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A Convergence Theorem for a Finite Family of Multivalued k-Strictly Pseudononspreading Mappings in $\mathbb{R}$-Trees[1]

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Abstract : In this paper, we introduce a new m-step iterative process for finite family of $k-$strictly pseudononspreading multivalued mappings in $\mathbb{R}$-trees. We obtain a strong convergence theorem of m-step iterative method to a common fixed point of a finite family of those multivalued mappings in $\mathbb{R}$-trees. Our results extend many known recent results in the literature. We close this work with the first examples of $k-$strictly pseudononspreading multivalued mappings in $\mathbb{R}$-trees.

Keywords : fixed point; multivalued mapping; $\mathbb{R}$-tree; $k-$strictly pseudononspreading; convergence theorems.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

Fixed point theory for single-valued mappings in $\mathbb{R}$-trees was first studied by Kirk [1]. He proved that every continuous single-valued mappings defined on a geodesically bounded complete $\mathbb{R}$-tree always has a fixed point. His works are followed by a series of new works by many authors(see, e.g., [2]-[6]). It is worth

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mentioning that fixed point theorems in $\mathbb{R}$-trees can be applied to graph theory, biology and computer science (see e.g., [7]-[10]).

In 2009, Shahzad and Zegeye [11] proved strong convergence theorems of the Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying the *end-point condition* in Banach spaces. Later in 2010, Puttasontiphot [12] obtained similar results in complete CAT(0) spaces. In 2012, Samanmit and Panyanak [13] introduced a condition on mappings in $\mathbb{R}$-trees which is weaker than the endpoint condition, is called the *gate condition*. They proved strong convergence theorems of a modified Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying such condition.

In 2011, Osilike and Isiogugu [14] introduced a new class of single-valued k-strictly pseudononspreading mappings in Hilbert space as follows: A mapping $T : E \to E$ is called *k-strictly pseudononspreading* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in E$. In a Hilbert space, the aboved inequality is equivalent to

$$(2 - k)\|Tx - Ty\|^2 \leq k\|x - y\|^2 + (1 - k)\|y - Tx\|^2 + (1 - k)\|x - Ty\|^2 + k\|x - Tx\|^2 + k\|y - Ty\|^2$$

for all $x, y \in E$. They proved weak and strong convergence theorems for those class of mappings in Hilbert spaces.

Recently, Phuengrattana[15] introduced a new class of multivalued k-strictly pseudononspreading mappings in $\mathbb{R}$-trees. He proved strong convergence theorems of a new two-step iterative process for two k-strictly pseudononspreading multivalued mappings having the gate condition.

In this paper, motivate by the above results, we introduce a new m-step iterative process for finite k-strictly pseudononspreading multivalued mappings in $\mathbb{R}$-trees. We also obtain the strong convergence theorem for approximating a common fixed point of those multivalued mappings in $\mathbb{R}$-trees by assuming the gate condition. Finally, we close this work with the first example for class of k-strictly pseudononspreading multivalued mappings in $\mathbb{R}$-trees.

2 Preliminaries

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic segment* joining x and y. When it is unique this geodesic is denoted by $[x, y]$. For $x, y \in X$ and $\alpha \in [0, 1]$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1-\alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points. If $x \in X$ and $E \subset X$, then the distance from x to E is defined by

$$\mathrm{d}(x, E) = \inf\{d(x, y) : y \in E\}.$$

The set E is called *proximinal* if for each $x \in X$, there exists an element $y \in E$ such that $d(x, y) = \mathrm{d}(x, E)$, and E is said to be *gated* if for any point $x \notin E$ there is a unique point y_x such that for any $z \in E$,

$$d(x, z) = d(x, y_x) + d(y_x, z).$$

Clearly gated sets in a complete geodesic space are always closed and convex. The point y_x is called the *gate* of x in E. It is easy to see that y_x is also the unique nearest point of x in E. We shall denote by $\mathcal{CB}(E)$ the family of nonempty closed bounded subsets of E, by $\mathcal{CC}(E)$ the family of nonempty closed convex subsets of E and by $\mathcal{KC}(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $\mathcal{CB}(E)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A} \mathrm{d}(a, B), \sup_{b\in B} \mathrm{d}(b, A)\right\}, \quad A, B \in \mathcal{CB}(E).$$

Let $T : E \to \mathcal{CB}(E)$ be a multivalued mapping. For each $x \in E$, we let

$$P_{Tx}(x) = \{u \in Tx : d(x, u) = \mathrm{d}(x, Tx)\}.$$

In the case of $P_{Tx}(x)$ is a singleton we will assume, without loss of generality, that $P_{Tx}(x)$ is a point in E. A point $x \in E$ is called a *fixed point* of T if $x \in Tx$. A point $x \in E$ is called an *endpoint* of T if x is a fixed point of T and $T(x) = \{x\}$[16]. We shall denote by $F(T)$ the set of all fixed points of T and by $E(T)$ the set of all endpoints of T. We see that for each mapping T, $E(T) \subseteq F(T)$ and the converse is not true in general. A mapping T is said to satisfies the *endpoint condition* if $E(T) = F(T)$.

An $\mathbb{R}-$tree is a special case of a CAT(0) space. For a thorough discussion of these spaces and their applications, see [17]. We now collect some basic properties of $\mathbb{R}-$trees.

Lemma 2.1. *Let X be a complete $\mathbb{R}-$tree and E be a nonempty subset of X. Then the following statements hold:*
(i) [18, page 1048] *the gate subsets of X are precisely its closed and convex subsets;*
(ii) [17, page 176] *if E is closed and convex, then for each $x \in X$, there exists a unique point $P_E(x) \in E$ such that*

$$d(x, P_E(x)) = \mathrm{d}(x, E).$$

That is, every nonempty closed convex subset of a complete $\mathbb{R}-$tree is proximinal.
(iii) [17, page 176] *if E is closed convex and x' belong to $[x, P_E(x)]$, then $P_E(x') = P_E(x)$;*

(iv) [6, Lemma 3.1] *if* A *and* B *are bounded closed convex subsets of* X, *then for any* $u \in X$,

$$d(P_A(u), P_B(u)) \leq H(A, B);$$

(v) [19, Lemma 2.5] *if* $x, y, z \in X$ *and* $\alpha \in [0,1]$, *then*

$$d^2((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d^2(x,z) + \alpha d^2(y,z) - \alpha(1-\alpha)d^2(x,y);$$

(vi) [19, Lemma 2.3] *if* $x, y, z \in X$, *then* $d(x,z) + d(z,y) = d(x,y)$ *if and only if* $z \in [x, y]$.

We state the following conditions in $\mathbb{R}$-trees:

A multivalued mapping $T : E \to \mathcal{CB}(E)$ is said to satisfy *condition (I)* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\mathrm{d}(x, Tx) \geq f(\mathrm{d}(x, F(T)))$ for all $x \in E$.

A finite family of multivalued mappings $\{T_i\}_{i=1}^m$ of E into $\mathcal{CB}(E)$ is said to satisfy *condition(m)* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\mathrm{d}(x, T_i x) \geq f(\mathrm{d}(x, \mathcal{F}))$ for some $i \in \{1, 2, ..., m\}$ and for all $x \in E$, where $\mathcal{F} = \bigcap_{i=1}^m F(T_i)$.

The following proposition is also needed.

Proposition 2.2 ([20])**.** *Let* (X, d) *be a complete metric space and* F *be a nonempty closed subset of* X. *Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_{n+1}, p) \leq d(x_n, p)$ *for all* $p \in F$ *and* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to some point in* F *if and only if* $\lim_{n\to\infty} \mathrm{d}(x_n, F) = 0$.

3 Main Results

Definition 3.1. Let E be a nonempty subset of a complete $\mathbb{R}$-tree X. A multivalued mapping $T : E \to \mathcal{CB}(E)$ is called

(i) *nonspreading* if

$$2H^2(Tx, Ty) \leq \mathrm{d}^2(y, Tx) + \mathrm{d}^2(x, Ty)$$

for all $x, y \in E$.

(ii) *k-strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$(2-k)H^2(Tx, Ty) \leq kd^2(x,y) + (1-k)\,\mathrm{d}^2(y, Tx) + (1-k)\,\mathrm{d}^2(x, Ty) + k\,\mathrm{d}^2(x, Tx) + k\,\mathrm{d}^2(y, Ty)$$

for all $x, y \in E$.

It is easy to see that every nonspreading multivalued mapping is 0-strictly pseudononspreading. Moreover, if T is k-strictly pseudononspreading with $F(T) \neq \emptyset$, then for all $x \in E$ and $p \in F(T)$ we have

$$H^2(Tx, Tp) \leq d^2(x,p) + k\,\mathrm{d}^2(x, Tx).$$

Thus T may not be quasi-nonexpansive. It is easy to show that if T is a k-strictly pseudononspreading multivalued mapping with $F(T) \neq \emptyset$, then $F(T)$ is closed.

The following result can be found in [15].

Lemma 3.1. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Assume that $T : E \to \mathcal{KC}(E)$ is a k-strictly pseudononspreading multivalued mapping. If $\{x_n\}$ is a sequence in E such that $x_n \to x$ and $\mathrm{d}(x_n, Tx_n) \to 0$ as $n \to \infty$, then $x \in Tx$.*

Now, we are ready to prove the main theorem.

Theorem 3.2. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T_1 : E \to \mathcal{KC}(E)$ be a k-strictly pseudononspreading multivalued mapping and $T_2, T_3, ..., T_m : E \to \mathcal{KC}(E)$ be k-strictly pseudononspreading and L-Lipschitzian multivalued mappings with $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Suppose that $T_1, T_2, ..., T_m$ satisfy the gate condition. Let $u_1, u_2, ..., u_m$ be keys of $T_1, T_2, ..., T_m$, respectively. For $x_1 \in E$, the sequence $\{x_n\}$ generated by*

$$y_n^{(1)} = \alpha_n^{(1)} z_n^{(1)} \oplus (1 - \alpha_n^{(1)}) x_n \text{ for all } n \in \mathbb{N},$$

where $z_n^{(1)}$ is the gate of u_1 in $T_1 x_n$, and

$$y_n^{(2)} = \alpha_n^{(2)} z_n^{(2)} \oplus (1 - \alpha_n^{(2)}) y_n^{(1)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(2)}$ is the gate of u_2 in $T_2 y_n^{(1)}$, and

$$\vdots$$

$$y_n^{(m-1)} = \alpha_n^{(m-1)} z_n^{(m-1)} \oplus (1 - \alpha_n^{(m-1)}) y_n^{(m-2)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(m-1)}$ is the gate of u_{m-1} in $T_{m-1} y_n^{(m-2)}$, and

$$x_{n+1} = \alpha_n^{(m)} z_n^{(m)} \oplus (1 - \alpha_n^{(m)}) y_n^{(m-1)} \text{ for all } n \in \mathbb{N},$$

where $z_n^{(m)}$ is the gate of u_m in $T_m y_n^{(m-1)}$. Let $\{\alpha_n^{(i)}\}$ be sequences in $[0,1]$ such that $0 < a \leq \alpha_n^{(i)} \leq b < 1 - k$ for each $i \in \{1, 2, ..., m\}$. If one of the following is satisfied:

(i) $\{T_i\}_{i=1}^m$ *satisfies condition*(m),

(ii) *one member of the family* $\{T_i\}_{i=1}^m$ *is hemicompact,*

then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Proof. Let $p \in \mathcal{F}$. By the gate condition and Lemma 2.1(v), we have

$$
\begin{aligned}
&d^2(x_{n+1}, p)\\
&= d^2(\alpha_n^{(m)} z_n^{(m)} \oplus (1-\alpha_n^{(m)}) y_n^{(m-1)}, p)\\
&\le (1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, p) + \alpha_n^{(m)} d^2(z_n^{(m)}, p) - \alpha_n^{(m)}(1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, p) + \alpha_n^{(m)} d^2(P_{T_m y_n^{(m-1)}}(u_m), P_{T_m p}(u_m))\\
&\quad - \alpha_n^{(m)}(1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, p) + \alpha_n^{(m)} H^2(T_m y_n^{(m-1)}, T_m p) - \alpha_n^{(m)}(1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, p) + \alpha_n^{(m)} (d^2(y_n^{(m-1)}, p) + k\, \mathrm{d}^2(y_n^{(m-1)}, T_m y_n^{(m-1)}))\\
&\quad - \alpha_n^{(m)}(1-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&= d^2(y_n^{(m-1)}, p) - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, p) + \alpha_n^{(m-1)} d^2(z_n^{(m-1)}, p) - \alpha_n^{(m-1)}(1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)})\\
&\quad - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, p) + \alpha_n^{(m-1)} d^2(P_{T_{(m-1)} y_n^{(m-2)}}(u_{(m-1)}), P_{T_{(m-1)} p}(u_{(m-1)}))\\
&\quad - \alpha_n^{(m-1)}(1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)}) - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, p) + \alpha_n^{(m-1)} H^2(T_{(m-1)} y_n^{(m-2)}, T_{(m-1)} p)\\
&\quad - \alpha_n^{(m-1)}(1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)}) - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\le (1-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, p) + \alpha_n^{(m-1)} (d^2(y_n^{(m-2)}, p) + k\, \mathrm{d}^2(y_n^{(m-2)}, T_{(m-1)} y_n^{(m-2)}))\\
&\quad - \alpha_n^{(m-1)}(1-k-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)}) - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&= d^2(y_n^{(m-2)}, p) - \alpha_n^{(m-1)}(1-k-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)})\\
&\quad - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)})\\
&\vdots\\
&\le d^2(x_n, p) - \alpha_n^{(1)}(1-k-\alpha_n^{(1)}) d^2(x_n, z_n^{(1)}) - \alpha_n^{(2)}(1-k-\alpha_n^{(2)}) d^2(y_n^{(1)}, z_n^{(2)}) - \ldots -\\
&\quad \alpha_n^{(m-1)}(1-k-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)}) - \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)}).
\end{aligned}
\tag{3.1}
$$

From $\alpha_n^{(i)} < 1-k$ for each $i \in \{1, 2, ..., m\}$, we obtain $d(x_{n+1}, p) \le d(x_n, p)$ for all $n \in \mathbb{N}$. This implies that $\{d(x_n, p)\}$ is nonincreasing and bounded below. Hence $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$. By (3.1), we have $\alpha_n^{(1)}(1-k-\alpha_n^{(1)}) d^2(x_n, z_n^{(1)}) + \alpha_n^{(2)}(1-k-\alpha_n^{(2)}) d^2(y_n^{(1)}, z_n^{(2)}) + \ldots + \alpha_n^{(m-1)}(1-k-\alpha_n^{(m-1)}) d^2(y_n^{(m-2)}, z_n^{(m-1)}) + \alpha_n^{(m)}(1-k-\alpha_n^{(m)}) d^2(y_n^{(m-1)}, z_n^{(m)}) \le d^2(x_n, p) - d^2(x_{n+1}, p)$.

Thus, by $0 < a \leq \alpha_n^{(i)} \leq b < 1 - k$ for each $i \in \{1, 2, ..., m\}$, we have

$$\lim_{n\to\infty} d(x_n, z_n^{(1)}) = 0 \text{ and } \lim_{n\to\infty} d(y_n^{(i-1)}, z_n^{(i)}) = 0 \text{ for each } i \in \{2, 3, ..., m\}. \tag{3.2}$$

Also, with $\mathrm{d}(x_n, T_1x_n) \leq d(x_n, z_n^{(1)})$, we have

$$\lim_{n\to\infty} \mathrm{d}(x_n, T_1x_n) = 0. \tag{3.3}$$

By using the definition of T_2, we have

$$\begin{aligned}
\mathrm{d}(x_n, T_2x_n) &\leq \mathrm{d}(x_n, T_2y_n^{(1)}) + H(T_2y_n^{(1)}, T_2x_n) \\
&\leq d(x_n, z_n^{(2)}) + Ld(y_n^{(1)}, x_n) \\
&\leq d(x_n, y_n^{(1)}) + d(y_n^{(1)}, z_n^{(2)}) + Ld(y_n^{(1)}, x_n) \\
&= (1 + L)d(x_n, y_n^{(1)}) + d(y_n^{(1)}, z_n^{(2)}) \\
&\leq (1 + L)(d(x_n, z_n^{(1)}) + d(z_n^{(1)}, y_n^{(1)})) + d(y_n^{(1)}, z_n^{(2)}) \\
&= (1 + L)(d(x_n, z_n^{(1)}) + (1 - \alpha_n^{(1)})d(z_n^{(1)}, x_n)) + d(y_n^{(1)}, z_n^{(2)}) \\
&= (1 + L)(2 - \alpha_n^{(1)})d(x_n, z_n^{(1)}) + d(y_n^{(1)}, z_n^{(2)}) \\
&\leq (1 + L)(2 - a)d(x_n, z_n^{(1)}) + d(y_n^{(1)}, z_n^{(2)}).
\end{aligned}$$

Thus by (3.2), we have

$$\lim_{n\to\infty} \mathrm{d}(x_n, T_2x_n) = 0. \tag{3.4}$$

Similaly, by using the definition of T_3, we have

$$\begin{aligned}
&\mathrm{d}(x_n, T_3x_n) \\
&\leq (1 + L_2)(2 - a)d(x_n, z_n^{(1)}) + (1 + L_1)(2 - a)d(y_n^{(1)}, z_n^{(2)}) + d(y_n^{(2)}, z_n^{(3)})
\end{aligned}$$

and also

$$\lim_{n\to\infty} \mathrm{d}(x_n, T_3x_n) = 0. \tag{3.5}$$

By using the same way, we have

$$\lim_{n\to\infty} \mathrm{d}(x_n, T_jx_n) = 0 \tag{3.6}$$

for all $j \in \{4, 5, ..., m\}$.

Case (i): $\{T_i\}_{i=1}^m$ satisfies condition(m). Then there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\mathrm{d}(x, T_i x) \geq f(\mathrm{d}(x,\mathcal{F}))$ for some $i \in \{1, 2, ..., m\}$ and for all $x \in E$, where $\mathcal{F} = \bigcap_{i=1}^m F(T_i)$.

If $\mathrm{d}(x, T_1 x) \geq f(\mathrm{d}(x,\mathcal{F}))$ for all $x \in E$. For eaah $n \in \mathbb{N}$, we have $x_n \in E$. By using (3.3), we obtain

$$0 = \lim_{n\to\infty} \mathrm{d}(x_n, T_1 x_n) \geq \lim_{n\to\infty}(f(\mathrm{d}(x_n,\mathcal{F}))) = f(\lim_{n\to\infty}(\mathrm{d}(x_n,\mathcal{F}))) \geq 0.$$

Hence $f(\lim_{n\to\infty}(\mathrm{d}(x_n,\mathcal{F}))) = 0$, therefore $\lim_{n\to\infty}(\mathrm{d}(x_n,\mathcal{F})) = 0$. Similarly in other cases, we can use (3.4), (3.5) and (3.6) to show that $\lim_{n\to\infty}(\mathrm{d}(x_n,\mathcal{F}))= 0$. By the closedness of $\mathcal{F}$ and Proposition 2.2, we have $\{x_n\}$ converges strongly to some point in $\mathcal{F}$.

Case (ii): One member of the family $\{T_i\}_{i=1}^m$ is hemicompact. Without loss of generality, we assume that T_1 is hemicompact. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\{x_{n_l}\}$ converges strongly to $z \in E$. By (3.3), (3.4), (3.5) and (3.6), it follows by Lemma 3.1 that $z \in \mathcal{F}$. Since $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$, it implies that $\{x_n\}$ converges strongly to $z \in \mathcal{F}$. □

As a direct consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X. Let $T_1 : E \to \mathcal{KC}(E)$ be a nonspreading multivalued mapping and $T_2, T_3, ..., T_m : E \to \mathcal{KC}(E)$ be nonspreading and L-Lipschitzian multivalued mappings with $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Suppose that $T_1, T_2, ..., T_m$ satisfy the gate condition. Let $u_1, u_2, ..., u_m$ be keys of $T_1, T_2, ..., T_m$, respectively. For $x_1 \in E$, the sequence $\{x_n\}$ generated by*

$$y_n^{(1)} = \alpha_n^{(1)} z_n^{(1)} \oplus (1 - \alpha_n^{(1)}) x_n \quad \text{for all } n \in \mathbb{N},$$

where $z_n^{(1)}$ is the gate of u_1 in $T_1 x_n$, and

$$y_n^{(2)} = \alpha_n^{(2)} z_n^{(2)} \oplus (1 - \alpha_n^{(2)}) y_n^{(1)} \quad \text{for all } n \in \mathbb{N},$$

where $z_n^{(2)}$ is the gate of u_2 in $T_2 y_n^{(1)}$, and

$$\vdots$$

$$y_n^{(m-1)} = \alpha_n^{(m-1)} z_n^{(m-1)} \oplus (1 - \alpha_n^{(m-1)}) y_n^{(m-2)} \quad \text{for all } n \in \mathbb{N},$$

where $z_n^{(m-1)}$ is the gate of u_{m-1} in $T_{m-1} y_n^{(m-2)}$, and

$$x_{n+1} = \alpha_n^{(m)} z_n^{(m)} \oplus (1 - \alpha_n^{(m)}) y_n^{(m-1)} \quad \text{for all } n \in \mathbb{N},$$

where $z_n^{(m)}$ is the gate of u_m in $T_m y_n^{(m-1)}$. Let $\{\alpha_n^{(i)}\}$ be sequences in $[0,1]$ such that $0 < a \leq \alpha_n^{(i)} \leq b < 1-k$ for each $i \in \{1,2,...,m\}$. If one of the following is satisfied:

(i) $\{T_i\}_{i=1}^m$ satisfies condition(m),

(ii) one member of the family $\{T_i\}_{i=1}^m$ is hemicompact,

then $\{x_n\}$ converges strongly to an element of $\mathcal{F}$.

Next, this is the first example of that class in the literature.

Example 3.4. (For k-strictly pseudononspreading multivalued mappings.)
(1.) Let $E=[0,\infty), k \in [0,1)$ and $T: E \to \mathcal{KC}(E)$ be defined by

$$Tx = [0, (\frac{k}{b})x] \text{ for all } x \in E \text{ and } b \geq 2.$$

Then T is $k-$strictly pseudononspreading.
(2.) Let $E=[0,\infty), k \in [0,1)$ and $S: E \to \mathcal{KC}(E)$ be defined by

$$S(x) = [((\frac{k}{b}) - (\frac{k}{b})^2)x, (\frac{k}{b})x] \text{ for all } x \in E \text{ and } b \geq 2.$$

Then S is $k-$strictly pseudononspreading.

Proof. (1.) We see that $H(Tx,Ty) = (\frac{k}{b})|x,y|$.
Since $b \geq 2$, we obtain $b^2 \geq 2b$ and $b > k$.
So we have $2b > 2k$ and this show that $b^2 \geq 2b > 2k \geq 2k - k^2$.
From $2k - k^2 < b^2$, we have

$$(2-k)k^2 = 2k^2 - k^3 \leq b^2 k.$$

This show that

$$(2-k)(\frac{k}{b})^2 \leq k. \tag{3.7}$$

From (3.7), we obtain

$$\begin{aligned}(2-k)H^2(Tx,Ty) &= (2-k)|(\frac{k}{b})x - (\frac{k}{b})y|^2 \\ &= (2-k)(\frac{k}{b})^2|x-y|^2 \\ &\leq k|x-y|^2 = kd^2(x,y) \\ &\leq kd^2(x,y) + (1-k)\mathrm{d}^2(y,Tx) + (1-k)\mathrm{d}^2(x,Ty) \\ &\quad + k\mathrm{d}^2(x,Tx) + k\mathrm{d}^2(y,Ty).\end{aligned}$$

Hence T is $k-$strictly pseudononspreading.

(2.) Similarly. □

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