Thai Journal of Mathematics
Volume 13 (2015) Number 3 : 527–537
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence Theorems of Fixed Point Iterative Methods Defined by Admissible Functions

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Abstract : The purpose of this paper is to prove some convergence theorems for fixed point iterative methods defined by means of the new concept of admissible function, introduced by Rus [1]. Moreover, we find some sufficient conditions for weak and strong convergences of general iterative methods for nonexpansive mappings and their generalizations. The results obtained in this paper extend and generalize many known results in the studied area.

Keywords : fixed point; nonexpansive mapping; quasi-nonexpansive mapping; admissible function; iterative method.
2010 Mathematics Subject Classification : 47H09; 47H10; 54H25.

1 Introduction

Let X be a nonempty set and $T : X \to X$ be a nonexpansive mapping. A *fixed point* of T is an element $x \in X$ which satisfies $T(x) = x$. The set of fixed points of T is denoted by $F(T)$.

The famous fixed point theorem for nonexpansive mappings have first studied by Browder [2] and Göhde [3] in Banach spaces as follows:

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Theorem 1.1 ([2],[3]). *Let X be a uniformly convex Banach space and C be a nonempty closed convex bounded subset of X. Then every nonexpansive mapping $T : C \to C$ has a fixed point.*

In case X and T are general enough, like in Theorem 1.1, when T has at least one fixed point, however, the Picard iteration defined by $x_1 \in X$ and

$$x_{n+1} = Tx_n = T^n x_1 \text{ for all } n \in \mathbb{N}$$

does not converge in general or, even if it converges, its limit is not a fixed point of T.

In such circumstances, it is necessary to consider more reliable fixed point iterative methods, like Krasnoselskij iteration, Mann iteration, S-iteration etc.

In 1953, Mann [4] introduced the following iteration method which was referred to as *Mann iteration* for approximating a fixed point of T.

Let C be a nonempty subset of Banach space X and $T : C \to C$ be a self-map. The sequence $\{x_n\}_{n=1}^{\infty}$ in C is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$. He proved a weak convergence for a nonexpansive mapping under the control conditions $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. For $\alpha_n = \lambda$ (constant), the Mann iteration (1.1) reduces to the so-called *Krasnoselskij iteration* [5] that is,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

In 2007, Agarwal, ORegan and Sahu [6] introduced the *S-iteration* process in a Banach space, a sequence $\{x_n\}_{n=1}^{\infty}$ in C is defined by $x_1 \in C$ and

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these for asymptotically nonexpansive mappings. (see [6], Proposition 3.1).

In 2012, Rus [1] introduced about a new approach of fixed point iterative methods, based on the concept of *admissible functions* (see Definition 1.2) of a self operator. The theory of admissible functions of an operator opened a new direction of research and unified the most important aspects of the iterative approximation of fixed point for single valued self operators.

Definition 1.2. ([1]) Let X be a nonempty set. A mapping $G : X \times X \to X$ is called an *admissible function* if it satisfies

(G1) $G(x, x) = x$, for all $x \in X$ and

(G2) $G(x, y) = x$ implies $y = x$, for $x, y \in X$.

In 2013, Berinde [7] introduced an iterative algorithm in terms of admissible functions, which is called *the Krasnoselskij algorithm corresponding to* G or *the GK-algorithm.*

Definition 1.3. ([7]) Let X be a nonempty set, $G : X \times X \to X$ be an admissible function and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\}$ in X given by $x_1 \in X$ and

$$x_{n+1} = G(x_n, T(x_n)), \quad n \in \mathbb{N}, \tag{1.4}$$

is called *the Krasnoselskij algorithm corresponding to* G or *the GK-algorithm.*

He proved some strong and weak convergence theorems for a Krasnoselskij type of a fixed point iterative method defined by an admissible function for non-expansive mapping on Hilbert spaces.

The following year, Berinde proved some convergence theorems for a GK-algorithm with an affine Lipschitzian property (see Definition 1.4) of a nonlinear φ-pseudocontractive operator defined on a Hilbert space and obtained the result in [8].

Definition 1.4. ([8]) Let $G : X \times X \to X$ be an admissible function on a normed space X. We say that G is *affine Lipschitzian* if there exist a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \le \|\mu(x_1 - x_2) + (1 - \mu)(y_1 - y_2)\|,$$

for all x_1, x_2, y_1 and y_2 in X.

In this paper, we give necessary and sufficient conditions for the convergence of the GK-algorithm, the Mann iterative method and S-iterative method defined by admissible function for nonexpansive mappings and their generizations on a Banach space.

2 Preliminaries

Recall that a mapping $T : C \to X$ is *demiclosed* at v if for any sequence $\{x_n\}$ in C, the following implication holds: $x_n \rightharpoonup u \in C$ and $Tx_n \longrightarrow v$ imply $Tu = v$.

Lemma 2.1. ([9]) *Let $r > 0$ be a fixed real number. Then a Banach space X is uniformly convex if and only if there is a continuous strictly increasing convex map $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for all $x, y \in B_r = \{x \in X : \|x\| \le r\}$,*

$$\|\lambda x + (1 - \lambda)y\|^2 \le \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $\lambda \in [0, 1]$.

Lemma 2.2. ([10]) *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence of real numbers in $(0,1)$ bounded away from 0 and 1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty}\|x_n\| \le a, \limsup_{n\to\infty}\|y_n\| \le a \text{ and } \limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a$$

for some $a \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.3. ([11]) *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex subset of X. If $T : C \longrightarrow C$ is a nonexpansive mapping, then $I - T$ is demiclosed with respect to zero.*

3 Main Results

3.1 Admissible Functions and Iterative Algorithms in Terms of Admissible Functions

In the previous sections, we have introduced the admissible functions. We will now give some of their examples:

Example 3.1. *rm Let $X = \mathbb{R}$ with usual metric d and $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by*

$$G(x,y) = \begin{cases} x & \text{if } x = y, \\ \frac{2x^2y}{x^2+y^2} & \text{if } x \neq y. \end{cases}$$

Then G is an admissible function.

Example 3.2. Let $(X,+,\mathbb{R})$ be a real vector space, a nonempty convex subset C of X, $\lambda \in (0,1)$ and $G : C \times C \to C$ defined by

$$G(x,y) = (1-\lambda)x + \lambda y, \quad x, y \in C.$$

It is easy to see that G satisfies conditions G1 and G2, then G is an admissible function.

Example 3.3. Let $(X,+,\mathbb{R})$ be a real vector space, a nonempty convex subset C of X, $\chi : C \times C \to (0,1)$ and $G : C \times C \to C$ defined by

$$G(x,y) = (1-\chi(x,y))x + \chi(x,y)y, \quad x, y \in X.$$

It is clear that G is an admissible function.

Example 3.4. Let $(X, +, \mathbb{R})$ be a real vector space, a nonempty convex subset C of X, for each $n \in \mathbb{N}$ let $G_n : C \times C \to C$ be defined

$$G_n(x, y) = (1 - \frac{1}{n})x + \frac{1}{n}y, \quad x, y \in C.$$

Thus, for each $n \in \mathbb{N}$, we have G_n is an admissible function.

Example 3.5. Let (X, d) be a metric space endowed with a W-convex structure of Takahashi [12]. Here $W : X \times X \times [0, 1] \to X$ is an operator with the following property

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \forall x, y, u \in X, \lambda \in [0, 1].$$

We additionally suppose that $\lambda \in (0, 1)$ and $G(x, y) := W(x, y, \lambda)$. Let $x, y \in X$ and $\lambda \in (0, 1)$, we have

$$d(u, W(x, x, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, x) = d(u, x).$$

Choose $u = x$, then $d(x, W(x, x, \lambda)) = 0$, that is $G(x, x) = W(x, x, \lambda) = x$.
Now we suppose $x = G(x, y) = W(x, y, \lambda)$, then

$$d(u, x) = d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

Thus,

$$(1 - \lambda)d(u, x) = (1 - \lambda)d(u, y),$$

and choose $u = x$, then $d(x, y) = 0$, that is $x = y$. Therefore $G(x, y) := W(x, y, \lambda)$ with $\lambda \in (0, 1)$ is an admissible function.

It is clear that the iterations in Example 3.1, 3.2, 3.3 and Example 3.5 are GK-algorithms. Now we will introduce another representation of iterative algorithms in terms of admissible functions.

Definition 3.6. (GM-algorithm) Let $G_n : X \times X \to X$ be an admissible function for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\}$ in X given by $x_1 \in X$ and

$$x_{n+1} = G_n(x_n, T(x_n)), \quad n \in \mathbb{N} \tag{3.1}$$

is called *the Mann algorithm corresponding to* G_n or *the GM-algorithm.*

It is easy to see that Example 3.4 is the GM-algorithm and in the particular case when C is a nonempty convex subset of a Banach space X and $G_n(x_n, Tx_n) = (1 - \lambda_n)x_n + \lambda_n Tx_n$ with $\{\lambda_n\}$ in $[0, 1]$ for $n \in \mathbb{N}$, we have that $\{x_n\}$ in C, where $x_{n+1} = G_n(x_n, Tx_n)$, is a usual Mann iteration.

Definition 3.7. (GS-algorithm) Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $T : X \to X$ be an operator. Then the iterative algorithm $\{x_n\}$ in X given by $x_1 \in X$ and

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(Tx_n, Ty_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.2}$$

is called *the S-algorithm corresponding to G_n^1 and G_n^2* or *the GS-algorithm.*

We see that when C is a nonempty convex subset of a Banach space X, $G_n^2(x_n, Tx_n) = (1-\beta_n)x_n + \beta_n Tx_n$ and $G_n^1(Tx_n, T(G_n^2(x_n, Tx_n))) = (1-\alpha_n)Tx_n + \alpha_n T(G_n^2(x_n, Tx_n))$ with $\{\alpha_n\}, \{\beta_n\}$ are sequences of real number in $[0, 1]$ for $n \in \mathbb{N}$. The sequence $\{x_n\}$ in C generated by $x_{n+1} = G_n^1(Tx_n, Ty_n)$, where $y_n = G_n^2(x_n, Tx_n)$ is an S-iteration.

3.2 Convergence Theorems for Fixed Point Iterative Methods Defined by Admissible Function

In this section, we find control conditions for iterative methods defined by admissible function to converge to fixed points.

First, recall that let C be a nonempty subset of a Banach space X and $T : C \to C$ be a self-map. A mapping T is called *demicompact* if every bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ is strongly convergent, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is strongly convergent.

We begin with the GK-algorithm of nonexpansive mapping in a uniformly convex Banach space.

Theorem 3.8. *Let C be a closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ be a nonexpansive and demicompact mapping. If $G : C \times C \to C$ is an affine Lipschitzian admissible function with constant $\lambda \in (0, 1)$. Then the GK-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges(strongly) to a fixed point of T in C.

Proof. By Theorem 1.1, $F(T)$ is a nonempty set. Let $p \in F(T)$. We first show that the sequence $\{x_n - Tx_n\}$ converges strongly to zero. Since G is an affine Lipschitzian admissible function and T is nonexpansive, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|G(x_n, Tx_n) - G(p, p)\| \\ &\leq \|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| \\ &\leq \lambda\|(x_n - p)\| + (1-\lambda)\|(Tx_n - p)\| \\ &\leq \lambda\|(x_n - p)\| + (1-\lambda)\|(x_n - p)\| \\ &= \|x_n - p\|. \end{aligned}$$

That is $\lim_{n\to\infty}\|x_n - p\|$ exists. Suppose that $\lim_{n\to\infty}\|x_n - p\| = a$, then

$$\limsup_{n\to\infty}\|Tx_n - p\| \le \lim_{n\to\infty}\|x_n - p\| = a$$

and since

$$a = \lim_{n\to\infty}\|x_{n+1} - p\| \le \lim_{n\to\infty}\|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| \le \lim_{n\to\infty}\|x_n - p\| = a,$$

we have

$$\lim_{n\to\infty}\|\lambda(x_n - p) + (1-\lambda)(Tx_n - p)\| = a$$

By Lemma 2.2, we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

This shows that $x_n - Tx_n \to 0$, and since T is demicompact, it follows that there exists a subsequence $\{x_{n_k}\} \subseteq C$ of $\{x_n\}$ and $q \in C$ such that

$$\lim_{n\to\infty} x_{n_k} = q.$$

But T is nonexpansive, hence T is continuous. This implies

$$\lim_{n\to\infty} Tx_{n_k} = Tq.$$

That is

$$0 = \lim_{n\to\infty}(x_{n_k} - Tx_{n_k}) = q - Tq.$$

This means that q is a fixed point of T and since $\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|x_{n_k} - q\| = 0$. Therefore, $\{x_n\}$ converges strongly to a fixed point of T in C. □

We now consider a class of mappings that properly includes the class of nonexpansive mappings with fixed points, that is quasi-nonexpansive mappings. A condition that ensures strong convergence of iterative sequences to fixed points of quasi-nonexpansive type mappings was introduced in [10] and [13].

Definition 3.9. ([10],[13]) Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$. Then T is said to satisfy *Condition I* if there exist a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(t) > t$ for $t \in (0,\infty)$ such that

$$\|x - Tx\| \ge f(d(x, F(T))),$$

where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Example 3.10. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = -x$ and $F(T) = \{0\}$, then the mapping $f : [0,\infty) \to [0,\infty)$ with $f(x) = 1.5x$ satisfies

$$\|x - Tx\| = 2x \ge 1.5x = f(d(x, F(T))).$$

That is, T satisfies Condition I.

Next, we introduce a new property for the algorithms.

Definition 3.11. Let $G_n : X \times X \to X$ be an admissible function on a normed space X for $n \in \mathbb{N}$. We say that $\{G_n\}$ is *sequentially affine Lipschitzian* if there exists a sequence of real numbers $\{\alpha_n\}$ in $[0,1]$ such that

$$\|G_n(x_1, y_1) - G_n(x_2, y_2)\| \leq \|\alpha_n(x_1 - x_2) + (1 - \alpha_n)(y_1 - y_2)\|,$$

for all x_1, x_2, y_1 and y_2 in X.

It is easy to see that admissible functions in Example 3.2 and 3.4 are sequentially affine Lipschitzian. In the particular case when $G_n(x, y) = (1 - \alpha_n)x + \alpha_n y$ with $\{\alpha_n\}$ in $[0,1]$ and $n \in \mathbb{N}$, we have $\{G_n\}$ is sequentially affine Lipschitzian.

We prove the strong convergence of the GM-iteration for quasi-nonexpansive mappings satisfying Condition I.

Theorem 3.12. *Let C be a closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a continuous quasi-nonexpansive mapping with satisfies Condition I. If $\{G_n\}$ is sequentially affine Lipschitzian with a sequence $\{\alpha_n\}$ which is bounded away from 0 and 1. Then the GM-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in C$ and*

$$x_{n+1} = G_n(x_n, Tx_n), \quad n \in \mathbb{N}$$

converges(strongly) to a fixed point of T in C.

Proof. Let $p \in F(T)$. Since $\{G_n\}$ is sequentially affine Lipschitzian and T is quasi-nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|G_n(x_n, Tx_n) - G_n(p, p)\| \\
&\leq \|\alpha_n(x_n - p) + (1 - \alpha_n)(Tx_n - p)\| \\
&\leq \alpha_n\|(x_n - p)\| + (1 - \alpha_n)\|(Tx_n - p)\| \\
&\leq \alpha_n\|(x_n - p)\| + (1 - \alpha_n)\|(x_n - p)\| \\
&= \|x_n - p\|.
\end{aligned}$$

Using the same proof as in Theorem 3.8, we can show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Because for $p \in F(T)$, $\|x_{n+1} - p\| \leq \|x_n - p\|$, it follows that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Since T satisfies condition I, we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))), n \geq 0.$$

This implies, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Then for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \epsilon/2 \text{ for all } n \geq n_0.$$

Consider, for $n, m \geq n_0$. So there is a $p \in F(T)$ such that $d(x_{n_0}, p) < \epsilon/2$, we have

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq 2\|x_{n_0} - p\| < \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence and by completeness of X, we have $\lim_{n\to\infty} x_n = q$ for some $q \in C$. Since T is continuous and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Therefore, $q \in F(T)$ implies that $\{x_n\}$ converges strongly to a fixed point of T in C. □

Next, we show that GS-algorithm with only sequentially affine Lipschitzian property converges weakly on uniformly convex Banach space.

Theorem 3.13. *Let X be a uniformly convex Banach space that satisfies Opials condition, C be a nonempty closed convex subset of X and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $G_n^1, G_n^2 : X \times X \to X$ be admissible functions for $n \in \mathbb{N}$ and $\{G_n^1\}, \{G_n^2\}$ are sequentially affine Lipschitzian with $\{\alpha_n\}$ and $\{\beta_n\}$ respectively. Suppose that $\limsup_{n\to\infty} \alpha_n < 1$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Then the GS-algorithm $\{x_n\}_{n=1}^{\infty}$ given by $x_1 \in X$ and*

$$\begin{cases} y_n = G_n^2(x_n, Tx_n), \\ x_{n+1} = G_n^1(Tx_n, Ty_n), \quad n \in \mathbb{N}, \end{cases}$$

converges weakly to a fixed point of T.

Proof. Let $p \in F(T)$. Since $\{G_n^2\}$ is sequentially affine Lipschitzian with $\{\beta_n\}$ and T is nonexpansive mapping, we have

$$\begin{aligned} \|y_n - p\| &= \|G_n^2(x_n, Tx_n) - G_n^2(p,p)\| \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned}$$

and since $\{G_n^1\}$ is sequentially affine Lipschitzian with $\{\alpha_n\}$, we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|G_n^1(Tx_n, Ty_n) - G_n^1(p,p)\| \\ &\leq \|\alpha_n(Tx_n - p) + (1-\alpha_n)(Ty_n - p)\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \|x_n - q\|. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \|x_n - p\|$ exists and implies that $\{x_n\}$ is bounded. By Lemma 2.1, there is a continuous strictly increasing convex mapping g with $g(0) = 0$ such that

$$\begin{aligned} \|y_n - p\|^2 &= \|G_n^2(x_n, Tx_n) - G_n^2(p,p)\|^2 \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|), \end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|G_n^1(Tx_n,Ty_n)-G_n^1(p,p)\|^2\\
&\le \|\alpha_n(Tx_n-p)+(1-\alpha_n)(Ty_n-p)\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2-\alpha_n(1-\alpha_n)g(\|Tx_n-Ty_n\|)\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)(\|x_n-p\|^2-\beta_n(1-\beta_n)g(\|x_n-Tx_n\|))\\
&= \|x_n-p\|^2-\beta_n(1-\alpha_n)(1-\beta_n)g(\|x_n-Tx_n\|).
\end{aligned}$$

Thus

$$\beta_n(1-\alpha_n)(1-\beta_n)g(\|x_n-Tx_n\|)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

By the control conditions on α_n and β_n, we get

$$\lim_{n\to\infty} g(\|x_n-Tx_n\|)=0.$$

Then $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Because $\{x_n\}$ is bounded in X, it follows that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to $p\in C$. By Lemma 2.3, $I-T$ is demiclosed at zero, from $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, we have $(I-T)p=0$, so that $p\in F(T)$.

Next, we show that $\{x_n\}$ converges weakly to a fixed point of T. Suppose there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $q\in F(T)$ such that $q\ne p$. Since X satisfies the Opials condition, we have

$$\lim_{n\to\infty}\|x_n-p\|=\lim_{j\to\infty}\|x_{n_j}-p\|<\lim_{j\to\infty}\|x_{n_j}-q\|=\lim_{n\to\infty}\|x_n-q\|,$$

and

$$\lim_{n\to\infty}\|x_n-q\|=\lim_{j\to\infty}\|x_{n_j}-q\|<\lim_{j\to\infty}\|x_{n_j}-p\|=\lim_{n\to\infty}\|x_n-p\|,$$

a contradiction, hence $p=q$. Therefore, $\{x_n\}$ converges weakly to $p\in F(T)$. □

Acknowledgement : I would like to thank the referee for his comments and suggestions on the manuscript. This work was supported by the National Research Council of Thailand and Mathematical Association of Thailand.

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(Received 29 September 2015)
(Accepted 12 December 2015)