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Generalized Quasilinearization Method and Cubical Convergence for Mixed Boundary Value Problems

Ramzi S. N. Alsaedi

Abstract : The generalized quasilinearization method for a non-linear second-order ordinary differential equation with mixed boundary conditions has been studied when the forcing function is the sum of two functions without require that any of the two functions involved to be 2-hyperconvex or 2-hyperconcave. Two sequences are developed under suitable conditions which converge to the unique solution of the boundary value problem. Furthermore, the convergence obtain here is of order 3.

Keywords : Generalized quasilinearization; Mixed BVP; Cubical convergence.
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1 Introduction

The method of quasilinearization [1] combined with the technique of lower and upper solutions is an excellent tool for solving a large class of nonlinear problems. This technique works fruitfully only for the problems involving convex/concave functions. Later after that the convexity assumption was relaxed and the method was generalized and extended in various directions to make it applicable to a large class of problems. It has referred to as a generalized quasilinearization method, see [8]. The method is extremely useful in scientific computations due to its accelerated rate of convergence as in [9, 10].

In [3, 13], the authors have obtained a higher order of convergence for initial value problems. They considered situations when the forcing function is either hyperconvex or hyperconcave. In [11], the authors have obtained the results of higher order of convergence for second-oredr boundary value problems when the forcing function is the sum of 2-hyperconvex and 2-hyperconcave functions with natural and coupled lower and upper solutions. The aim of this paper is to cosider and study the existence and approximation of solutions for second-order ordinary differential equation with Dirichlet boundary conditions, by taking the forcing function to be the sum of two functions without require any of the two functions involved to be 2-hyperconvex or 2-hyperconcave. We have proved the existence of the unique solution of the nonlinear problem with Dirichlet boundary conditions using natural lower and upper solutions. We demonstrate the iterates converge cubically to the unique solution of the nonlinear problem.

2 Preliminaries

It is well known that the following mixed BVP

$$-\psi''(t) = \lambda\psi(t), \quad t \in J = [0, \pi],$$
$$\psi(0) = \psi'(\pi) = 0, \tag{2.1}$$

has a nontrivial solution if and only if $\lambda = [(2m-1)/2]^2 \ \ (m = 1, 2, 3, \ldots)$. In consequence, if $\lambda \neq [(2m-1)/2]^2 \ \ (m = 1, 2, 3, \ldots)$, and $\sigma(t) \in C[0, \pi]$, the unique solution of the mixed boundry value problem

$$-\psi''(t) - \lambda\psi = \sigma(t), \quad t \in J = [0, \pi],$$
$$\psi(0) = \psi'(\pi) = 0, \tag{2.2}$$

is given by

$$\psi(t) = \int_0^\pi K_\lambda(t, s)\sigma(s)ds.$$

Here, $K_\lambda(t, s)$ is the Green's function, where $K_\lambda(t, s)$ for $\lambda > 0$, is given by

$$\begin{aligned} & [\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)]^{-1}[\cos(\sqrt{\lambda}(\pi - t))\sin(\sqrt{\lambda}s)], \\ 0 \ \leq \ & s \leq t \leq \pi, \\ & [\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)]^{-1}[\sin(\sqrt{\lambda}t)\cos(\sqrt{\lambda}(\pi - s))], \\ 0 \ \leq \ & t \leq s \leq \pi. \end{aligned}$$

And $K_\lambda(t, s)$ for $\lambda < 0$, is given by

$$\begin{aligned} & [\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\pi)]^{-1}[(\cosh\sqrt{-\lambda}(\pi - t))\sinh(\sqrt{-\lambda}s)], \\ 0 \ \leq \ & s \leq t \leq \pi, \\ & [\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\pi)]^{-1}[\sinh(\sqrt{-\lambda}t)\cosh(\sqrt{-\lambda}(\pi - s))], \\ 0 \ \leq \ & t \leq s \leq \pi. \end{aligned}$$

Finally, when $\lambda = 0$, then

$$K_0(t, s) = \begin{cases} s, & 0 \leq s \leq t \leq \pi; \\ t, & 0 \leq t \leq s \leq \pi. \end{cases}$$

Thus, we have the following comparison result.

Here some definitions and notations will be given to facilitate later explanations.

Definition 2.1 The functions $\alpha_0, \beta_0 \in C^2[J, R]$ are said to be *natural lower and upper solutions* of (2.1) if

$$-\alpha_0'' \leq f(t, \alpha_0) + g(t, \alpha_0), \quad \alpha_0(0) \leq 0, \quad \alpha_0'(\pi) \leq 0 \quad \text{on } J,$$
$$-\beta_0'' \geq f(t, \beta_0) + g(t, \beta_0), \quad \beta_0(0) \geq 0, \quad \beta_0'(\pi) \geq 0 \quad \text{on } J. \tag{2.3}$$

One can define coupled lower and upper solutions of the other types in the same manner. See [13, 14] for details.

Definition 2.2 A function $h : A \to B$, $A, B \subset R$ is called *m-hyperconvex*, $m \geq 0$, if $h \in C^{m+1}[A, B]$ and $d^{m+1}h/du^{m+1} \geq 0$ for $u \in A$; h is called *m-hyperconcave* if the inequality is reversed.

In this paper, we use the maximum norm of u over J, that is,

$$\| u \| = \max\Big\{ \mid u(t) \mid : t \in J \Big\}.$$

Also throughout this paper the following notation

$$\frac{\partial^k f(t, u)}{\partial u^k} = f^{(k)}(t, u)$$

has been used for any function $f(t, u)$ and for $k = 0, 1, 2, \ldots$.

The next corollary is a special case of [8, Theorem 3.1.3].

Corollary 2.3 *Assume that $\alpha_0, \beta_0 \in C^2[J, R]$ are lower and upper solutions of (2.1) respectively such that $\alpha_0(t) \leq \beta_0(t)$ on J. Then there exists a solution u for the BVP (2.1) such that $\alpha_0 \leq u \leq \beta_0$ on J.*

Corollary 2.4 (Comparison Result) *Let $\lambda < 0$ on J and $p \in C^2[J, R]$. If*

$$-p'' \geq \lambda p, \quad p(0) \geq 0, \;\; p'(\pi) \geq 0.$$

Then $p(t) \geq 0$ on J. If the inequalities are reversed, then $p(t) \leq 0$ on J.

3 Main Results

Consider the following BVP

$$-u'' = f(t, u) + g(t, u), \;\; \psi(0) = \psi'(\pi) \;=\; 0, \; t \in J = [0, \pi], \tag{3.1}$$

where $f, g \in C[\Omega, R]$ and $\Omega = [(t, u) \in J \times R : \alpha_0(t) \leq u(t) \leq \beta_0(t)]$, and $\alpha_0, \beta_0 \in C^2[J, R]$ with $\alpha_0(t) \leq \beta_0(t)$ on J.

Theorem 3.1 *Assume that*

(A_1) $\alpha_0, \beta_0 \in C^2[J, R]$ *are lower and upper solutions of (3.1), respectively, such that* $\alpha_0(t) \le \beta_0(t)$ *on* J,

(A_2) $f, g \in C^3[\Omega, R]$ *such that* $f(t,u)$ *is nondecreasing,* $g(t,u)$ *is nonincreasing and* $f_u(t,u) + g_u(t,u) < 0$ *for every* $(t,u) \in \Omega$.

Then there exists monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$, $n \ge 0$ *which converges uniformly and monotonically to the unique solution of (3.1) and the convergence is of order 3.*

Proof. Take

$$\phi(t,u) = F(t,u) - f(t,u) \ ; \ \psi(t,u) = G(t,u) - g(t,u) \quad \text{on} \ \ \Omega, \tag{3.2}$$

where $F, G \in C^3[\Omega, R]$ such that F is a 2-hyperconvex function in u and G is a 2-hyperconcave function in u on J [i.e., $F^{(3)}(t,u) \ge 0, \ G^{(3)}(t,u) \le 0$ for $(t,u) \in \Omega$].

In view of $F^{(3)}(t,u) \ge 0$, for $(t,u) \in \Omega$, we see that

$$F(t,x) \ge \sum_{i=0}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!}, \quad x \ge y, \tag{3.3}$$

$$F(t,x) \le \sum_{i=0}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!}, \quad x \le y. \tag{3.4}$$

Similarly, in view of $G^{(3)}(t,u) \le 0$ for $(t,u) \in \Omega$, we have

$$G(t,x) \ge \sum_{i=0}^{1} \frac{G^{(i)}(t,y)(x-y)^i}{i!} + \frac{G^{(2)}(t,x)(x-y)^2}{2!}, \quad x \ge y, \tag{3.5}$$

$$G(t,x) \le \sum_{i=0}^{1} \frac{G^{(i)}(t,y)(x-y)^i}{i!} + \frac{G^{(2)}(t,x)(x-y)^2}{2!}, x \le y. \tag{3.6}$$

Therefore, (3.3), (3.4), (3.5) and (3.6) can be written in following form

$$f(t,x) \ge f(t,y) + \sum_{i=1}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!} - [\phi(t,x) - \phi(t,y)], \ x \ge y, \tag{3.7}$$

$$f(t,x) \le f(t,y) + \sum_{i=1}^{2} \frac{F^{(i)}(t,y)(x-y)^i}{i!} - [\phi(t,x) - \phi(t,y)], \ x \le y, \tag{3.8}$$

$$\begin{aligned} g(t,x) \ge g(t,y) + G^{(1)}(t,y)(x-y) + \frac{G^{(2)}(t,x)(x-y)^2}{2!} \\ - [\psi(t,x) - \psi(t,y)], \ x \ge y, \end{aligned} \tag{3.9}$$

$$g(t,x) \le g(t,y) + G^{(1)}(t,y)(x-y) + \frac{G^{(2)}(t,x)(x-y)^2}{2!} - [\psi(t,x)-\psi(t,y)],\ x \le y, \tag{3.10}$$

respectively. Let first consider the following BVPs :

$$\begin{aligned} -w'' &= \chi(t,\alpha,\beta;w) \\ &= f(t,\alpha) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha)(w-\alpha)^i}{i!} - l[w^3 - \alpha^3] \\ &\quad + g(t,\alpha) + G^{(1)}(t,\alpha)(w-\alpha) + \frac{G^{(2)}(t,\beta)(w-\alpha)^2}{2!} \\ &\quad - [\psi(t,w) - \psi(t,\alpha)], \\ w(0) = w'(\pi) &= 0; \end{aligned} \tag{3.11}$$

$$\begin{aligned} -v'' &= \omega(t,\alpha,\beta;v) \\ &= f(t,\beta) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta)(v-\beta)^i}{i!} - [\phi(t,v) - \phi(t,\beta)] \\ &\quad + g(t,\beta) + G^{(1)}(t,\beta)(v-\beta) + \frac{G^{(2)}(t,\alpha)(v-\beta)^2}{2!} \\ &\quad - [\psi(t,v) - \psi(t,\beta)], \\ v(0) = v'(\pi) &= 0. \end{aligned} \tag{3.12}$$

Now by using the above BVPs (3.11) and (3.12) to develop the sequences$\{\alpha_n\}$ and $\{\beta_n\}$ respectively. Initially, to prove (α_n, β_0) are the lower and upper solutions of (3.11) and (3.12) respectively, let us consider natural lower and upper solutions of the equation (3.1) :

$$\begin{aligned} -\alpha_0'' &\le f(t,\alpha_0) + g(t,\alpha_0), \quad \alpha_0(0) \le 0, \quad \alpha_0'(\pi) \le 0 \quad \text{on } J, \\ -\beta_0'' &\ge f(t,\beta_0) + g(t,\beta_0), \quad \beta_0(0) \ge 0, \quad \beta_0'(\pi) \ge 0 \quad \text{on } J. \end{aligned} \tag{3.13}$$

where $\alpha_0(t) \le \beta_0(t)$. The inequalities (3.7), (3.9) and (3.13) imply

$$\begin{aligned} -\alpha_0'' &\le f(t,\alpha_0) + g(t,\alpha_0) \\ &= \chi(t,\alpha_0,\beta_0;\alpha_0), \quad \alpha_0(0) \le 0, \quad \alpha_0'(\pi) \le 0; \\ &\quad - \beta_0'' \ge f(t,\beta_0) + g(t,\beta_0) \\ &\ge f(t,\alpha_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_0)(\beta_0-\alpha_0)^i}{i!} - [\phi(t,\beta_0) - \phi(t,\alpha_0)] \\ &\quad + g(t,\alpha_0) + G^{(1)}(t,\alpha_0)(\beta_0-\alpha_0) + \frac{G^{(2)}(t,\beta_0)(\beta_0-\alpha_0)^2}{2!} \\ &\quad - [\psi(t,\beta_0) - \psi(t,\alpha_0)] \\ &= \chi(t,\alpha_0,\beta_0;\beta_0), \quad \beta_0(0) \ge 0, \quad \beta_0'(\pi) \ge 0. \end{aligned} \tag{3.14}$$

By apply Corollary 2.3 together (3.14) conclude that there exists a solution $\alpha_1(t)$ of (3.11) with $\alpha = \alpha_0$ and $\beta = \beta_0$ such that $\alpha_0(t) \le \alpha_1(t) \le \beta_0(t)$ on J.

Using the inequalities (3.8), (3.10) and (3.13), we can get

$$\begin{aligned}
-\beta_0'' &\ge f(t,\beta_0) + g(t,\beta_0) \\
&= \omega(t,\alpha_0,\beta_0;\beta_0), \quad \beta_0(0) \ge 0, \quad \beta_0'(\pi) \ge 0; \\
-\alpha_0'' &\le f(t,\alpha_0) + g(t,\alpha_0) \\
&\le f(t,\beta_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta_0)(\alpha_0-\beta_0)^i}{i!} - [\phi(t,\alpha_0) - \phi(t,\beta_0)] \\
&\quad + g(t,\beta_0) + G^{(1)}(t,\beta_0)(\alpha_0-\beta_0) + \frac{G^{(2)}(t,\alpha_0)(\alpha_0-\beta_0)^2}{2!} \\
&\quad - [\psi(t,\alpha_0) - \psi(t,\beta_0)] \\
&= \omega(t,\alpha_0,\beta_0;\alpha_0), \quad \alpha_0(0) \le 0, \quad \alpha_0'(\pi) \le 0. \qquad (3.15)
\end{aligned}$$

Hence α_0, β_0 are lower and upper solutions of (3.12) with $\alpha_0(t) \le \beta_0(t)$. Apply Corollary 2.3 together (3.15) conclude that there exists a solution $\beta_1(t)$ of (3.12) with $\alpha = \alpha_0$ and $\beta = \beta_0$ such that $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$ on J.

Now to prove that $\alpha_1(t)$ is the unique solution of (3.11), we need to prove that $\partial\chi(t,\alpha_0,\beta_0;\alpha_1)/\partial\alpha_1 < 0$. Since $F(t,u)$ is a 2-hyperconvex function in u and $G(t,u)$ is a 2-hyperconcave function in u on J with $f_u(t,u) + g_u(t,u) < 0$ on Ω, we have

$$\begin{aligned}
\frac{\partial\chi(t,\alpha_0,\beta_0;\alpha_1)}{\partial\alpha_1} &= f^{(1)}(t,\alpha_1) - \frac{F^{(3)}(t,\xi_1)(\alpha_1-\alpha_0)^2}{2} \\
&\quad + g^{(1)}(t,\alpha_1) + G^{(3)}(t,\eta_1)(\alpha_1-\alpha_0)(\beta_0-\xi_2) \\
&\le f^{(1)}(t,\alpha_1) + g^{(1)}(t,\alpha_1) < 0, \qquad (3.16)
\end{aligned}$$

where $\alpha_0 \le \xi_1, \xi_2 \le \alpha_1$ and $\xi_2 \le \eta_1 \le \beta_0$. Hence by Corollary 2.4, we can conclude that α_1 is the unique solution of (3.11). Similarly, one can prove that β_1 is the unique solution of (3.12).

Using the nonincreasing property of $G^{(2)}(t,u)$, (3.7), (3.8), (3.9) and (3.10) with $\alpha_0(t) \le \alpha_1(t) \le \beta_0(t)$, $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$ we have

$$\begin{aligned}
-\alpha_1'' &= \chi(t,\alpha_0,\beta_0;\alpha_1) \\
&= f(t,\alpha_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_0)(\alpha_1-\alpha_0)^i}{i!} - [\phi(t,\alpha_1) - \phi(t,\alpha_0)] \\
&\quad + g(t,\alpha_0) + G^{(1)}(t,\alpha_0)(\alpha_1-\alpha_0) + \frac{G^{(2)}(t,\beta_0)(\alpha_1-\alpha_0)^2}{2!} \\
&\quad - [\psi(t,\alpha_1) - \psi(t,\alpha_0)] \\
&\le f(t,\alpha_1) + g(t,\alpha_1), \quad \alpha_1(0) \le 0, \quad \alpha_1'(\pi) \le 0; \qquad (3.17)
\end{aligned}$$

$$
\begin{aligned}
-\beta_1'' &= \omega(t,\alpha_0,\beta_0;\beta_1) \\
&= f(t,\beta_0) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta_0)(\beta_1-\beta_0)^i}{i!} - [\phi(t,\beta_1)-\phi(t,\beta_0)] \\
&\quad + g(t,\beta_0) + G^{(1)}(t,\beta_0)(\beta_1-\beta_0) + \frac{G^{(2)}(t,\alpha_0)(\beta_1-\beta_0)^2}{2!} \\
&\quad - [\psi(t,\beta_1)-\psi(t,\beta_0)] \\
&\geq f(t,\beta_1) + g(t,\beta_1), \quad \beta_1(0) \geq 0, \quad \beta_1'(\pi) \geq 0. \qquad (3.18)
\end{aligned}
$$

Since α_1, β_1 are lower and upper solutions of (3.1), we can apply Corollary 2.3 to obtain $\alpha_1 \leq \beta_1$ on J. Thus we have $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on J.

Assume now that α_n and β_n are solutions of BVPs (3.11) and (3.12), respecively, with $\alpha = \alpha_{n-1}$ and $\beta = \beta_{n-1}$ such that $\alpha_{n-1} \leq \alpha_n \leq \beta_n \leq \beta_{n-1}$ on J and

$$
\begin{aligned}
-\alpha_n'' &\leq f(t,\alpha_n) + g(t,\alpha_n), \quad \alpha_n(0) \leq 0, \quad \alpha_n'(\pi) \leq 0 \quad \text{on } J, \\
-\beta_n'' &\geq f(t,\beta_n) + g(t,\beta_n), \quad \beta_n(0) \geq 0, \quad \beta_n'(\pi) \geq 0 \quad \text{on } J. \qquad (3.19)
\end{aligned}
$$

We need to show that $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ on J, where α_{n+1} and β_{n+1} are solutions of BVPs (3.11) and (3.12), respecively, with $\alpha = \alpha_n$ and $\beta = \beta_n$.

The inequalities (3.7), (3.9) and (3.19) imply

$$
\begin{aligned}
-\alpha_n'' &\leq f(t,\alpha_n) + g(t,\alpha_n) \\
&= \chi(t,\alpha_n,\beta_n;\alpha_n), \quad \alpha_n(0) \leq 0, \quad \alpha_n'(\pi) \leq 0; \\
-\beta_n'' &\geq f(t,\beta_n) + g(t,\beta_n) \\
&\geq f(t,\alpha_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_n)(\beta_n-\alpha_n)^{(i)}}{i!} - [\phi(t,\beta_n)-\phi(t,\alpha_n)] \\
&\quad + g(t,\alpha_n) + G^{(1)}(t,\alpha_n)(\beta_n-\alpha_n) + \frac{G^{(2)}(t,\beta_n)(\beta_n-\alpha_n)^{(2)}}{2!} \\
&\quad - [\psi(t,\beta_n)-\psi(t,\alpha_n)] \\
&= \chi(t,\alpha_n,\beta_n;\beta_n), \quad \beta_n(0) \geq 0, \quad \beta_n'(\pi) \geq 0. \qquad (3.20)
\end{aligned}
$$

This prove that α_n, β_n are lower and upper solutions of (3.11) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Hence using (3.20) and Corollary 2.3, we can conclude that there exists a solution α_{n+1} of (3.11) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \leq \alpha_{n+1} \leq \beta_n$ on J.

The inequalities (3.8), (3.10) and (3.19) imply

$$
\begin{aligned}
-\beta_n'' &\geq f(t,\beta_n) + g(t,\beta_n) \\
&= \omega(t,\alpha_n,\beta_n;\beta_n), \quad \beta_n(0) \geq 0, \quad \beta_n'(\pi) \geq 0;
\end{aligned}
$$

$$
\begin{aligned}
-\alpha_n'' &\leq f(t,\alpha_n) + g(t,\alpha_n) \\
&\leq f(t,\beta_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta_n)(\alpha_n-\beta_n)^{(i)}}{i!} - [\phi(t,\alpha_n)-\phi(t,\beta_n)] \\
&\quad + g(t,\beta_n) + G^{(1)}(t,\beta_n)(\alpha_n-\beta_n) + \frac{G^{(2)}(t,\alpha_n)(\alpha_n-\beta_n)^{(2)}}{2!} \\
&\quad - [\psi(t,\alpha_n)-\psi(t,\beta_n)] \\
&= \omega(t,\alpha_n,\beta_n;\alpha_n), \quad \alpha_n(0)\leq 0, \quad \alpha_n'(\pi)\leq 0. \qquad (3.21)
\end{aligned}
$$

Hence α_n, β_n are lower and upper solutions of (3.12) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Applying *Corollary 2.3* we can conclude that there exists a solution β_{n+1} of (3.12) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \leq \beta_{n+1} \leq \beta_n$ on J. In view of assumptions on f and g, α_n, β_n are unique by Corollary 2.4.

Furthermore, by (3.7), (3.8), (3.9) and (3.10) with $\alpha_n \leq \alpha_{n+1} \leq \beta_n$, $\alpha_n \leq \beta_{n+1} \leq \beta_n$, and $G^{(2)}(t,u)$ nonincreasing in u, we have

$$
\begin{aligned}
-\alpha_{n+1}'' &= \chi(t,\alpha_n,\beta_n;\alpha_{n+1}) \\
&= f(t,\alpha_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^{(i)}}{i!} - [\phi(t,\alpha_{n+1})-\phi(t,\alpha_n)] \\
&\quad + g(t,\alpha_n) + G^{(1)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n) + \frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!} \\
&\quad - [\psi(t,\alpha_{n+1})-\psi(t,\alpha_n)] \\
&\leq f(t,\alpha_{n+1}) + g(t,\alpha_{n+1}), \quad \alpha_{n+1}(0)\leq 0, \quad \alpha_{n+1}'(\pi)\leq 0; \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
-\beta_{n+1}'' &= \omega(t,\alpha_n,\beta_n;\beta_{n+1}) \\
&= f(t,\beta_n) + \sum_{i=1}^{2} \frac{F^{(i)}(t,\beta_n)(\beta_{n+1}-\beta_n)^{(i)}}{i!} - [\phi(t,\beta_{n+1})-\phi(t,\beta_n)] \\
&\quad + g(t,\beta_n) + G^{(1)}(t,\beta_n)(\beta_{n+1}-\beta_n) + \frac{G^{(2)}(t,\alpha_n)(\beta_{n+1}-\beta_n)^{(2)}}{2!} \\
&\quad - [\psi(t,\beta_{n+1})-\psi(t,\beta_n)] \\
&\geq f(t,\beta_{n+1}) + g(t,\beta_{n+1}), \quad \beta_{n+1}(0)\geq 0, \quad \beta_{n+1}'(\pi)\geq 0. \qquad (3.23)
\end{aligned}
$$

Since $\alpha_{n+1}, \beta_{n+1}$ are lower and upper solutions of (3.1), we can apply Corollary 2.4 to obtain $\alpha_{n+1} \leq \beta_{n+1}$ on J. This proves $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ on J. Thus by induction, we have

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } J.$$

By the fact that α_n, β_n are lower and upper solutions of (3.1) with $\alpha_n \leq \beta_n$ and Corollary 2.3 we can conclude that there exists a solution $u(t)$ of (3.1) such that $\alpha_n \leq u \leq \beta_n$ on J. So we have

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq u \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 \quad \text{on } J. \qquad (3.24)$$

Using Green's function, we can write $\alpha_n(t)$ and $\beta_n(t)$ as follows:

$$\alpha_n(t) = \int_0^1 K_0(t,s)\chi(s,\alpha_{n-1}(s),\beta_{n-1}(s);\alpha_n(s))ds,$$
$$\beta_n(t) = \int_0^1 K_0(t,s)\omega(s,\alpha_{n-1}(s),\beta_{n-1}(s);\beta_n(s))ds. \tag{3.25}$$

We can prove that the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are equicontnuous and uniformly bounded. Now applying Ascoli-Arzela's theorem, we can show that there exist subsequences $\{\alpha_{n,j}(t)\}$ and $\{\beta_{n,j}(t)\}$, such that $\alpha_{n,j}(t) \to \rho(t)$ and $\beta_{n,j}(t) \to r(t)$ with $\rho(t) \le u(t) \le r(t)$ on J. Since the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are monotone, we have $\alpha_n(t) \to \rho(t)$ and $\beta_n(t) \to r(t)$. Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \le u(t) \le r(t) = \lim_{n\to\infty} \beta_n(t).$$

Next we show that $r(t) \le \rho(t)$. From BVPs (3.11) and (3.12) we get

$$-\rho(t)'' = f(t,\rho(t)) + g(t,\rho(t)), \quad \rho(0) = \rho'(\pi) \ = \ 0,$$
$$-r(t)'' = f(t,r(t)) + g(t,r(t)), \quad r(0) = r'(\pi) \ = \ 0. \tag{3.26}$$

Set $p(t) = r(t) - \rho(t)$ and note that $p(0) = p'(\pi) = 0$, we have

$$\begin{aligned} -p''(t) &= -r''(t) - (-\rho''(t)) \\ &= f(t,r(t)) + g(t,r(t)) - f(t,\rho(t)) - g(t,\rho(t)) \\ &= f_u(t,\xi)(r(t)-\rho(t)) + g_u(t,\eta)(r(t)-\rho(t)) \\ &= (f_u(t,\xi) + g_u(t,\eta))p \end{aligned} \tag{3.27}$$

where ξ, η are between r and ρ. This implies that $-p'' \le \lambda p$, where $f_u + g_u \le \lambda < 0$. Now applying Corollary 2.4, we get $r(t) \le \rho(t)$ on J. This proves $r(t) = \rho(t) = u(t)$. Hence $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of (3.1).

Let us consider the order of convergence of $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to the unique solution $u(t)$ of (3.1). To obtain this, set

$$p_n(t) = u(t) - \alpha_n(t) \ge 0,$$
$$q_n(t) = \beta_n(t) - u(t) \ge 0, \tag{3.28}$$

for $t \in J$ with $p_n(0) = p_n'(\pi) = 0$, $q_n(0) = q_n'(\pi) = 0$. therefore, we can write

$$p_{n+1}(t) = \int_0^1 K_0(t,s)[f(s,u) + g(s,u) - \chi(s,\alpha_n(s),\beta_n(s);\alpha_{n+1}(s))]ds,$$
$$q_{n+1}(t) = \int_0^1 K_0(t,s)[\omega(s,\alpha_n(s),\beta_n(s);\beta_{n+1}(s)) - f(s,u) - g(s,u)]ds.$$

Now using the Taylor series expansion with Lagrange remainder, the mean value theorem together with (A_2) of the hypothesis and the properties on F and G, we obtain

$$
\begin{aligned}
0 &\le p_{n+1}(t) \\
&= \int_0^1 K_0(t,s)\Big\{ f(s,u)+g(s,u)-[f(t,\alpha_n)+g(t,\alpha_n) \\
&\quad + \sum_{i=1}^{2} \frac{F^{(i)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n)^{(i)}}{i!} - [\phi(t,\alpha_{n+1})-\phi(t,\alpha_n)] \\
&\quad + G^{(1)}(t,\alpha_n)(\alpha_{n+1}-\alpha_n) + \frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!} \\
&\quad - [\psi(t,\alpha_{n+1})-\psi(t,\alpha_n)]]\Big\} ds \\
&= \int_0^1 K_0(t,s)\Big\{ f(s,u)+g(s,u)-[f(t,\alpha_{n+1})+g(t,\alpha_{n+1}) \\
&\quad - \frac{F^{(3)}(t,\xi_1)(\alpha_{n+1}-\alpha_n)^{(3)}}{3!} - \frac{G^{(2)}(t,\xi_2)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!} \\
&\quad + \frac{G^{(2)}(t,\beta_n)(\alpha_{n+1}-\alpha_n)^{(2)}}{2!}]\Big\} ds \\
&\le \int_0^1 K_0(t,s)\Big\{ f_u(s,\eta_1)(u-\alpha_{n+1}) + g_u(s,\eta_2)(u-\alpha_{n+1}) \\
&\quad + \frac{F^{(3)}(t,\xi_1)(u-\alpha_n)^{(3)}}{3!} - \frac{G^{(3)}(t,\eta_3)(\beta_n-\xi_2)(u-\alpha_n)^{(2)}}{2!}\Big\} ds \\
&= \int_0^1 K_0(t,s)\Big\{ [f_u(s,\eta_1)+g_u(s,\eta_2)]p_{n+1} + \frac{F^{(3)}(t,\xi_1){p_n}^{(3)}}{3!} \\
&\quad - \frac{G^{(3)}(t,\eta_3)(q_n+p_n){p_n}^{(2)}}{2!}\Big\} ds,
\end{aligned}
$$

where $\alpha_n \le \xi_1,\, \xi_2 \le \alpha_{n+1} \le \eta_1,\, \eta_2 \le u$ and $\xi_2 \le \eta_3 \le \beta_n$. It fallows by ($A_2$) that there exists $\lambda < 0$ and an integer N such that $f_u(s,\eta_1)+g_u(s,\eta_2) < \lambda$, $t \in [0,\pi]$ for $n \ge N$. Therefore, the error p_{n+1} satisfies the BVP

$$
\begin{aligned}
-p''_{n+1}(t) - \lambda p_{n+1}(t) &= f(t,u)+g(t,u)-\chi(t,\alpha_n(t),\beta_n(t);\alpha_{n+1}(t)) - \lambda p_{n+1}(t), \\
p_{n+1}(0) &= p'_{n+1}(\pi) = 0.
\end{aligned}
$$

This means that

$$
\begin{aligned}
p_{n+1}(t) \le \int_0^1 K_\lambda(t,s)\Big\{ &[f_u(s,\eta_1)+g_u(s,\eta_2)-\lambda]p_{n+1} + \frac{F^{(3)}(t,\xi_1){p_n}^{(3)}}{3!} \\
&- \frac{G^{(3)}(t,\eta_3)(q_n+p_n){p_n}^{(2)}}{2!}\Big\} ds.
\end{aligned}
$$

Let $| K_\lambda(t,s) | \leq A_1$, $| f_u(s,u) + g_u(s,\nu) - \lambda | \leq A_2$, $| F^{(3)}(t,u)/3! | \leq A_3$ and $| G^{(3)}(t,u)/2! | \leq A_4$. Then we have

$$\| p_{n+1} \| \leq k_1 \| p_n \|^3 + k_2 \| p_n \|^2 (\| q_n \| + \| p_n \|), \tag{3.29}$$

where $k_1 = A_1 A_3/(1 - A_1 A_2)$ and $k_2 = A_1 A_4/(1 - A_1 A_2)$.

Similarly, we can show

$$\| q_{n+1} \| \leq k_1 \| q_n \|^3 + k_2 \| q_n \|^2 (\| q_n \| + \| p_n \|), \tag{3.30}$$

where $k_1 = A_1 A_3/(1 - A_1 A_2)$ and $k_2 = A_1 A_4/(1 - A_1 A_2)$.

Hence combining (3.29) and (3.30) we obtain

$$\| p_{n+1} \| + \| q_{n+1} \| \leq C \, [\| p_n \| + \| q_n \|]^3$$

where C is an appropriate positive constant. This completes the proof. □

We note that the unique solution we have obtained is the unique solution of (3.1) in the sector determined by the lower and upper solutions.

Similar results can be obtained for the other coupled upper and lower solutions of (3.1) which are given by

$$\begin{aligned}
-\alpha_0'' &\leq f(t,\beta_0) + g(t,\alpha_0), \quad \alpha_0(0) \leq 0, \quad \alpha_0(\pi) \leq 0,\\
-\beta_0'' &\geq f(t,\alpha_0) + g(t,\beta_0), \quad \beta_0(0) \geq 0, \quad \beta_0(\pi) \geq 0,\\
-\alpha_0'' &\leq f(t,\alpha_0) + g(t,\beta_0), \quad \alpha_0(0) \leq 0, \quad \alpha_0(\pi) \leq 0,\\
-\beta_0'' &\geq f(t,\beta_0) + g(t,\alpha_0), \quad \beta_0(0) \geq 0, \quad \beta_0(\pi) \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
-\alpha_0'' &\leq f(t,\beta_0) + g(t,\beta_0), \quad \alpha_0(0) \leq 0, \quad \alpha_0(\pi) \leq 0,\\
-\beta_0'' &\geq f(t,\alpha_0) + g(t,\alpha_0), \quad \beta_0(0) \geq 0, \quad \beta_0(\pi) \geq 0.
\end{aligned}$$

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Ramzi S. N. Alsaedi
Department of Mathematics
King Abdul Aziz University
Jeddah P.O.Box 80203, Saudia Arabia.
e-mail : `ramzialsaedi@yahoo.co.uk`