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# Common Coupled Fixed Points of Compatible Pair of Maps Satisfying Condition (B) with a Rational Expression in Partially Ordered Metric Spaces[1]

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**Abstract :** The aim of this paper is to prove the existence of common coupled coincidence points and common coupled fixed points for a pair of compatible maps satisfying condition (B) with a rational expression in partially ordered metric spaces. Further, we discuss the importance of rational expression in condition (B). Our results extend and improve the results of Luong and Thuan [N.L. Luoung, N.X. Thuan, Coupled fixed point theorems in partially ordered metric spaces, Bull. Math. Anal. Appl. 2 (2010) 14–24], generalize the results of Lakshmikantham and Ciric [V. Lakshmikantham, Lj.B. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349], and Saud and Abdullah [S.M. Alsulami, A. Alotaibi, Coupled cpoincidence

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point theorems for compatible mappings in partially ordered metric spaces, Bull. Math. Anal. Appl. 4 (2) (2012) 129–138].

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# 1 Introduction

Existence of fixed points in partially ordered metric spaces was initiated by Ran and Reurings [1]. Afterwards in 2006, Bhaskar and Lakshmikantam [2] proved some coupled fixed point theorems for mixed monotone mappings and discussed the existence and uniqueness of solutions for a periodic boundary value problem. Later in 2009, the results of Lakshmikantham and Bhaskar were extended to two mappings by Lakshmikantham and Ciric [3].

In 2010, Choudhury and Kundu [4] introduced the notion of compatibility in the concept of coupled coincidence points and generalized the results of Lakshmikantham and Ciric [3]. Recently, Abbas *et al.* [5] have introduced the concept of $w$-compatible maps and obtained coupled coincidence points for non-linear contractive mappings in cone metric spaces. For more literature on the existence of coupled fixed points, we refer [1, 3, 6–23].

**Definition 1.1.** Let $X$ be a nonempty set. A partial order is a binary relation $\preceq$ over $X$ which is reflexive, anti-symmetric and transitive. A set $X$ together with the binary relation $\preceq$ is called a partially ordered set, which is denoted by $(X, \preceq)$.

**Definition 1.2.** Let $(X, \preceq)$ be a partially ordered set. A selfmap $g$ on $X$ is said to be

$(i)$ nondecreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $g(x_1) \preceq g(x_2)$;

$(ii)$ nonincreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $g(x_1) \succeq g(x_2)$.

**Definition 1.3** ([2])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. $F$ is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in $x$ and monotone nonincreasing in $y$ *i.e.*, for any $x_1,\ x_2 \in X$ with $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ and for all $y$ in $X$ and for any $y_1,\ y_2 \in X$ with $y_1 \preceq y_2$ implies $F(x, y_1) \succeq (F(x, y_2)$ for all $x$ in $X$.

**Definition 1.4** ([2])**.** Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

**Definition 1.5** ([3])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say that $F$ has mixed g-monotone property if $g$ is nondecreasing in its first argument and nonincreasing in second argument. *i.e.*, for any $x_1,\ x_2 \in X$

with $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for all $y \in X$ and for any $y_1,\ y_2 \in X$ with $gy_1 \preceq gy_2$ implies $F(x, y_1) \succeq F(x, y_2)$ for all $x$ in $X$.

**Definition 1.6** ([3])**.** Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.

**Definition 1.7** ([3])**.** Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a common coupled fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

**Definition 1.8** ([4])**.** Let $(X, d)$ be metric space. $F : X \times X \to X$ and $g : X \to X$ be two mappings. $F$ and $g$ are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$.

**Definition 1.9** ([5])**.** Let $(X, d)$ be metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. $F$ and $g$ are said to be $w$-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Clearly, compatibility implies $w-$compatible but its converse need not be true.

**Example 1.10.** *Let $X = [0, 1]$ with the usual metric. We define $F : X \times X \to X$ and $g : X \to X$ by*

$$F(x,y) = \begin{cases} \frac{x}{4} + \frac{y}{4} & if\ x \in [0, \frac{1}{2}]\ and\ y \in [0, 1] \\ \frac{x}{3} + \frac{y}{3} & if\ x \in (\frac{1}{2}, 1]\ and\ y \in [0, 1] \end{cases} \quad and\ gx = \begin{cases} \frac{x}{2} & if\ x \in [0, \frac{1}{2}] \\ \frac{2x}{3} & if\ x \in (\frac{1}{2}, 1]. \end{cases}$$

*We consider the following four cases to show $F$ and $g$ are $w$-compatible on $X$.*

*Case (1): Let $x \in [0, \frac{1}{2}],\ y \in [0, \frac{1}{2}]$. Then $F(x, y) = \frac{x}{4} + \frac{y}{4} = gx = \frac{x}{2}$ and $F(y, x) = \frac{y}{4} + \frac{x}{4} = gy = \frac{y}{2}$ implies $x = y$. Also, $g(F(x, x)) = \frac{x}{4}$ and $F(gx, gx) = F(\frac{x}{2}, \frac{x}{2}) = \frac{x}{4}$. Hence $g(F(x, x)) = F(gx, gy)$.*

*Case (2): Let $x \in [0, \frac{1}{2}],\ y \in [\frac{1}{2}, 1]$. Then $F(x, y) = \frac{x}{4} + \frac{y}{4} = gx = \frac{x}{2}$ and $F(y, x) = \frac{y}{3} + \frac{x}{3} = gy = \frac{2y}{3}$ implies $x = y = \frac{1}{2}$. Now $g(F(\frac{1}{2}, \frac{1}{2})) = \frac{1}{8} = F(g\frac{1}{2}, g\frac{1}{2})$.*

*Case (3): Let $x \in (\frac{1}{2}, 1], y \in (\frac{1}{2}, 1]$. Then $F(x, y) = \frac{x}{3} + \frac{y}{3} = gx = \frac{2x}{3}$ and $F(y, x) = \frac{y}{3} + \frac{x}{3} = gy = \frac{2y}{3}$ implies $x = y$.Now $g(F(x, x)) = g(\frac{x}{3} + \frac{x}{3}) = \frac{4x}{9}$ and $F(gx, gx) = \frac{4x}{9}$.*

*Case (4): Let $x \in (\frac{1}{2}, 1], y \in [0, \frac{1}{2}]$. Then $F(x, y) = \frac{x}{3} + \frac{y}{3} = gx = \frac{2x}{3}$ and $F(y, x) = \frac{y}{4} + \frac{x}{4} = gy = \frac{y}{2}$ implies $x = y$. Hence $F$ and $g$ are $w$-compatible.*

*From the above four cases, we have $F$ and $g$ are $w$-compatible on $X$.*

*We observe that $F$ and $g$ are not compatible. Let $x_n = \frac{1}{2}+\frac{1}{n}$ and $y_n = \frac{1}{2}-\frac{1}{n}$. Then $F(x_n, y_n) = F(\frac{1}{2}+\frac{1}{n}, \frac{1}{2}-\frac{1}{n}) = \frac{\frac{1}{2}+\frac{1}{n}+\frac{1}{2}-\frac{1}{n}}{3} = \frac{1}{3}$ and $gx_n = g(\frac{1}{2}+\frac{1}{n}) = \frac{2}{3}(\frac{1}{2}+\frac{1}{n}) = \frac{1}{3}+\frac{2}{3n}$. Hence $\lim\limits_{n\to\infty} = F(x_n, y_n) = \lim\limits_{n\to\infty} gx_n = \frac{1}{3}$. Also, $F(y_n, x_n) = F(\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n}) = \frac{(\frac{1}{2}-\frac{1}{n}+\frac{1}{2}+\frac{1}{n})}{4} = \frac{1}{4}$ and $gy_n = g(\frac{1}{2}-\frac{1}{n}) = \frac{1}{2}(\frac{1}{2}-\frac{1}{n}) = \frac{1}{4}-\frac{1}{2n}$. Hence $\lim\limits_{n\to\infty} = F(y_n, x_n) = \lim\limits_{n\to\infty} gy_n = \frac{1}{4}$. Now $\lim\limits_{n\to\infty} d(F(gx_n, gy_n), g(F(x_n, y_n)) = \lim\limits_{n\to\infty} d(\frac{\frac{1}{3}+\frac{2}{2n}+\frac{1}{4}-\frac{1}{2n}}{4}, g(\frac{1}{3})) = \lim\limits_{n\to\infty} |\frac{15}{48}+\frac{1}{2n}| \neq 0$.*

The following theorem was proved by Bhaskar and Lakshmikantham [2].

**Theorem 1.1** ([2])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Suppose that $F : X \times X \to X$ is mapping such that $F$ has mixed monotone property on $X$. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) = \frac{k}{2}[d(x, u) + d(y, v)] \tag{1.1}$$

*for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

*Suppose that either*

*(i) $F$ is continuous, or*

*(ii) $X$ has the following property:*

*(a) if $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ then $x_n \preceq x$ for all $n$.*

*(b) if $\{y_n\}$ is a non-increasing sequence with $y_n \to y$ then $y \preceq y_n$ for all $n$.*

*Then $x = F(x, y)$ and $y = F(y, x)$. i.e., $F$ has a coupled fixed point in $X$.*

In 2008, Babu *et al.* [24] considered the following class of mappings satisfying *condition* $(B)$.

Let $(X, d)$ be a metric space. A map $T : X \to X$ is said to satisfy *condition* $(B)$, if there exist $\delta \in (0, 1)$ for some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Lmin\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

Using an analogue of condition (B) Luong and Thuan [17] generalized the results of Bhaskar and Lakshmikantham [2] and proved the following fixed point theorem.

**Theorem 1.2** ([17]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Suppose that $F : X \times X \to X$ is a mapping such that $F$ has mixed monotone property on $X$ and assume that there exist $\alpha$, $\beta \in [0, 1)$ and $L \geq 0$ with $\alpha + \beta < 1$ such that*

$$d(F(x,y), F(u,v)) \leq \alpha d(x,u) + \beta d(y,v) + L \min\{d(F(x,y),u), d(F(u,v),x), d(F(x,y),x), d(F(u,v),u)\} \quad (1.2)$$

*for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. If there exists $x_0, y_0 \in X$ such that*

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

*Suppose that either*

*$(i)$ $F$ is continuous, or*

*$(ii)$ $X$ has the following property:*

*$(a)$ if $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$ then $x_n \preceq x$ for all $n$.*

*$(b)$ if $\{y_n\}$ is a non-increasing sequence such that $y_n \to y$ then $y \preceq y_n$ for all $n$.*

*Then $x = F(x, y)$ and $y = F(y, x)$. i.e., $F$ has a coupled fixed point in $X$. Moreover, $F$ has a unique coupled fixed point if $(x, y) \in X \times X$ is comparable with $(u, v) \in X \times X$.*

**Remark 1.11.** *In Theorem* 1.1 *by taking $\alpha = \beta$ and $L = 0$, we get Theorem* 1.2.

In 2009, Lakshmikantham and Ciric [3] extended Theorem 1.1 to two commuting mappings and with mixed g-monotone property of $F$ and obtained the following theorem as a corollary (Corollary 2.1, [3]).

**Theorem 1.3** ([3]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that $F$ has mixed $g-$monotone property and assume that there exists $k \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) = \frac{k}{2}[d(gx, gu) + d(gy, gv)] \quad (1.3)$$

*for all $x, y, u, v, \in X$ with $gx \succeq gu$ and $gy \preceq gv$. If there exists $x_0, y_0 \in X$ such that*

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

*Suppose $F(X \times X) \subseteq gX$, $g$ is continuous and commutes with $F$. Also, assume that either*

*$(i)$ $F$ is continuous, or*

*$(ii)$ $X$ has the following property:*

*$(a)$ if $\{x_n\}$ is a nondecreasing sequence with $x_n \to x$ then $x_n \preceq x$ for all $n$.*

*$(b)$ if $\{y_n\}$ is a nonincreasing sequence with $y_n \to y$ then $y \preceq y_n$ for all $n$.*

*Then $gx = F(x,y)$ and $gy = F(y,x)$. Then $F$ and $g$ have a coupled coincidence point in $X$. Moreover, $F$ and $g$ have a unique coupled fixed point if $(x,y) \in X \times X$ is comparable with $(u,v) \in X \times X$.*

In 2012, Saud and Abdulla [21] extended Theorem 1.2 to two mappings in the following way.

**Theorem 1.4** ([21])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are such that $F$ has mixed $g-$monotone property on $X$ and assume that there exist $\alpha$, $\beta \in [0,1)$ and $L \geq 0$ with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) &\leq \alpha d(gx,gu) + \beta d(gy,gv) \\ &\quad + L\min\{d(F(x,y),gu), d(F(u,v),gx), d(F(x,y),gx), d(F(u,v),gu)\} \end{aligned} \tag{1.4}$$

*for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. If there exists $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$. Suppose $F(X \times X) \subseteq gX$ . Also suppose that*

*$(i)$ $g$ is a continuous and monotonically increasing on $X$.*

*$(ii)$ $F$ and $g$ are compatible.*

*Either*

*$(iii)$ $(a)$ $F$ is continuous, or*

*$(iii)$ $(b)$ $X$ has the following property:*

*(1) if $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ then $x_n \preceq x$ for all $n$.*

*(2) if $\{y_n\}$ is a nonincreasing sequence such that $y_n \to y$ then $y \preceq y_n$ for all $n$.*

*Then $gx = F(x,y)$ and $gy = F(y,x)$ .i.e, $F$ and $g$ have a coupled coincidence point in $X$.*

Recently, Karapinar et al. [16] introduced a more general contraction condition (1.5) than (1.4) and proved the existence of coupled coincidence points.

**Theorem 1.5** (Theorem 2.1 and Theorem 2.2, [16])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are such that $F$ has mixed $g-$monotone property and assume that there exists $\varphi : [0,\infty) \to [0,\infty)$ which is* *continuous, $\varphi(t) < t$, for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$ and $L \geq 0$ such that*

$$d(F(x,y), F(u,v)) \leq \varphi(max\{d(gx,gu), d(gy,gv)\} \\ +L\min\{d(F(x,y),gu), d(F(u,v),gx), d(F(x,y),gx), d(F(u,v),gu)\} \quad (1.5)$$

*for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. If there exists $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Suppose $F(X \times X) \subseteq gX$ . Also suppose that*

*$(i)$ $g$ is a continuous $X$.*

*$(ii)$ $F$ and $g$ are compatible.*

*Either*

*$(iii)$ $(a)$ $F$ is continuous, or*

*$(iii)(b)$ $X$ has the following property:*

*(1) if $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$ then $gx_n \preceq gx$ for all $n$.*

*(2) if $\{y_n\}$ is a nonincreasing sequence such that $y_n \to y$ then $gy \preceq gy_n$ for all $n$.*

*Then $gx = F(x,y)$ and $gy = F(y,x)$ i.e, $F$ and $g$ have a coupled coincidence point in $X$.*

Here we observe that condition (1.4) is a special case of (1.5) by choosing $\varphi(t) = (\alpha + \beta)t$, where $\alpha + \beta < 1$.

Hence, under the hypotheses of Theorem 1.5, it is possible to apply Theorem 1.4 upto the existence of coupled coincidence points. In fact, we can conclude more with the hypotheses of Theorem 1.4, *i.e.*, the existence of common coupled fixed points too. We prove it by introducing the following contractive condition with a rational expression which is more general than condition (1.4).

**Definition 1.12.** Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric $d$ on $X$ such that $(X, d)$ is a metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are mappings satisfying the following condition:

if there exist $\alpha$, $\beta$, $\gamma \in [0, 1)$ and $L \geq 0$ with $\alpha + \beta + \gamma < 1$ such that

$$d(F(x,y), F(u,v)) \leq \alpha d(gx,gu) + \beta d(gy,gv) \\ +\frac{\gamma}{2}\frac{[d(gx,F(x,y)) + d(gy,F(y,x))][d(gu,F(u,v)) + d(gv,F(v,u)]}{1 + d(gx,gu) + d(gy,gv)} \\ +L\min\{d(F(x,y),gu), d(F(u,v),gx), d(F(x,y),gx), d(F(u,v),gu)\} \quad (1.6)$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, then we say that $F$ and $g$ satisfy 'condition (B) with a rational expression'.

It is trivial to see that the condition (1.4) implies (1.6). But, the following example shows that its converse need not be true so that condition (1.6) is more general than (1.4).

**Example 1.13.** *Let* $X = \{0, \frac{1}{2}, 2\}$ *with the usual metric, and* $\leq := \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (2,2), (0,2)\}$. *We write* $A = \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, 2), (2,2), (2,0), (0,2)\}$, $B = \{(\frac{1}{2}, 0), (2, \frac{1}{2})(\frac{1}{2}, \frac{1}{2})\}$ *We define* $F : X \times X \to X$ *and* $g : X \to X$ *by*

$$F(x,y) = \begin{cases} 0 & if \quad (x,y) \in A \\ \\ \frac{1}{2} & if \quad (x,y) \in B \end{cases} \quad and \quad g0 = 0, g\tfrac{1}{2} = 2 \ and \ g2 = \tfrac{1}{2}.$$

*We take* $x, y, u, v \in X$, *such that* $gx \geq gu$ *and* $gy \leq gv$, *then inequality* (1.6) *holds with* $\alpha, \beta = \frac{1}{4}, \gamma = \frac{3}{8}$ *and* $L = 1$. *But condition* (1.4) *fails to hold at* $x = 2, y = \frac{1}{2}, u = 0$ *and* $v = \frac{1}{2}$ *for any* $\alpha \geq 0$, $\beta \geq 0$, *with* $\alpha + \beta < 1$. *Indeed, since we have*

$$\begin{aligned} d(F(x,y), F(u,v)) = \tfrac{1}{2} \nleq \tfrac{\alpha}{2} &= \alpha d(gx, gu) + \beta.0 + L.0 \\ &= \alpha d(gx, gu) + \beta d(gy, gv) \\ &\quad + L \min\{d(F(x,y), gu), d(F(u,v), gx), \\ &\qquad d(F(x,y), gx), d(F(u,v), gu)\} \end{aligned}$$

*for any* $\alpha \geq 0$, $\beta \geq 0$, *with* $\alpha + \beta < 1$ *and* $L \geq 0$.

*Also, for the above chosen values of* $x$, $y$, $u$ *and* $v$, *the inequality* (1.5) *fails to hold for any* $\varphi(t) < t$, *since*

$$\begin{aligned} d(F(x,y), F(u,v)) = \tfrac{1}{2} \nleq \varphi(max\{d(gx, gu), 0\}) &+ L.0 \\ = \varphi(max\{d(gx, gu), d(gy, gv)\}) & \\ + L \min\{d(F(x,y), gu), d(F(u,v), gx), & \\ d(F(x,y), gx), d(F(u,v), gu)\} & \end{aligned}$$

*for any* $\varphi$ *which is continuous , and* $\varphi(0) = 0$ *and* $\varphi(t) < t$ *for all* $t > 0$.

Hence it is our interest to find the existence of coupled coincidence points and further existence of common coupled fixed points for the maps $F$ and $g$ satisfying condition (1.6).

The aim of this paper is to prove the existence of coupled coincidence points and then the existence and uniqueness of common coupled fixed point for a pair of maps $F$ and $g$ satisfying a more general condition 'condition (B) with a rational expression'. Further, we discuss the importance of rational expression in condition (B). Our results generalize the results of Lakshmikantam and Ciric [3], Saud and Abdullah [21] and extends the results of Luoung and Thuan [17].

# 2 Main Results

**Theorem 2.1.** *Let* $(X, \preceq, d)$ *be a partially ordered metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are such that* $F$ *has mixed* $g-$*monotone property.*

*Assume that $F$ and $g$ satisfy condition (B) with a rational expression. If there exists $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Further, suppose that $F(X \times X) \subseteq gX$ . Let $\{x_n\}$ and $\{y_n\}$ defined by $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. Then $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in $X$.*

*Proof.* Let $x_0,\ y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq gX$, we define $\{x_n\}$ and $\{y_n\}$ such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{2.1}$$

We first show that

$$gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1} \text{ for all } n \geq 0, \tag{2.2}$$

by mathematical induction. By our assumption we have $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Thus $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Thus, (2.2) is true for $n = 0$. Suppose (2.2) is true for some $n = m$ *i.e.*

$$gx_m \preceq gx_{m+1} \text{ and } gy_m \succeq gy_{m+1}. \tag{2.3}$$

We shall prove that (2.2) is true for some $n = m + 1$. By mixed $g$-monotone property of $F$, using (2.3), we have

$$gx_{m+2} = F(x_{m+1}, y_{m+1}) \succeq F(x_m, y_{m+1}) \succeq F(x_m, y_m) = gx_{m+1} \tag{2.4}$$

and

$$gy_{m+2} = F(y_{m+1}, x_{m+1}) \preceq F(y_{m+1}, x_m) \preceq F(y_m, x_m) = gy_{m+1}, \tag{2.5}$$

so that (2.2) is true for some $n = m + 1$. Thus from (2.3), (2.4) and (2.5) we conclude that (2.2) is true for all $n \geq 0$ by induction. Since $gx_n \succeq gx_{n-1}$ and $gy_n \preceq gy_{n-1}$, using condition (1.6), we have

$$\begin{aligned}
&d(gx_{n+1}, gx_n)\\
&= d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))\\
&\leq \alpha d(gx_n, gx_{n-1}) + \beta d(gy_n, gy_{n-1})\\
&\quad + \frac{\gamma}{2} \frac{[d(gx_n, F(x_n, y_n)) + d(gy_n, F(y_n, x_n))][d(gx_{n-1}, F(x_{n-1}, y_{n-1})) + d(gy_{n-1}, F(y_{n-1}, x_{n-1}))]}{1 + d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}\\
&\quad + L \min\{d(F(x_n, y_n), gx_{n-1}), d(F(x_{n-1}, y_{n-1}), gx_n), d(F(x_n, y_n), gx_n), d(F(x_{n-1}, y_{n-1}), gx_{n-1})\}\\
&\leq \alpha d(gx_n, gx_{n-1}) + \beta d(gy_n, gy_{n-1}) + \frac{\gamma}{2}[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})].
\end{aligned} \tag{2.6}$$

Similarly since $gy_{n-1} \succeq gy_n$ and $gx_{n-1} \preceq gx_n$, using condition (1.6), we have

$$\begin{aligned}
d(gy_n, gy_{n+1}) \leq{}& \alpha d(gy_n, gy_{n-1}) + \beta d(gx_n, gx_{n-1})\\
&+ \frac{\gamma}{2}[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})].
\end{aligned} \tag{2.7}$$

Thus, form (2.6) and (2.7), we have

$$d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1}) \leq (\alpha + \beta)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]. \quad (2.8)$$

Let $d_n = d(gx_{n+1}, gx_n) + d(gy_n, gy_{n+1})$ and $\delta = \frac{(\alpha+\beta)}{1-\gamma}$. Then from (2.8), we have $d_n \leq \delta d_{n-1}$. Hence it follows that $d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq \cdots \leq \delta^n d_0$. Taking limits $n \to \infty$, we have $\lim_{n\to\infty} d_n = 0$, which implies implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n-1}) = 0 \text{ and } \lim_{n\to\infty} d(gy_n, gy_{n-1}) = 0. \quad (2.9)$$

Now for $m \geq n$, we have

$$d(gx_m, gx_n) \leq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \cdots + d(gx_{n+1}, gx_n)$$

and

$$d(gy_m, gy_n) \leq d(gy_m, gy_{m-1}) + d(gy_{m-1}, gy_{m-2}) + \cdots + d(gy_{n+1}, gy_n).$$

Therefore,

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\leq d_{m-1} + d_{m-2} + \cdots + d_n \\ &\leq \delta^{m-1} d_0 + \delta^{m-2} d_0 + \delta^{m-3} d_0 + \cdots + \delta^n d_0 \\ &\leq \frac{\delta^n}{1-\delta} d_0, \end{aligned}$$

which tends to 0 as $n \to \infty$. Hence $\lim_{n\to\infty} d(gx_m, gx_n) = 0$ and $\lim_{n\to\infty} d(gy_m, gy_n) = 0$. Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $X$. □

**Theorem 2.2.** *In addition to the hypotheses of Theorem 2.1, suppose that:*

*(i) $(X, d)$ is complete.*

*(ii) $g$ is a continuous and monotonically increasing on $X$.*

*(iii) $F$ and $g$ are compatible.*

*Either*

*(iv) (a) $F$ is continuous, or*

*(b) $X$ has the following property:*

*(1) if $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x$ then $x_n \preceq x$ for all $n$.*

*(2) if $\{y_n\}$ is a nonincreasing sequence such that $y_n \to y$ then $y \preceq y_n$ for all $n$.*

*Then $gx = F(x, y)$ and $gy = F(y, x)$. i.e, $F$ and $g$ have a coupled coincidence point in $X$.*

*Proof.* By the proof of Theorem 2.1, we have $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $X$. Since $X$ is complete there exist $x$ and $y \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = x \text{ and } \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = y. \tag{2.10}$$

We now prove that $(x, y)$ is a coupled coincidence point of $F$ and $g$. Since $F$ and $g$ are compatible, from (2.10), we have

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \tag{2.11}$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0. \tag{2.12}$$

Suppose $(iv)(a)$ holds. Now for all $n \geq 0$, we have

$$d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n)).$$

On taking limits as $n \to \infty$, using continuity of $g$ and (2.11), we get $d(gx, F(x, y)) = 0$. Similarly, using continuity of $F$, $g$ and (2.12), we get $d(gy, F(y, x)) = 0$. Thus $F$ and $g$ have a coupled coincidence point.

Next suppose $(iv)(b)$ holds. By Theorem 2.1, we have $\{gx_n\}$ and $\{gy_n\}$ are increasing and decreasing sequences respectively in $X$ and $\lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} gy_n = y$. Then, by condition $(iii)$, we have

$$gx_n \preceq x \text{ and } gy_n \succeq y. \tag{2.13}$$

Since $F$ and $g$ are compatible and $g$ is continuous, by (2.11) and (2.12), we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) \tag{2.14}$$

and

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{2.15}$$

Now we have

$$d(gx, F(x, y)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y)).$$

On taking limits as $n \to \infty$, using (2.14), we have

$$d(gx, F(x, y)) \leq \lim_{n\to\infty} d(g(gx_{n+1}), F(x, y)).$$

Since the mapping $g$ is monotonically increasing, using (1.6) and (2.13), we have

$$\begin{aligned} d(gx, F(x, y)) &\leq \lim_{n\to\infty} [d(F(gx_n, gy_n), F(x, y)) \\ &\leq \alpha d(ggx_n, gx) + \beta d(ggy_n, gy) \end{aligned}$$

$$+\frac{\gamma}{2}\frac{[d(ggx_n, F(gx_n, gy_n)) + d(ggy_n, F(gy_n, gx_n))][d(gx, F(x,y)) + d(gy, F(gy, gx)]}{1 + d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}$$
$$+ L\min\{d(F(gx_n, gy_n), gx), d(F(x,y), ggx_n), d(F(gx_n, gy_n), ggx_n), d(F(x,y), gx)\}],$$

using (2.14) and (2.15), we have $d(gx, F(x,y) \leq \lim_{n\to\infty} d(F(gx_n, gy_n), F(x,y)) \leq 0$. Therefore $gx = F(x,y)$. Similarly, we can obtain $gy = F(y,x)$. Thus $(x,y)$ is a coupled coincidence point of $F$ and $g$. □

The following example shows that Theorem 2.2 is a generalization of Theorem 1.4.

**Example 2.3.** *Let $X = \{0, \frac{1}{2}, 2\}$ with the usual metric, and $\leq := \{(0,0), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2}), (2,2), (0,2)\}$. We write $A = \{(0,0), (0,\frac{1}{2}), (\frac{1}{2},2), (2,2), (2,0), (0,2)\}$, $B = \{(\frac{1}{2},0), (2,\frac{1}{2})\}$ and $C = \{(\frac{1}{2},\frac{1}{2})\}$. We define $F : X\times X \to X$ and $g : X \to X$ by*

$$F(x,y) = \begin{cases} 0 & if \quad (x,y)\in A \\ \frac{1}{2} & if \quad (x,y)\in B \\ 2 & if \quad (x,y)\in C \end{cases}$$

*and $g0 = 0, g\frac{1}{2} = 2$ and $g2 = \frac{1}{2}$. Now, it is easy to see that $F$ has a mixed $g$-monotone property, $F(X\times X) \subseteq gX$ and $F$ and $g$ are continuous functions. By choosing $x_0 = \frac{1}{2}$ and $y_0 = \frac{1}{2}$, we have $gx_0 \preceq F(\frac{1}{2},\frac{1}{2})$ and $gy_0 \succeq F(\frac{1}{2},\frac{1}{2})$. Also $F$ and $g$ are compatible on $X$.*

*We take $x,y,u,v \in X$, such that $gx \geq gu$ and $gy \leq gv$. Now we show that the inequality (1.6) holds with $\alpha,\beta = \frac{1}{4}, \gamma = \frac{3}{8}$ and $L = 1$.*

*$Case(a)$ : If $(x,y) = (u,v)$, then we have $d(F(x,y), F(u,v)) = 0$ and hence the condition (1.6) holds.*
*$Case(b)$ : If $(x,y), (u,v) \in A$ or $B$ or $C$ then we have*
*$d(F(x,y), F(u,v)) = 0$ and hence the condition (1.6) holds.*

*$Case(c)$ : If $(x,y) = (0,0)$ then $(u,v) \in \{(0,0), (0,2)\}$, so that the condition (1.6) holds.*

*$Case(d)$ : If $(x,y) = (0,\frac{1}{2})$ then $(u,v) \in \{(0,\frac{1}{2})\}$, so that the condition (1.6) holds.*
*$Case(e)$ : If $(x,y) = (0,2)$ then $(u,v) \in \{(0,2)\}$, so that the condition (1.6) holds.*

*$Case(f)$ : If $(x,y) = (\frac{1}{2},\frac{1}{2})$ then $(u,v) \in \{(\frac{1}{2},\frac{1}{2})\}$, so that the condition (1.6) holds.*

*$Case(g)$ : If $(x,y) = (\frac{1}{2},0)$ then $(u,v) \in \{(\frac{1}{2},0), (\frac{1}{2},2)\}$.*

*If $(u,v) = (\frac{1}{2},2)$ then*
*$d(F(x,y), F(u,v)) = \frac{1}{2} \leq \frac{\beta}{2} + \gamma + L(\frac{3}{2}) = 2$, so that the condition (1.6) holds.*

*$Case(h)$ : If $(x,y) = (\frac{1}{2},2)$ then $(u,v) \in \{(\frac{1}{2},2)\}$,so that the condition (1.6) holds.*

*Case*$(i)$ : *If* $(x,y)=(2,\frac{1}{2})$ *then* $(u,v)\in\{(0,\frac{1}{2}),(2,\frac{1}{2})\}$.

*If* $(u,v)=(0,\frac{1}{2})$ *then*
$d(F(x,y),F(u,v))=\frac{1}{2}\le\frac{\alpha}{2}+\gamma=\frac{1}{2}$, *so that the condition* (1.6) *holds.*

*Case*$(j)$ : *If* $(x,y)=(2,2)$ *then* $(u,v)\in\{(2,2)\}$, *so that the condition* (1.6) *holds.*

*Case*$(k)$ : *If* $(x,y)=(2,0)$ *then* $(u,v)\in\{(0,2),(2,2),(0,0),(2,0)\}$, *so that the condition* (1.6) *holds.*

*Here, the importance of the rational expression in the inequality* (1.6) *is shown in Case* $(i)$ *with* $\gamma=\frac{3}{8}$.

*But, the inequality* (1.6) *fails to hold if we remove the rational term in* (1.6). *Consequently condition (1.4) fails to hold; for, by choosing* $x=2, y=\frac{1}{2}, u=0$ *and* $v=\frac{1}{2}$ *, we have*

$$\begin{aligned} d(F(x,y),F(u,v))=\tfrac{1}{2}\nleq\tfrac{\alpha}{2}&=\alpha d(gx,gu)+\beta.0+L.0\\ &=\alpha d(gx,gu)+\beta d(gy,gv)\\ &\quad+L\min\{d(F(x,y),gu),d(F(u,v),gx),\\ &\qquad d(F(x,y),gx),d(F(u,v),gu)\}\end{aligned}$$

*for any* $\alpha\ge 0$, $\beta\ge 0$, *with* $\alpha+\beta<1$ *and* $L\ge 0$.
*Thus, Theorem* 2.2 *is a generalization of Theorem* 1.4.

*Here we note that* $(0,0),(\frac{1}{2},\frac{1}{2})$ *are two coincidence points of* $F$ *and* $g$. *Hence, we observe that if any two elements of the set of all coupled coincidence points of* $F$ *and* $g$ *are not comparable with any element of* $X\times X$ *then the uniqueness of the coupled fixed point fails.*

To prove the uniqueness we define the following order relation.

We define relation $\preceq$ on $X\times X$ by $(x,y)\preceq(u,v)\Leftrightarrow x\preceq u, y\succeq v$, for $x,y,u,v\in X$. Then $(X\times X,\preceq)$ is a poset, using this notation we prove the following theorem.

**Theorem 2.4.** *In addition to the hypotheses of Theorem* 2.2, *suppose that*

$(i)$ *g is one-one,*

$(ii)$ *for every* $(x,y),(z,t)\in X\times X$, *there is a* $(u,v)\in X\times X$ *which is comparable with* $(x,y)$ *and* $(z,t)$.

*Then* $F$ *and* $g$ *have a unique common coupled fixed point in* $X$.

*Proof.* From Theorem 2.2, the set of coupled coincidence points of $F$ and $g$ is non-empty. Suppose $(x,y)$ and $(z,t)$ be coupled coincidence points of $F$ and $g$, *i.e.*,

$$gx=F(x,y) \text{ and } gy=F(y,x) \tag{2.16}$$

and

$$gz = F(z,t) \text{ and } gt = F(t,z). \tag{2.17}$$

We now show that $gx = gz$ and $gy = gt$. By assumption there exists $(u,v) \in X \times X$ which is comparable with $(x,y)$ and $(z,t)$. With out loss of generality we assume that $(F(x,y), F(y,x)) \preceq (F(u,v), F(v,u))$ and $(F(z,t), F(t,z)) \preceq (F(u,v), F(v,u))$. Let $u_0 = u$, $v_0 = v$. Since $F(X \times X) \subseteq gX$, there exists $u_1,\ v_1 \in X$ such that $F(u_0,v_0) = gu$ and $F(v_0,u_0) = gv$, continuing this process we obtain sequences $\{gu_n\}$ and $\{gv_n\}$ such that $F(u_n,v_n) = gu_{n+1}$ and $F(v_n,u_n) = gv_{n+1}$ for all $n$. We now show that

$$gx \preceq gu_n \text{ and } gy \succeq gv_n \text{ for all } n, \tag{2.18}$$

by using mathematical induction, we have

$$gx = F(x,y) \preceq F(u,v) = (u_0,v_0) = gu_1 \tag{2.19}$$

and $gy = F(y,x) \succeq F(v,u) = (v_0,u_0) = gv_1$. Therefore (2.18) is true for $n = 1$. Assume that (2.18) is true for some $m \in N$, *i.e.*,

$$gx \preceq gu_m \text{ and } gy \succeq gv_m. \tag{2.20}$$

By using the mixed $g$-monotone property of $F$ and using (2.20), we have $gu_{m+1} = F(u_m,v_m) \preceq F(x,v_m) \preceq F(x,y) = gx$ and $gv_{m+1} = F(v_m,u_m) \succeq F(y,v_m) \succeq F(y,x) = gy$. Thus by induction (2.18) is true for all $n$. Since from (2.18), we have $gx \preceq gu_m$ and $gy \succeq gv_m$, using inequality (1.6), we have

$$\begin{aligned}
d(gu_{n+1}, gx) &= d(F(x,y), F(u_n,v_n)) \\
&\leq \alpha d(gx, gu_n) + \beta d(gy, gv_n) \\
&\quad + \frac{\gamma}{2}\frac{[d(gx,F(x,y)) + d(gy,F(y,x))][d(gu_n,F(u_n,v_n)) + d(gv_n,F(gv_n,gu_n))]}{1 + d(gx,gu_n) + d(gy,gv_n)} \\
&\quad + L\min\{d(F(x,y),gu_n), d(F(u_n,v_n),gx), d(F(x,y),gx), d(F(u_n,v_n),gu_n)\} \\
&\leq \alpha d(gx,gu_n) + \beta d(gy,gv_n).
\end{aligned} \tag{2.21}$$

Similarly,

$$d(gv_{n+1}, gy) \leq \alpha d(gx,gu_n) + \beta d(gy,gv_n). \tag{2.22}$$

From (2.21) and (2.22), we have

$$\begin{aligned}
d(gx, gu_{n+1}) + d(gv_{n+1}, gy) &\leq (\alpha+\beta)[d(gx,gu_n) + d(gy,gv_n)] \\
&\leq (\alpha+\beta)^2[d(gx,gu_{n-1}) + d(gy,gv_{n-1})] \\
&\leq \cdots \leq (\alpha+\beta)^{n+1}[d(gx,gu_0) + d(gy,gv_0)].
\end{aligned}$$

On taking limits as $n \to \infty$, we have

$$\lim_{n\to\infty} d(gx, gu_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gy, gv_{n+1}) = 0. \tag{2.23}$$

Similarly,

$$\lim_{n\to\infty} d(gz, gu_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(gt, gv_{n+1}) = 0. \tag{2.24}$$

Thus, from (2.23) and (2.24), it follows that

$$gx = gz \text{ and } gy = gt. \tag{2.25}$$

Thus, $F$ and $g$ have a unique coupled coincidence point. Since $F$ and $g$ are compatible on $X$, they are w-compatible and since $g$ is one-one, by using (2.16) and (2.17), we have

$$g(gx) = g(F(x, y)) = F(gx, gy) \text{ and } g(gy) = g(F(y, x)) = F(gy, gx). \tag{2.26}$$

Let $gx = z$, and $gy = w$, from (2.26), we have $gz = F(z, w)$ and $gw = F(w, z)$. Therefore $(z, w)$ is a coupled coincidence point of $F$ and $g$. From (2.25), we have $gz = gx$ and $gy = gw$, this implies $gz = z$ and $gw = w$. Therefore $(z, w)$ is a common coupled fixed point of $F$ and $g$. Now we show that $(z, w)$ is a unique coupled fixed point of $F$ and $g$. Let $(p, q)$ be another coupled fixed point of $F$ and $g$. Hence $p = gp$ and $q = gq$. By (2.25), we have $p = gp = gx = z$ and $q = gq = gy = w$. Therefore $(z, w)$ is a unique common coupled fixed point of $F$ and $g$. $\square$

**Remark 2.5.**

*(i) By choosing $\gamma = 0$ in Theorem 2.2, we get Theorem 1.4 as a corollary to Theorem 2.2.*

*(ii) By choosing $g = I_X$, the identity map on $X$ and $\gamma = 0$ in Theorem* 2.4, *we get Theorem 1.2 as a corollary to Theorem 2.4..*

*(iii) By choosing $\alpha = \beta = \frac{\alpha}{2}$, $L = \gamma = 0$ in Theorem 2.4, we get Theorem 1.3.*

**Example 2.6.** *Let $X = [0, 1]$ with the usual metric. Then $(X, \leq)$ is a partially ordered set. We define $F : X \times X \to X$ and $g : X \to X$ by*

$$F(x, y) = \begin{cases} \frac{x^2 - y^2}{6} & if\ x \geq y \\ 0 & if\ x < y \end{cases} \quad \text{and } gx = \frac{x^2}{2}.$$

*Clearly, $F$ and $g$ are continuous functions on $X$, $F$ has mixed $g$-monotone property, $g$ is monotonically increasing and one-one. We choose $x_0 = 0$ and $y_0 = c$ where $c \in (0, 1]$ then $gx_0 = 0$ and $F(0, c) = 0 = F(x_0, y_0)$ and $gy_0 = gc = \frac{c^2}{2} \succeq F(c, 0) = \frac{c^2}{6} = F(y_0, x_0)$. Also, $F(X \times X) = [0, 1) \subseteq gX$. Let $\{x_n\}$ and $\{y_n\}$*

*are sequences in* $X$ *such that* $\lim_{n\to\infty} gx_n = F(x_n, y_n) = a$ *and* $\lim_{n\to\infty} gy_n = F(y_n, x_n) = b$.

*Case(i): Suppose* $x_n \geq y_n$.

*Now for all* $n \geq 0$, *we have* $\lim_{n\to\infty} gx_n = \frac{x_n^2}{2} = F(x_n, y_n) = \frac{x_n^2-y_n^2}{6} = a$ *and* $\lim_{n\to\infty} gy_n = \frac{y_n^2}{2} = F(y_n, x_n) = \frac{y_n^2-x_n^2}{6} = b$. *Obviously,* $a = 0$ *and* $b = 0$. *Then* $\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$ *and* $\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0$. *Thus* $F$ *and* $g$ *are compatible.*

*Case(ii): Suppose* $x_n < y_n$ *then* $\lim_{n\to\infty} gx_n = \frac{x_n^2}{2} = F(x_n, y_n) = \frac{x_n^2-y_n^2}{6} = a$ *and* $\lim_{n\to\infty} gy_n = \frac{y_n^2}{2} = F(y_n, x_n) = \frac{y_n^2-x_n^2}{6} = b$ *implies* $a = 0$ *and* $b = 0$. *Hence* $F$ *and* $g$ *are compatible. Now we verify the inequality* (1.6) *with* $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}, \gamma = \frac{1}{2}$ *and* $L = 0$. *Let* $x, y, u, v \in X$, *such that* $gx \geq gu$ *and* $gy \leq gv$ *that is* $x^2 \geq u^2$ *and* $y^2 \leq v^2$.

*Case* $-1$ : $x \geq y$ *and* $u \geq v$. *Then*

$$\begin{aligned}
d(F(x,y), F(u,v)) &= d\Big(\frac{x^2-y^2}{6}, \frac{u^2-v^2}{6}\Big) \\
&= \Big|\frac{(x^2-y^2)-(u^2-v^2)}{6}\Big| \\
&= \Big|\frac{(x^2-u^2)-(y^2-v^2)}{6}\Big| \\
&\leq \frac{(x^2-u^2)+(y^2-v^2)}{6} \\
&\leq \frac{1}{3}\frac{(x^2-u^2)}{2} + \frac{1}{3}\frac{(y^2-v^2)}{2} = \frac{1}{3}d(gx,gu) + \frac{1}{3}d(gy,gv).
\end{aligned}$$

*Case* $-2$: $x \geq y$ *and* $u < v$. *Then*

$$\begin{aligned}
d(F(x,y), F(u,v)) = d\Big(\frac{x^2-y^2}{6}, 0\Big) &= \frac{x^2-y^2}{6} \\
&= \frac{u^2+x^2-y^2-u^2}{6} \\
&= \frac{(u^2-y^2)-(u^2-y^2)}{6} \\
&\leq \frac{(v^2-y^2)+(u^2-x^2)}{6} \\
&= \frac{(u^2-x^2)-(y^2-v^2)}{6} \\
&= \frac{1}{3}\frac{(x^2-u^2)}{2} + \frac{1}{3}\frac{(y^2-v^2)}{2} \\
&= \frac{1}{3}d(gx,gu) + \frac{1}{3}d(gy,gv).
\end{aligned}$$

*Case* $-$ 3: $x < y$ *and* $u \geq v$. *This implies* $gx \geq gu$ *and* $gy \geq gv$. *This case does not arise.*

*Case* $-$ 4: $x < y$ *and* $u < v$.

*Then* $F(x,y) = 0$ *and* $F(u,v) = 0$ *and* $d(F(x,y), F(u,v)) = 0$ *so that inequality (1.6) holds. From the above four cases, it follows that* $F$ *and* $g$ *satisfy inequality (1.6) with* $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{6}$ *and* $L = 0$.

*Thus it is verified that* $F$ *and* $g$ *satisfy all the conditions of Theorem 2.4,* $(0,0)$ *is a coupled fixed point of* $F$ *and* $g$.

*Also, we note that* $F$ *and* $g$ *are not commuting. Thus Theorem 1.3 is not applicable. This example suggests that Theorem 2.4 generalizes Theorem 1.3.*

One more example in this direction is the following.

**Example 2.7.** *Let* $X = \{0, \frac{1}{2}, 5\}$ *with the usual metric, and* $\leq := \{(0,0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (5,5)\}$. *Clearly,* $\leq$ *is a poset on* $X$. *We define* $F : X \times X \to X$ *and* $g : X \to X$ *by*

$$F(x,y) = \begin{cases} 0 & if \quad (x,y) \in A \\ \\ \frac{1}{2} & if \quad (x,y) \in B \end{cases}$$

*where* $A = \{(0,0), (0, \frac{1}{2}), (5, \frac{1}{2}), (5,5), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$, $B = \{(5,0), (0,5), (\frac{1}{2}, 5)\}$ *and* $g0 = 0, g\frac{1}{2} = \frac{1}{2}$ *and* $g5 = 5$.

*Clearly,* $F$ *and* $g$ *are continuous on* $X$, $g$ *is one-one,* $F$ *has mixed* $g$*-monotone property and* $F(X \times X) = \{0, \frac{1}{2}\} \subseteq gX = \{0, \frac{1}{2}, 5\}$. *Clearly,* $gx_0 = 0 \preceq F(x_0, y_0) = (0,0)$ *and* $gy_0 = 0 \succeq F(y_0, x_0) = (0,0)$. *We consider* $x, y, u, v \in X$, *such that* $gx \succeq gu$ *and* $gy \preceq gv$. *Thus, the inequality (1.6) holds with* $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, \gamma = \frac{1}{4}$ *and* $L = 1$.

*Hence, all the conditions of Theorem 2.4 holds and* $(0,0)$ *is a unique common fixed point of* $F$ *and* $g$.

*But the condition (1.3.1) of Theorem 1.3 fails to hold for any* $k \in [0,1)$. *Indeed, since for* $x = 5, y = 0, u = 5, y = \frac{1}{2}$, *we have* $d(gy, gv) = 0$ *and*

$$d(F(x,y), F(u,v)) = \frac{1}{2} \nleq \frac{k}{2}\frac{1}{2} = \frac{k}{2}[d(gx,gu) + d(gy,gv)]$$

*for any* $k \in [0,1)$. *Thus, Theorem* 1.3 *is not applicable.*

*Hence, Theorem 2.4 is a generalization of Theorem 1.3.*

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