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Existence and Algorithm for Generalized Mixed Equilibrium Problem with a Relaxed Monotone Mapping

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Abstract : In this paper, we introduce a new generalized mixed equilibrium problem with a relaxed monotone mapping. By using KKM theorem, we establish an existence theorem for this problem in a Hausdorff topological vector space. Moreover, we introduce an iterative sequence and prove a weak convergence theorem for a generalized mixed equilibrium problem with a relaxed monotone mapping in Hilbert spaces. The results presented in this paper can be viewed as generalization and extension of many results in literature.

Keywords : Relaxed monotone mapping; generalized mixed equilibrium problem; KKM theorem; Hausdorff topological vector space.
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1 Introduction

Let E be a Hausdorff topological vector space with dual space E^*, K a nonempty compact convex subset of E. The equilibrium problem is to find $x \in K$ such that

$$g(x, y) \geq 0 \quad \text{for all} \quad y \in K, \tag{1.1}$$

where $g : K \times K \to \mathbb{R}$ a bifunctions (see in Takahashi [1, Lemma 1]). It is known that many problems of physics optimization, engineering and economics can be

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described by some suitable equilibrium problems. The equilibrium problem is a one important topics of mathematical sciences such as optimization problems, problems of Nash equilibrium, variational inequality problems, complementary problems, fixed point problems; it unifies the above problems in a very convenient way. Recently, Blum and Oettli [2] was introduced a generalized equilibrium problem for g and h is to find $x \in K$ such that

$$g(x,y) + h(x,y) \geq 0 \quad \text{for all} \quad y \in K, \tag{1.2}$$

where K a nonempty compact convex subset of E and $g, h : K \times K \to \mathbb{R}$ a bifunctions. In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme and proved a strong convergence theorem for finding the best approximation to the initial data when the solution set of (1.2), which denoted by $EP(g,h)$, is nonempty. Using this result, S. Takahashi and W. Takahashi [4] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space. (see also in [5, 6, 7, 8, 9, 10, 11]).

Very recently, Wang et. al [12] introduced generalized equilibrium problem with a relaxed monotone mapping, that is the problem, to find $x \in K$ such that

$$g(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle \geq 0 \quad \text{for all} \quad y \in K, \tag{1.3}$$

where K be a nonempty closed convex subset of a real Hilbert space H, $A : K \to H$ a λ-inverse-strongly mapping, and $g : K \times K \to \mathbb{R}$ a bifunction. The generalized monotonicity plays an important role in the literature of variational inequalities and equilibrium problems. There is a substantial number of papers on existence results for solving variational inequalities and equilibrium problems based on different relaxed monotonicity notions such as monotonicity, quasimonotonicity, pseudomonotonicity, relaxed monotonicity (see; [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]).

In this paper, we consider the following generalized mixed equilibrium problem with a relaxed monotone mapping: Finding $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle \geq 0 \quad \text{for all} \quad y \in K. \tag{1.4}$$

The set of solution of (1.4) is denoted by $GMEPRM(g,h,T,A)$. In this paper, we introduce a new generalized mixed equilibrium problem with a relaxed monotone mapping in a Hausdorff topological vector space. By using KKM theorem, we establish an existence theorem for this problem in a Hausdorff topological vector space. Moreover, we introduce an iterative sequence and prove a weak convergence theorem for a generalized mixed equilibrium problem with a relaxed monotone mapping in Hilbert spaces. The results presented in this paper can be viewed as generalization and extension of many results in literature.

2 Preliminaries

Let K be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, we note that there exists a unique nearest point in K, denoted by $P_K(x)$

such that

$$\|x - P_K(x)\| \leq \|x - y\|$$

for all $y \in K$. We know that $u \in VI(K, A) \Leftrightarrow u = P_K(u - \lambda Au)$ for all $\lambda > 0$. The mapping P_K is called the metric projection of H onto K. It is well-known that P_K is a nonexpansive mapping from H onto K. It is also known that $P_K x \in K$ and

$$\langle x - P_K(x), P_K(x) - y\rangle \geq 0 \tag{2.1}$$

for all $x \in H$ and $y \in K$. It is obvious that (2.1) is equivalent to

$$\|x - y\|^2 \geq \|x - P_K(x)\|^2 + \|y - P_K(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in K$. A mapping $T : K \to H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. We denote by $F(T)$ the set of fixed points of T. A mapping $A : K \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in K$. A mapping $A : K \to H$ is called λ-inverse-strongly monotone if there exists a real number $\lambda > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \lambda\|Ax - Ay\|^2$$

for all $x, y \in K$. A mapping $A : K \to H$ is called k-Lipschitz-continuous if there exists a real number $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|$$

for all $x, y \in K$. It is obvious to see that if A is λ-inverse-strongly monotone then A is monotone and Lipschitz-continuous.

A mapping $T : K \to E^*$ is said to be relaxed η-α monotone if there exist a mapping $\eta : K \times K \to K$ and a function $\alpha : E \to \mathbb{R}$ positively homogeneous of degree p, i.e., $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \quad \forall x, y \in K,$$

where $p > 1$ is a constant. In the case of $\eta(x, y) = x - y$ for all $x, y \in K$, T is said to be relaxed α-monotone. Moreover, every monotone mapping is relaxed η-α monotone with $\eta(x, y) = x - y$ for all $x, y \in K$ and $\alpha \equiv 0$.

Definition 2.1. [21] *Let E be a Banach space with the dual space E^* and K be a nonempty subset of E. Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. The mapping $T : K \to E^*$ is said to be η-hemicontinuous if, for any fixed $x, y \in K$, the function $f : [0, 1] \to (-\infty, \infty)$ defined by $f(t) = \langle T((1 - t)x + ty), \eta(x, y)\rangle$ is continuous at 0^+.*

Definition 2.2. *Let E be a Banach space with the dual space E^* and K be a nonempty subset of E. Let $A : K \to E^*$ be mapping. The mapping $A : K \to E^*$ is said to be hemicontinuous if for any $x, y \in K$, the mapping $f : [0,1] \to E^*$ defined by $f(t) = \langle A((1-t)x + ty), z\rangle$ is continuous, for all $z \in E^*$.*

Definition 2.3. [26] *Let E be a topological vector space and B a nonempty subset of E. A multivalued mapping $G : B \to 2^E$ is said to be a KKM mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq B$, we have*

$$conv\{x_1, x_2, \ldots, x_n\} \subseteq \cup_{i=1}^n G(x_i)$$

where $conv\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Lemma 2.4. [26] *Let B be a nonempty subset of a Hausdorff topological vector space E and $G : B \to 2^E$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\cap_{x\in B} G(x) \neq \emptyset$.*

Lemma 2.5. [27] *Let K be a nonempty closed convex subset of a strictly convex Banach space X and $S : K \to K$ a nonexpansive mapping with $F(S) \neq 0$. Then $F(S)$ is closed convex.*

3 Existence

In this section, we consider the following generalized mixed equilibrium problem with a relaxed monotone mapping. Let K be a nonempty closed convex subset of a Hausdorff topological vector space E, $g, h : K \times K \to \mathbb{R}$, $A : K \to E^*$ be a monotone mapping, and $T : K \to E^*$ a relaxed η-α monotone mapping. For prove our main result, let us give the following assumptions:

$(A1)$ $g(x,x) = 0$ for all $x \in K$;

$(A2)$ g is monotone, i.e. $g(x,y) + g(y,x) \leq 0$ for all $x, y \in K$;

$(A3)$ for each $x \in K$, $y \mapsto g(x,y)$ is convex and lower semicontinuous;

$(A4)$ for each $x, y, z \in K$, $\limsup_{t\to 0} g(tz + (1-t)x, y) \leq g(x,y)$;

$(B1)$ $h(x,x) = 0$ for all $x \in K$;

$(B2)$ for each $x \in K$, $y \mapsto h(x,y)$ is lower semicontinuous;

$(B3)$ for each $x \in K$, $y \mapsto h(x,y)$ is convex;

$(B4)$ for each $x, y, z \in K$, $\limsup_{t\to 0} h(tz + (1-t)x, y) \leq h(x,y)$;

$(C1)$ $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$;

$(C2)$ for each $u, v \in K$, $z \mapsto \langle Tv, \eta(z,u)\rangle$ is convex and lower semicontinuous and $z \mapsto \langle Tu, \eta(v,z)\rangle$ is lower semicontinuous;

$(C3)$ for each $x, y \in K$, $\alpha(x-y) + \alpha(y-x) \geq 0$;

$(C4)$ for each $u, v, x, z \in K$, $\limsup_{t\to 0}\langle Tu, \eta(v, tx + (1-t)z)\rangle \le \langle Tu, \eta(v, z)\rangle$;

$(D1)$ for each $u, v \in K$, $z \mapsto \langle Av, z - u\rangle$ is convex and lower semicontinuous and $z \mapsto \langle Au, v - z\rangle$ is lower semicontinuous;

$(D2)$ for each $u, v, x, z \in K$, $\limsup_{t\to 0}\langle Au, v - (tx + (1-t)z)\rangle \le \langle Au, v - z\rangle$;

$(D3)$ $\langle Tx, \eta(y, x)\rangle + \langle Ty, \eta(x, y)\rangle + \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \le 0$ for all $x, y \in K$.

The idea of the proof of the next theorem is contained in the paper of Peng and Yao [7], Wang et.al [12], and Combettes and Hirstoaga [28].

Lemma 3.1. *Let K be a nonempty closed convex subset of Hausdorff topological vector space E. Let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$ and $(A3)$, and $h : K \times K \to \mathbb{R}$ satisfying $(B1)$ and $(B3)$. Let $T : K \to E^*$ be an η-hemicontinuous and relaxed η-α monotone mapping which satisfying $(C2)$. Let $A : K \to E^*$ be a monotone and hemicontinuous mapping which satisfying $(D1)$ and assume that $\eta(x, x) = 0$ for all $x \in K$. Then for all $r > 0$ and $z \in K$ the following problems are equivalent;*

(i) find $x \in K$ such that

$$g(x, y) + h(x, y) + \langle Tx, \eta(y, x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \ge 0,$$

for all $y \in K$;

(ii) find $x \in K$ such that

$$g(x, y) + h(x, y) + \langle Ty, \eta(y, x)\rangle + \langle Ay, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \ge \alpha(y - x),$$

for all $y \in K$.

Proof. Let $x \in K$ be a solution of the problem (i). Since T is relaxed η-α monotone and A is monotone, we get

$$\begin{aligned} & g(x, y) + h(x, y) + \langle Ty, \eta(y, x)\rangle + \langle Ay, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \\ & \quad \ge g(x, y) + h(x, y) + \langle Tx, \eta(y, x)\rangle + \alpha(y - x) + \langle Ax, y - x\rangle \\ & \qquad + \frac{1}{r}\langle y - x, x - z\rangle \\ & \quad \ge \alpha(y - x), \quad \text{for all} \quad y \in K. \end{aligned}$$

Hence x is a solution of the problem (ii).
Conversely, let $x \in K$ be a solution of the problem (ii). Setting $y_t = (1-t)x + ty$ for all $t \in (0, 1)$, then $y_t \in K$. Thus, it follows that

$$\begin{aligned} g(x, y_t) + h(x, y_t) + \langle Ty_t, \eta(y_t, x)\rangle + \langle Ay_t, y_t - x\rangle + \frac{1}{r}\langle y_t - x, x - z\rangle & \\ \ge \ \alpha(y_t - x) & \\ = \ t^p\alpha(y - x). & \quad (3.1) \end{aligned}$$

From the conditions $(A1)$, $(A3)$, $(B1)$, $(B3)$, $(C2)$, and $(D1)$, we obtain

$$g(x, y_t) \leq (1-t)g(x,x) + tg(x,y) = tg(x,y), \tag{3.2}$$

$$h(x, y_t) \leq (1-t)h(x,x) + th(x,y) = th(x,y), \tag{3.3}$$

$$\begin{aligned} \langle Ty_t, \eta(y_t, x)\rangle &\leq (1-t)\langle Ty_t, \eta(x,x)\rangle + t\langle Ty_t, \eta(y,x)\rangle \\ &= t\langle T(x+t(y-x)), \eta(y,x)\rangle, \end{aligned} \tag{3.4}$$

and

$$\langle Ay_t, y_t - x\rangle = \langle Ay_t, x + t(y-x) - x\rangle = t\langle A(x+t(y-x)), y-x\rangle. \tag{3.5}$$

Since

$$\langle y_t - x, x - z\rangle = \langle x + t(y-x) - x, x - z\rangle = t\langle y - x, x - z\rangle, \tag{3.6}$$

it follows from (3.1)-(3.6) that

$$\begin{aligned} g(x,y) + h(x,y) + \langle T(x+t(y-x)), \eta(y,x)\rangle + \langle A(x+t(y-x)), y-x\rangle& \\ + \frac{1}{r}\langle y-x, x-z\rangle \ \geq \ t^{p-1}\alpha(y-x),& \end{aligned} \tag{3.7}$$

for all $y \in K$. Letting $t \to 0$ in (3.7), we get

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \tag{3.8}$$

for all $y \in K$. Hence x is a solution of the problem (i). This completes the proof. □

Theorem 3.2. *Let K be a nonempty compact convex subset of Hausdorff topological vector space E. Let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$ and $(A3)$ and let $h : K \times K \to \mathbb{R}$ satisfying $(B1)$ and $(B3)$. Let $T : K \to E^*$ be an η-hemicontinuous and relaxed η-α monotone mapping which satisfying $(C1)$-$(C3)$. Let $A : K \to E^*$ be a monotone and hemicontinuous mapping which satisfying $(D1)$. Then, for all $r > 0$ and $z \in K$ there exists $x \in K$ such that*

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K.$$

Proof. Let z be any given point in K and let $r > 0$. We will show that $T_r(z) \neq \emptyset$. Define $M_z, N_z : K \to 2^K$ by

$$\begin{aligned} M_z(y) = \{x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle& \\ + \frac{1}{r}\langle y-x, x-z\rangle \geq 0\}, \quad \text{for all} \quad y \in K& \end{aligned}$$

and

$$N_z(y) = \{x \in K : g(x,y) + h(x,y) + \langle Ty, \eta(y,x)\rangle + \langle Ay, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq \alpha(y-x)\}, \quad \text{for all} \quad y \in K.$$

Note that, for each $y \in K$, $M_z(y)$ is nonempty because $y \in M_z(y)$. We claim that M_z is a KKM mapping. Assume that M_z is not a KKM mapping. Then there exists $\{y_1, y_2, ..., y_n\} \subset K$ and $t_i > 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^n t_i = 1$ such that $\widehat{z} = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n M_z(y_i)$ for each $i = 1, 2, ..., n$. This implies that

$$g(\widehat{z}, y_i) + h(\widehat{z}, y_i) + \langle T\widehat{z}, \eta(y_i, \widehat{z})\rangle + \langle A\widehat{z}, y_i - \widehat{z}\rangle + \frac{1}{r}\langle y_i - \widehat{z}, \widehat{z} - z\rangle < 0,$$

for each $i = 1, 2, ..., n$. By $(A1)$, $(A3)$, $(B1)$, $(B3)$, $(C2)$, and $(D1)$, we have

$$\begin{aligned}
0 &= g(\widehat{z}, \widehat{z}) + h(\widehat{z}, \widehat{z}) \\
&= g\left(\widehat{z}, \sum_{i=1}^n t_i y_i\right) + h\left(\widehat{z}, \sum_{i=1}^n t_i y_i\right) + \left\langle T\widehat{z}, \eta\left(\sum_{i=1}^n t_i y_i, \widehat{z}\right)\right\rangle \\
&\quad + \left\langle A\widehat{z}, \sum_{i=1}^n t_i y_i - \widehat{z}\right\rangle \\
&\leq \sum_{i=1}^n t_i g(\widehat{z}, y_i) + \sum_{i=1}^n t_i h(\widehat{z}, y_i) + \sum_{i=1}^n t_i \langle T\widehat{z}, \eta(y_i, \widehat{z})\rangle + \sum_{i=1}^n t_i \langle A\widehat{z}, y_i - \widehat{z}\rangle \\
&< \sum_{i=1}^n t_i \frac{1}{r}\langle \widehat{z} - y_i, \widehat{z} - z\rangle \\
&= 0,
\end{aligned}$$

which is a contradiction. Hence M_z is a KKM mapping. We now show that $M_z(y) \subset N_z(y)$ for all $y \in K$. For any $y \in K$, we let $x \in M_z(y)$. Thus, we have

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0.$$

Since T is relaxed η-α monotone and A is monotone, we get

$$\begin{aligned}
& g(x,y) + h(x,y) + \langle Ty, \eta(y,x)\rangle + \langle Ay, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \\
&\quad \geq g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \alpha(y-x) + \langle Ax, y-x\rangle \\
&\qquad + \frac{1}{r}\langle y-x, x-z\rangle \\
&\quad \geq \alpha(y-x).
\end{aligned}$$

This implies that $x \in N_z(y)$ and hence $M_z(y) \subset N_z(y)$ for all $y \in K$. Since $z \mapsto \langle Ty, \eta(y,z)\rangle$ and $z \mapsto \langle Ay, y-z\rangle$ are lower semicontinuous function, we have

$z \mapsto \langle Ty, \eta(y,z)\rangle$ and $z \mapsto \langle Ay, y-z\rangle$ are weakly lower semicontinuous. Thus $M_z(y)$ is weakly closed for all $y \in K$ implies that $M_z(y)$ is closed for all $y \in K$. Since K is compact, we have $M_z(y)$ is compact in K for all $y \in K$. By Lemma 3.1 and Lemma 2.4, we get

$$\bigcap_{y\in K} M_z(y) = \bigcap_{y\in K} N_z(y) \neq \emptyset.$$

Therefore, there exists $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0.$$

□

Theorem 3.3. *Let K be a nonempty bounded closed convex subset of H. Let $g: K\times K \to \mathbb{R}$ satisfying $(A1)$-$(A3)$ and let $h: K\times K\to\mathbb{R}$ be a monotone mapping and satisfying $(B1)$-$(B3)$. Let $T: K \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping which satisfying $(C1)$-$(C3)$. Let $A: K\to H$ be a λ-inverse-strongly monotone and hemicontinuous mapping which satisfying $(D1)$. For $r > 0$ and $z \in K$, define $T_r : K \to 2^K$ by*

$$T_r(z) = \{x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K\}.$$

Then, the following results holds:

(1) dom $T_r = H$;

(2) T_r is single-valued;

(3) T_r is firmly nonexpansive i.e., for any $x, y \in K$, $\|T_r(x)-T_r(y)\|^2 \leq \langle T_r(x)-T_r(y), x-y\rangle$;

(4) $Fix(T_r) = GMEPRM(g,h,T,A)$;

(5) $GMEPRM(g,h,T,A)$ is closed and convex.

Proof. **Step 1.** We first show that dom $T_r = H$. Since K is bounded closed and convex, we note that K is weakly compact. Hence, for every $r > 0$ and $z \in K$ there exists $x \in K$ such that

$$g(x,y)+h(x,y)+\langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K.$$

Step 2. We will show that T_r is single-valued. For each $z \in K$ and $r > 0$, let $x_1, x_2 \in T_r(z)$. Thus, we have

$$g(x_1,x_2) + h(x_1,x_2) + \langle Tx_1, \eta(x_2,x_1)\rangle + \langle Ax_1, x_2-x_1\rangle + \frac{1}{r}\langle x_2-x_1, x_1-z\rangle \geq 0$$

and

$$g(x_2,x_1)+h(x_2,x_1)+\langle Tx_2,\eta(x_1,x_2)\rangle+\langle Ax_2,x_1-x_2\rangle+\frac{1}{r}\langle x_1-x_2,x_2-z\rangle\geq 0.$$

Adding the two inequalities, we obtain

$$\begin{aligned}g(x_1,x_2)+h(x_1,x_2)+g(x_2,x_1)+h(x_2,x_1)+\langle Tx_1-Tx_2,\eta(x_2,x_1)\rangle\\+\langle Ax_1-Ax_2,x_2-x_1\rangle+\frac{1}{r}\langle x_2-x_1,x_1-x_2\rangle\geq 0.\end{aligned}$$

From the monotonicity of H and $(A2)$, we have

$$\langle Tx_1-Tx_2,\eta(x_2,x_1)\rangle+\langle Ax_1-Ax_2,x_2-x_1\rangle+\frac{1}{r}\langle x_2-x_1,x_1-x_2\rangle\geq 0.$$

This implies that

$$\frac{1}{r}\langle x_2-x_1,x_1-x_2\rangle\geq\langle Tx_2-Tx_1,\eta(x_2,x_1)\rangle+\langle x_2-x_1,Ax_2-Ax_1\rangle. \tag{3.9}$$

Since T is relaxed η-α monotone, A is λ-inverse-strongly monotone and $r>0$, it follows that

$$\langle x_2-x_1,x_1-x_2\rangle\geq r[\alpha(x_2-x_1)+\lambda\|Ax_2-Ax_1\|^2]\geq r\alpha(x_2-x_1). \tag{3.10}$$

By exchanging the position of x_1 and x_2 in (3.9), we get

$$\frac{1}{r}\langle x_1-x_2,x_2-x_1\rangle\geq\langle Tx_1-Tx_2,\eta(x_1,x_2)\rangle+\langle x_1-x_2,Ax_1-Ax_2\rangle\geq\alpha(x_1-x_2).$$

Hence $\langle x_1-x_2,x_2-x_1\rangle\geq r\alpha(x_1-x_2)$ and therefore

$$\langle x_2-x_1,x_1-x_2\rangle=\langle x_1-x_2,x_2-x_1\rangle\geq r\alpha(x_1-x_2). \tag{3.11}$$

Adding the inequalities (3.10) and (3.11) and using $(C3)$, we have

$$-2\|x_1-x_2\|^2=2\langle x_2-x_1,x_1-x_2\rangle\geq 0.$$

Hence $x_1=x_2$ and therefore T_r is a single-valued mapping.

Step 3. We will show that T_r is a firmly nonexpansive mapping. For $x,y\in H$, we note that

$$\begin{aligned}g(T_r(x),T_r(y))+h(T_r(x),T_r(y))+\langle TT_r(x),\eta(T_r(y),T_r(x))\rangle\\+\langle AT_r(x),T_r(y)-T_r(x)\rangle+\frac{1}{r}\langle T_r(y)-T_r(x),T_r(x)-x\rangle\geq 0\end{aligned}$$

and

$$\begin{aligned}g(T_r(y),T_r(x))+h(T_r(y),T_r(x))+\langle TT_r(y),\eta(T_r(x),T_r(y))\rangle\\+\langle AT_r(y),T_r(x)-T_r(y)\rangle+\frac{1}{r}\langle T_r(x)-T_r(y),T_r(y)-y\rangle\geq 0.\end{aligned}$$

By $(A2)$, $(C1)$, $r > 0$, and h is monotone we obtain

$$\langle TT_r(x) - TT_r(y), \eta(T_r(y), T_r(x))\rangle + \langle AT_r(x) - AT_r(y), T_r(y) - T_r(x)\rangle + \frac{1}{r}\langle T_r(y) - T_r(x), T_r(x) - x - T_r(y) + y\rangle \geq 0.$$

Thus, we have

$$\begin{aligned} &\frac{1}{r}\langle T_r(y) - T_r(x), T_r(x) - T_r(y) + y - x\rangle \\ &\qquad \geq \langle TT_r(y) - TT_r(x), \eta(T_r(y), T_r(x))\rangle \\ &\qquad\quad + \langle T_r(y) - T_r(x), AT_r(y) - AT_r(x)\rangle \\ &\qquad \geq \alpha(T_r(y) - T_r(x)) + \lambda\|AT_r(y) - AT_r(x)\|^2 \\ &\qquad \geq \alpha(T_r(y) - T_r(x)). \end{aligned} \tag{3.12}$$

By exchanging the position of x and y in (3.12), we note that

$$\frac{1}{r}\langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle \quad \geq \quad \alpha(T_r(x) - T_r(y)). \tag{3.13}$$

From (3.12) and (3.13), we get

$$2\langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle \geq r[\alpha(T_r(y) - T_r(x)) + \alpha(T_r(x) - T_r(y))].$$

By $(C3)$, we obtain

$$\begin{aligned} \langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle &= \langle T_r(x) - T_r(y), T_r(y) - T_r(x)\rangle \\ &\quad + \langle T_r(x) - T_r(y), x - y\rangle \\ &\geq 0. \end{aligned}$$

Thus, we have $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$. Hence T_r is a firmly nonexpansive mapping.

Step 4. We will show that $Fix(T_r) = GMEPRM(g, h, T, A)$. Indeed, we have the following

$$\begin{aligned} u \in Fix(T_r) \quad &\Leftrightarrow \quad u = T_r(u) \\ &\Leftrightarrow \quad g(u, y) + h(u, y) + \langle Tu, \eta(y, u)\rangle + \langle Au, y - u\rangle \geq 0, \\ &\qquad\ \ \text{for all} \quad y \in K \\ &\Leftrightarrow \quad u \in GMEPRM(g, h, T, A). \end{aligned}$$

Step 5. We will show that $GMEPRM(g, h, T, A)$ is closed and convex. Since T_r is firmly nonexpansive, it follows by Lemma 2.5 that $GMEPRM(g, h, T, A)$ is closed and convex. This completes the proof. □

Corollary 3.4. [12] *Let K be a nonempty bounded closed convex subset of H. Let $T : K \to H$ be η-hemicontinuous and relaxed η-α monotone and satisfying*

$(C1)$-$(C3)$ and let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$-$(A3)$. For $r > 0$ and $z \in K$, define $\widetilde{T}_r : K \to 2^K$ by

$$\widetilde{T}_r(z) = \{x \in K : g(x, y) + \langle Tx, \eta(y, x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \text{for all} \quad y \in K\}.$$

Then, the following results holds:

(1) $\widetilde{T}_r$ is single-valued;

(2) $\widetilde{T}_r$ is firmly nonexpansive i.e., for any $x, y \in K$, $\|\widetilde{T}_r(x) - \widetilde{T}_r(y)\|^2 \leq \langle \widetilde{T}_r(x) - \widetilde{T}_r(y), x - y\rangle$;

(3) $Fix(\widetilde{T}_r) = GEP(g, T)$;

(4) $GEP(g, T)$ is closed and convex.

Proof. It is easy to see by setting $h \equiv 0$ and $A \equiv 0$ in Theorem 3.3. □

4 Weak convergence theorem

In the section, we introduce an iterative sequence and prove weak convergence theorem for solving a generalized mixed equilibrium problem with a relaxed monotone mapping.

Definition 4.1. *For any $r > 0$, the resolvent of a bifunction $g, h : K \times K \to \mathbb{R}$ is the set-valued operator $T_r : H \to 2^K$ defined by*

$$T_r(z) = \{x \in K : g(x, y) + h(x, y) + \langle Tx, \eta(y, z)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle x - z, y - x\rangle \geq 0, \text{for all} \quad y \in K\}. \tag{4.1}$$

We note that $domT_r = H$ under certain condition in Theorem 3.3.

Lemma 4.2. *Let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$-$(A4)$, and $h : K \times K \to \mathbb{R}$ be a monotone mapping satisfying $(B1)$-$(B4)$. Let $T : K \to H$ be η-hemicontinuous and relaxed η-α monotone and satisfying $(C2)$ and $(C4)$. Let $A : K \to H$ be a monotone and hemicontinuous mapping satisfying $(D1)$ and $(D2)$ and assume that $\eta(x, x) = 0$ for all $x \in K$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H, $(r_n)_{n\in\mathbb{N}}$ a sequence in $(0, +\infty)$, and (T_{r_n}) a sequence of mapping defined in (4.1) which dom $T_{r_n} = H$ for all $n \geq 1$. Define*

$$z_n = T_{r_n}x_n \quad \text{and} \quad u_n = x_n - z_n, \quad \forall n \in \mathbb{N}, \tag{4.2}$$

and suppose that

$$z_n \rightharpoonup x \quad \text{and} \quad u_n \to u. \tag{4.3}$$

Then, for all $r > 0$, $x \in K$ and

$$g(x, y) + h(x, y) + \langle Tx, \eta(y, x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle u, x - y\rangle \geq 0, \quad \text{for all} \quad y \in K.$$

Proof. Since dom $T_r = H$, we note that the sequence $(z_n)_{n\in\mathbb{N}}$ is well defined in K. It follows from $(A3)$, $(B2)$, $(C2)$, and $(D1)$ that $y \mapsto g(x,y)$, $y \mapsto h(x,y)$, $z \mapsto \langle Tv, \eta(z,u)\rangle$, and $z \mapsto \langle Av, z-u\rangle$ are weak lower semicontinuous for every $y \in K$. Therefore, we derive from g, h, A are monotone, T is relaxed η-α monotone, (4.1), (4.2), and (4.3) that

$$\begin{aligned}
g(y,x) &+ h(y,x) + \langle Ty, \eta(x,y)\rangle + \langle Ay, x-y\rangle \\
&\le \liminf g(y,z_n) + \liminf h(y,z_n) + \liminf \langle Ty, \eta(z_n,y)\rangle \\
&\quad + \liminf \langle Ay, z_n - y\rangle \\
&\le \liminf_{n\to\infty} [g(y,z_n) + h(y,z_n) + \langle Ty, \eta(z_n,y)\rangle + \langle Ay, z_n - y\rangle] \\
&\le \liminf_{n\to\infty} [-g(z_n,y) - h(z_n,y) - \langle Tz_n, \eta(y,z_n)\rangle - \langle Az_n, y - z_n\rangle] \\
&\le \frac{1}{r} \liminf_{n\to\infty} \langle u, zn_n - y\rangle \\
&= \frac{1}{r}\langle u, x-y\rangle. \qquad (4.4)
\end{aligned}$$

Fix $y \in K$ and define $x_t = (1-t)x + ty$ for all $t \in (0,1)$, then $x_t \in K$. Thus, by $(A1)$, $(B1)$, $(A3)$, $(B3)$, $(C2)$, $(D1)$, and (4.4), we have that

$$\begin{aligned}
0 &= g(x_t,x_t) + h(x_t,x_t) + \langle Tx_t, \eta(x_t,x_t)\rangle + \langle Ax_t, x_t - x_t\rangle \\
&\le (1-t)g(x_t,x) + tg(x_t,y) + (1-t)h(x_t,x) + th(x_t,y) \\
&\quad +(1-t)\langle Tx_t, \eta(x,x_t)\rangle + t\langle Tx_t, \eta(y,x_t)\rangle \\
&\quad +(1-t)\langle Ax_t, x - x_t\rangle + t\langle Ax_t, y - x_t\rangle \\
&= (1-t)[g(x_t,x) + h(x_t,x) + \langle Tx_t, \eta(x,x_t)\rangle + \langle Ax_t, x - x_t\rangle] \\
&\quad +t[g(x_t,y) + h(x_t,y) + \langle Tx_t, \eta(y,x_t)\rangle + \langle Ax_t, y - x_t\rangle] \\
&\le (1-t)\frac{1}{r}\langle u, x - x_t\rangle + t[g(x_t,y) + h(x_t,y) \\
&\quad +\langle Tx_t, \eta(y,x_t)\rangle + \langle Ax_t, y - x_t\rangle] \\
&= t(1-t)\frac{1}{r}\langle u, x-y\rangle + t[g(x_t,y) + h(x_t,y) \\
&\quad +\langle Tx_t, \eta(y,x_t)\rangle + \langle Ax_t, y - x_t\rangle]. \qquad (4.5)
\end{aligned}$$

Hence,

$$g(x_t,y) + h(x_t,y) + \langle Tx_t, \eta(y,x_t)\rangle + \langle Ax_t, y - x_t\rangle \ge (1-t)\frac{1}{r}\langle u, y-x\rangle.$$

By $(A4)$, $(B4)$, $(C4)$, and $(D2)$, we obtain that

$$\begin{aligned}
g(x,y) &+ h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle \\
&\ge \limsup g(x_t,y) + \limsup h(x_t,y) + \limsup \langle Tx_t, \eta(y,x_t)\rangle \\
&\quad + \limsup \langle Ax_t, y - x_t\rangle \\
&\ge \frac{1}{r}\langle u, y-x\rangle.
\end{aligned}$$

□

Theorem 4.3. *Let K be a nonempty bounded closed convex subset of H. Let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$-$(A4)$, and $h : K \times K \to \mathbb{R}$ be a monotone mapping and satisfying $(B1)$-$(B4)$. Let $T : K \to H$ satisfying $(C2)$ and $(C4)$, $A : K \to H$ satisfying $(D1)$-$(D3)$ and that the set $GMEPRM(g, h, T, A)$ of solutions (1.4) is nonempty. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence generated by the form*

$$x_0 \in K \quad \textit{and} \quad x_{n+1} = T_{r_n} x_n, \quad \textit{where} \quad r_n \in (0, +\infty), \quad \textit{for all} \quad n \in \mathbb{N}, \tag{4.6}$$

where $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $GMEPRM(g, h, T, A)$.

Proof. Since $GMEPRM(g, h, T, A) \neq \emptyset$, it follows that $domT_{r_n} = H$ for all $n \geq 1$. For any $n \in \mathbb{N}$, we note from (4.6) and (4.1) that

$$\begin{cases} 0 \leq g(x_{n+1}, x_{n+2}) + h(x_{n+1}, x_{n+2}) + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle \\ \qquad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle + \frac{1}{r_n}\langle x_{n+1} - x_n, x_{n+2} - x_{n+1}\rangle \\ 0 \leq g(x_{n+2}, x_{n+1}) + h(x_{n+2}, x_{n+1}) + \langle Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ \qquad + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle + \frac{1}{r_{n+1}}\langle x_{n+2} - x_{n+1}, x_{n+1} - x_{n+2}\rangle. \end{cases} \tag{4.7}$$

Setting $z_n = T_{r_n} x_n$ and $u_n = (x_n - z_n)/r_n$. Then (4.7) yields

$$\begin{cases} \langle u_n, x_{n+2} - x_{n+1}\rangle \leq g(x_{n+1}, x_{n+2}) + h(x_{n+1}, x_{n+2}) + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle \\ \qquad\qquad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle \\ \langle u_{n+1}, x_{n+1} - x_{n+2}\rangle \leq g(x_{n+2}, x_{n+1}) + h(x_{n+2}, x_{n+1}) + \langle Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ \qquad\qquad + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle \end{cases} \tag{4.8}$$

and by $(A2)$, $(D3)$, and the monotonicity of h that

$$\begin{aligned} &\langle u_n - u_{n+1}, x_{n+2} - x_{n+1}\rangle \\ &\quad \leq \; g(x_{n+1}, x_{n+2}) + g(x_{n+2}, x_{n+1}) + h(x_{n+1}, x_{n+2}) + h(x_{n+2}, x_{n+1}) \\ &\qquad + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle + Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ &\qquad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle \leq 0. \end{aligned} \tag{4.9}$$

Thus $\langle u_{n+1} - u_n, u_{n+1}\rangle \leq 0$ and, by Cauchy-Schwarz, $\|u_{n+1}\| \leq \|u_n\|$. Therefore

$$(\|u_n\|)_{n\in\mathbb{N}} \;\; \text{converges.} \tag{4.10}$$

Since T_{r_n} is firmly nonexpansive, it follows by Theorem 2.6 in [28] that $\sum_{n\in\mathbb{N}} r_n^2 \|u_n\|^2 = \sum_{n\in\mathbb{N}} r_n^2 \|z_n - x_n\|^2 < +\infty$. Since $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$, we have $\liminf_n \|u_n\| = 0$

and, consequently, (4.10) yields $u_n \to 0$. Since (x_n) is bounded, we may assume that there exist a sequence (x_{k_n}) of (x_n) such that $x_{k_n} \rightharpoonup x$ and

$$u_{k_n} \to 0. \tag{4.11}$$

On the other hand, since $z_n - x_n \to 0$, we have

$$z_{k_n} \rightharpoonup x. \tag{4.12}$$

Combining (4.11), (4.12), and Lemma 4.2, we conclude that x is a solution of (1.4). □

In the case of $h \equiv 0$, $T \equiv 0$, and $A \equiv 0$ in (1.4), $GMEPRM(g, h, T, A)$ deduced to equilibrium problem (for short, $EP(g)$) is to find $x \in K$ such that

$$g(x, y) \geq 0 \quad \text{for all} \quad y \in K. \tag{4.13}$$

Corollary 4.4. [28] *Let K be a nonempty bounded closed convex subset of H. Let $g : K \times K \to \mathbb{R}$ satisfying $(A1)$-$(A4)$ and that the set $EP(g)$ of solutions to* (4.13) *is nonempty. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence generated by the form*

$$x_0 \in K \quad \text{and} \quad x_{n+1} = J_{r_n} x_n, \quad \text{where} \quad r_n \in (0, +\infty), \quad \text{for all} \quad n \in \mathbb{N}, \tag{4.14}$$

where $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in $EP(g)$.

Proof. It follows from Theorem 4.3 by setting $h \equiv 0$, $T \equiv 0$, and $A \equiv 0$. □

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