preceq z \quad \textit{for all} \quad n \geq 0.$$

*Then, there exists an element $x^*$ in $A$ such that*

$$d(x^*, fx^*) = d(A,B).$$

Since, for any nonempty subset $A$ of $X$, the pair $(A, A)$ has the $\mathcal{P}$-property, also one can see that, for a self-mapping, the notion of $g$-proximally increasing mapping reduces to that of $g$-non-decreasing mapping, we can deduce the following result, due to Choudhury and Kundu [23], by taking $A = B$ in Theorem 3.4.

**Corollary 3.7** ([23, Theorem 2.1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $f, g : X \longrightarrow X$ be such that $f(X) \subseteq g(X)$, $f$ is $g$-non-decreasing, $g(X)$ is closed and*

$$\psi(d(fx, fy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)) \ \text{ for all } \ x, y \in X \ \text{ such that } \ gx \preceq gy,$$

*where $\psi, \alpha, \beta : [0, \infty) \longrightarrow [0, \infty)$ are such that, $\psi$ is continuous and monotone non-decreasing, $\alpha$ is continuous, $\beta$ is lower semi-continuous,*

$$\begin{aligned} &\psi(t) = 0 \quad \text{if and only if} \quad t = 0, \alpha(0) = \beta(0) = 0 \\ &\text{and} \quad \psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all} \quad t > 0. \end{aligned}$$

*Also, if any nondecreasing sequence $\{x_n\}$ in $X$ converges to $z$, then we assume*

$$x_n \preceq z \quad \text{for all} \quad n \geq 0.$$

*If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then $f$ and $g$ have a coincidence point.*

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