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Applications of Student Activity Problems for Some Kirkman Type Problems[1]

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Abstract : In this paper we apply the ideas from recent works on Student Activity Problems in proposing a theorem on some Kirkman type problems. That is, we find that for any prime number $p \geq 3$ it is possible for a school teacher to take p^2 school girls on a walk each day of the $p+1$ days, walking with p rows of p girls each, in such a way that each pair of girls walk together in the same row on exactly one day.

Keywords : Kirkman school girl problems, Steiner triple systems, student activity problems.

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1 Introduction to Student Activity Problems

Student Activity Problems (SAP) and School Activity Algorithm (SAA) are discussed in details in [1]. The problems are about arranging joint activities for students from $n \geq 2$ schools. There are $p > n$ different activity rooms $r_1, r_2, ..., r_p$. Each school provide p students to join the activities for p days. In each day, each student has to join an activity in one of the p rooms with a condition that in each room there are n students from n different schools. It is required that each student participates all of the p activities in p days and each has to join exactly once with every student from all other schools. Two main questions arise from the problems. For given $n \geq 2$ and $p > n$, is it always possible to have the arrangements that satisfy the conditions of SAP? The other question is about how to do such arrangements when it is possible. Theorem 1.1 which is adapted from [1] can provide sufficient conditions for the arrangements.

Theorem 1.1. *Let $n \geq 2$ be number of schools each of which provide $p > n$ students to participate p joint activities. For any prime number $p > n$, the arrangements that satisfy School Activity Problem (SAP) conditions are possible and can be arranged by using the School Activity Algorithm (SAA) provided.*

For better understanding about Theorem 1.1 we consider the case when $n = 4$ and $p = 5$. That is, we have 4 schools s_1, s_2, s_3, and s_4 each of which provide 5 students to join 5 activities in rooms r_1, r_2, r_3, r_4, and r_5. Let S_1, S_2, S_3, and S_4 be sets of 5 students of schools s_1, s_2, s_3, and s_4 respectively.

$$S_1 = \{1, 2, 3, 4, 5\},\ S_2 = \{6, 7, 8, 9, 10\}$$
$$S_3 = \{11, 12, 13, 14, 15\},\ S_4 = \{16, 17, 18, 19, 20\}.$$

Let $R_1, R_2, ..., R_5$ be sets of students doing activities in rooms $r_1, r_2, ..., r_5$ respectively. For the first day, called Day 1, we can arrange students for each room r_i as follows:

$$R_1 = \{1, 6, 11, 16\},\ R_2 = \{2, 7, 12, 17\}$$
$$R_3 = \{3, 8, 13, 18\},\ R_4 = \{4, 9, 14, 19\},\ R_5 = \{5, 10, 15, 20\}.$$

Using the algorithm SAA, we can arrange students to join activites in table form as follows:

	Day 1				Day 2				Day 3				Day 4				Day 5			
R_1	1	6	11	16	5	9	13	17	4	7	15	18	3	10	12	19	2	8	14	20
R_2	2	7	12	17	1	10	14	18	5	8	11	19	4	6	13	20	3	9	15	16
R_3	3	8	13	18	2	6	15	19	1	9	12	20	5	7	14	16	4	10	11	17
R_4	4	9	14	19	3	7	11	20	2	10	13	16	1	8	15	17	5	6	12	18
R_5	5	10	15	20	4	8	12	16	3	6	14	17	2	9	11	18	1	7	13	19

Figure 1.1

We shall briefly explain how to obtain the arrangements in Figure 1.1, see [1] for more details.

Once we obtain the arrangements of Day 1, consider students 1,2,3,4,5 in the first column of the arrangements of Day 1. We can obtain the arrangements of 1,2,3,4,5 for Day 2 by shifting, in circular manner each student of Day 1 to the next room, i.e. from r_1 to r_2, r_2 to r_3, r_3 to r_4, r_4 to r_5, and r_5 to r_1. We can obtain the arrangements of 1,2,3,4,5 for Day 3 by shifting, in circular manner, each student of Day 2 to the next room, i.e. from r_2 to r_3, r_3 to r_4, r_4 to r_5, r_5 to r_1, and r_1 to r_2. We can obtain the arrangements of Day 4 from Day 3, and the arrangements of Day 5 from Day 4, by similar ways of shifting. For students 6,7,8,9,10 in the second column in Day 1, we can obtain the arrangements of Day 2, Day 3, Day 4, and Day 5 by using similar ideas for the shifting of the students in the first column but instead of shifting each student to the next room, we shift each of them to the next second room. For students 11,12,13,14,15 in the third column of Day 1, we can obtain the arrangements of Day 2, Day 3, Day 4, and Day 5 by using similar ideas but here we shift each of them to the next third room. For students 16,17,18,19,20 in Day 1, we can obtain the arrangements of Day 2, Day 3, Day 4, and Day 5 by using similar ideas of shifting but now we shift each of them to the next fourth room, etc.

In section 2, we discuss about Kirkman type problems, and how to solve some of these problems by applying Theorem 1.1 and the algorithm SAA.

2 Kirkman Triple Systems and Kirkman Type Problems

There are some studies involving with arrangements or partitions of students(or elements of sets) with some conditions. Studies on the well known Steiner triple systems, see [2],[3] for examples, and on Kirkman school girl problems provide some questions and answers for some arrangements. There still are many varieties of interesting arrangements that have not been investigated.

In 1850 Reverend Thomas Kirkman posted a problem; later called Kirkman school girl problem, see [4],[5]. The problem say; is it possible for a school to take 15 school girls on a walk each day of the 7 days of a week, walking with 5 rows, of 3 girls each, in such a way that each pair of girls walk in the same row on exactly one day. Let the 15 school girls be denoted by 1,2,3,...,15 respectively. Kirkman showed one possible arrangement which is similar to the arrangement in Figure 2.1.

Day 1			Day 2			Day 3			Day 4			Day 5			Day 6			Day 7		
7	8	9	1	8	10	2	8	11	4	6	9	5	7	10	2	3	9	1	5	9
2	6	10	3	7	11	1	4	12	3	8	12	4	8	13	1	6	11	3	4	10
4	5	11	5	6	12	6	7	13	2	5	13	3	6	14	5	8	14	2	7	12
1	3	13	3	4	14	3	5	15	1	7	14	1	2	15	4	7	15	6	8	15
12	14	15	9	13	15	9	10	14	10	11	15	9	11	2	10	12	13	11	13	14

Figure 2.1

Later, number of works have been done in order to ask and answer for more general forms of Kirkman school girl problem. For another simple example, when there are 21 school girls it can be shown that it is possible to take the girls on walk each of the 10 days, walking with 7 rows of 3 girls each in such a way that each pair of girls walk in the same row on exactly one day. With similar condition for the arrangement of the walks as in the above two examples, necessary and sufficient conditions about the problems were later studied, see [2],[3] for examples. Most studies of these types of arrangements were involved with three girls(or three elements of sets) in a row. These happen to be particular cases of studies in Steiner triple systems, see [2],[3] for examples. Some studies have been done on quadruple systems which are equivalent to the problem of Kirkman school girl problem but with 4 school girls in a row instead of 3 girls. We shall refer to these kind of arrangements of school girls for walks as Kirkman type problems of which little have been investigated for the cases when the number of school girls in each row are greater than three.

In this section, we propose a question, on Kirkman type problem, whether it is possible to take $p^2 (p \geq 3)$ school girls on a walk each day of the $p+1$ days, walking with p rows of p girls each, in such a way that each pair of girls walk in the same row on exactly one day. We find that for some values of p, it is possible to arrange such walks. In fact, with the application of Theorem 1.1 in section 1, we can have Theorem 2.1 which provide sufficient conditions for the arrangements of p^2 school girls for the required arrangements.

Theorem 2.1. *For any prime number $p \geq 3$, it is possible to arrange p^2 school girls for a walk for $p+1$ days such that each day of the walks arranged, there are p rows with p girls in each row with the condition that any pair of girls has a chance to walk in the same row on exactly one day.*

Proof. Let the p^2 school girls be denoted by $1, 2, 3, ..., p^2$. We can arrange a walk for the starting day as follows:

$$\begin{array}{r} (1, 2, 3, ..., p) \\ (p+1, p+2, p+3, ..., 2p) \\ (2p+1, 2p+2, 2p+3, ..., 3p) \\ \vdots \qquad\qquad\qquad \vdots \\ ((p-2)p+1, (p-2)p+2, (p-2)p+3, ..., (p-1)p) \\ ((p-1)p+1, (p-1)p+2, (p-1)p+3, ..., (p)p) \end{array} \tag{2.1}$$

There are p rows with p school girls in each row. Consider the first $p-1$ rows of (2.1). In order to use Theorem 1.1, we treat the first $p-1$ rows as $p-1$ schools each of which has p students. Using Theorem 1.1 and the algorithm SAA, we can have the arrangements, similar to Figure 1.1, for Day 1, Day 2,..., Day p. In each day there are activities described by sets $R_1, R_2, R_3, ..., R_p$ of which we can regard as a walk for the day that have p rows of $p-1$ girls from $p-1$ different schools.

So, we have arranged such walks for p days such that no pair of girls has a chance to walk together twice.

Next, consider the last row of (2.1). Now, we insert the student $(p-1)p+1$ in each row of Day 1, insert the student $(p-1)p+2$ in each row of Day 2, insert the student $(p-1)p+3$ in each row of Day 3,..., insert the student $(p)p = p^2$ in each row of Day p.

Now we obtain a new table. The activity in each R_k of Day 1, Day 2,..., Day p can provide a walk for a day such that there are p rows each of which has p girls. Now we have arranged walks for p days. Together with the walk on the starting day, see(2.1), we have arranged walks for $p+1$ days that satisfy the condition of SAP □

To verify Theorem 2.1, we consider the case when $p = 5$, i.e. we have $p^2 = 25$ students. Suppose the students are 1,2,3,...,25. We can arrange a walk for the starting day as follows:

$$\begin{array}{r} (1,2,3,4,5) \\ (6,7,8,9,10) \\ (11,12,13,14,15) \\ (16,17,18,19,20) \\ (21,22,23,24,25) \end{array} \tag{2.2}$$

Consider the students in the first 4 row of (2.2). Using Theorem 1.1 and SAA with the first 4 rows, treating them as 4 different schools each of which has 5 students. We can then obtain the arrangement as in Figure 1.1. Consider the last row of (2.2). We insert 21 in each row of Day 1 of Figure 1.1, insert 22 in each row of Day 2, insert 23 in each row of Day 3, insert 24 in each row of Day 4, and insert 25 in each row of Day 5. Now, we have obtain a new table as in Figure 2.2

	Day 1					Day 2					Day 3					Day 4					Day 5				
R_1	1	6	11	16	21	5	9	13	17	22	4	7	15	18	23	3	10	12	14	24	2	8	14	20	25
R_2	2	7	12	17	21	1	10	14	18	22	5	8	11	19	23	4	6	13	20	24	3	9	15	16	25
R_3	3	8	13	18	21	2	6	15	19	22	1	9	12	20	23	5	7	14	16	24	4	10	11	17	25
R_4	4	9	14	19	21	3	7	11	20	22	2	10	13	16	23	1	8	15	17	24	5	6	12	18	25
R_5	5	10	15	20	21	4	8	12	16	22	3	6	14	17	23	2	9	11	18	24	1	7	13	19	25

Figure 2.2

The activities in R_1 of all Day 1, Day 2,...,Day 5 can represent a walk, called Walk 1, of 5 rows each of which has 5 girls. The activities in R_2, R_3, R_4, R_5 can similarly represent walks, called Walk 2, Walk 3, Walk 4, Walk 5 respectively. See Figure 2.3 for the five walks.

Walk 1 (R_1)					Walk 2 (R_2)					Walk 3 (R_3)					Walk 4 (R_4)					Walk 5 (R_5)				
1	6	11	16	21	2	7	12	17	21	3	8	13	18	21	4	9	14	19	21	5	10	15	20	21
5	9	13	17	22	1	10	14	18	22	2	6	15	19	22	3	7	11	20	22	4	8	12	16	22
4	7	15	18	23	5	8	11	19	23	1	9	12	20	23	2	10	13	16	23	3	6	14	17	23
3	10	12	19	24	4	6	13	20	24	5	7	14	16	24	1	8	15	17	24	2	9	11	18	24
2	8	14	20	25	3	9	15	16	25	4	10	11	17	25	5	6	12	18	25	1	7	13	19	25

Figure 2.3

Five walks in Figure 2.3, and another walk from (2.2) provide (5+1)=6 walks as predicted by Theorem 2.1.

In Theorem 2.1, it is required that $p \geq 3$ is any prime number. However, Theorem 2.1 may be held for some other values of p. These are open for further studies.

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