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A Simple Proof of the Brouwer Fixed Point Theorem

S. Dhompongsa[†,1] **and J. Nantadilok**[‡]

[†]Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : sompong.d@cmu.ac.th

[‡]Department of Mathematics, Faculty of Science
Lampang Rajabhat University, Lampang, Thailand
e-mail : jamnian52@lpru.ac.th

Abstract : We give a new proof of the Brouwer fixed point theorem which is more elementary than all known ones. The only tool we use is the Tietze (continuous) extension theorem. The idea of the proof suggests some successful computation of a fixed point.

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1 Introduction

Brouwer Fixed Point Theorem: For the unit cube $[0,1]^d$ of the Euclidean space $\mathbb{R}^d$, let $T : [0,1]^d \to [0,1]^d$ be a continuous function. Then T has a fixed point, i.e., a point $x \in [0,1]^d$ with $T(x) = x$.

The proof is by induction on the dimension d and its idea of the proof can be extended from the one of $d = 2$.

For a complete survey on the development of the Brouwer fixed point theorem, we refer the reader to Sehie Park [1].

[1]Corresponding author.

2 Preliminaries

The only tool we use in the proof in section 3 is the following theorem:
Tietze Extension Theorem: For a closed subset K of the Euclidean space $\mathbb{R}^d$, let $T : K \to \mathbb{R}$ be a continuous function. Then T has a continuous extension over $\mathbb{R}^d$, i.e., a continuous function $\bar{T} : \mathbb{R}^d \to \mathbb{R}$ such that the restriction of $\bar{T}$ over K is T.

3 Proof of Main Result

Let K be a nonempty compact convex subset of the Euclidean space $\mathbb{R}^d$. Any continuous function $T : K \to K$ has a fixed point. We prove the Theorem for K of the form $K = [0,1]^d$. Let us recall some familiar notations. For $j = 1, \ldots, d$, write $e_j = (\delta_{ji})_{i=1}^d$ where the Kronecker delta δ_{ji} is defined to be 1 or 0 according to $j = i$ or $j \neq i$. Thus $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{R}^d$. For $j = 1, \ldots, d$, let $\square_j = \left\{\sum_{i=1.i\neq j}^d x_i e_i : 0 \leq x_i \leq 1, i = 1, \ldots, d, i \neq j\right\}$ and $\square_j^+ = \square_j + e_j$. Set $\bar{O} = (0, \ldots, 0), \bar{1} = (1, \ldots, 1) \in \mathbb{R}^d$, and for $0 \leq u \leq \sqrt{d}$, let H_u be the hyperplane passing through $(u, \ldots, u)$ and having $\bar{1}$ as its normal vector and put $\Delta u = K \cap H_u$.

Note that Δu, for $u \leq 1/\sqrt{d}$, is a simplex $co(ae_i : i = 1, \ldots, d)$ for some $a \in [0,1]$. Every point in $[0,1]^d$ lies in a simplex $co(ae_i : i = 1, \ldots, d)$ for some $a \in [0,d]$.

We will consider the projection along e_j defined by

$$\pi_j : (x_1, \ldots, x_d) \mapsto \sum_{i=1, i\neq j}^{d} x_i e_i\ , \quad \text{for } (x_1, \ldots, x_d) \text{ in } K.$$

Set $\square_{uj} = \pi_j(\Delta_u), \text{◩}_{uj} = \square_j \backslash \square_{uj}$, and $\text{◪}_{uj} = \text{◩}_{uj}^+ = \text{◩}_{uj} + e_j$. Above the face $\square_j$, let S_{uj} be the continuous surface consisting of Δu together with ◩_{uj} or ◪_{uj}. For example, Figure 1 shows S_{u1} and S_{v1} in $\mathbb{R}^2$.

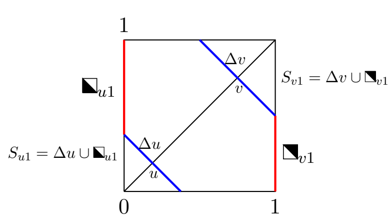

Figure 1:

Note that $\pi_j : S_{uj} \to \square_{uj}$ is a bijection.

Write $T = (f_1, \ldots, f_d)$ where $f_j : K \to [0,1]$ is continuous for each j. For each

u, draw the graph of f_j restricted to S_{uj} via the formula

$$g_{uj} : (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_d) \mapsto (x_1, \dots, x_{j-1}, f_j(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d), x_{j+1}, \dots, x_d)$$

for each $(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d)$ in S_{uj}.

Observe that $(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_d) = \pi_j(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d)$. Thus the graph of f_j at u means the set of points

$$(x_1, \dots, x_{j-1}, f_j(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d), x_{j+1}, \dots, x_d)$$

for $(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d)$ in S_{uj}. Figure 2 demonstrates the graphs of f_1 and f_2 for a given u.

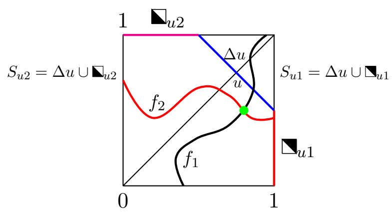

Figure 2:

The first result is fundamental whose proof based on the Brouwer fixed point theorem for $[0, 1]^{d-1}$.

Lemma 3.1. *For each u, the graphs of $f_1, \dots, f_d$ intersect at a point.*

Proof. Consider the function

$$S := \pi_1 g_{ud} \pi_d \cdots g_{u3} \pi_3 g_{u2} \pi_2 g_{u1} : \square_1 \to \square_1.$$

By the Brouwer fixed point theorem, there is a point $v \in \square_1$ such that $v = S(v)$. Clearly, the point $g_1(v)$ is a desired point of intersection. $\square$

A point of intersection in the proof can be found by successive projections on the graph of $f_1, \dots, f_d$. It is the limit of each convergent subsequence.

In the sequence, we will refer to "a point of intersection of the graphs of" $f_1, \dots, f_d$ shortly as "a point of intersection of" $f_1, \dots, f_d$.

Remark 3.2. *Note from Lemma 3.1 that*

(1) a point $w = (w_1, \dots, w_d)$ is a point of intersection if and only if there are points $w^{(j)} = (w_1^j, \dots, w_d^j)$ in S_{uj} for $j = 1, \dots, d$ such that $w_i^j = w_i$ for $i \neq j$ and $f_j(w^{(j)}) = w_j$. Thus,

(2) if w lies on Δu , then $T(w) = w$, i.e., w is a fixed point of T.

(3) *Suppose for each* $u_n \in (0,1), w^n = (w_{n1}, \dots, w_{nd})$ *is a point of intersection and points* $w^{n(j)} = (w^j_{n1}, \dots, w^j_{nd})$ *are being in* S_{u_nj} *such that, for each* $n, w^{n(j)}_i = w_{ni}$ *for* $i \neq j$ *and* $f_j(w^{n(j)}) = w_{nj}$ *for each* $j = 1, \dots, d$. *Also suppose that* $u_n \to u, w^n \to w = (w_1, \dots, w_d), w^{n(j)} \to w^{(j)} = (w^j_1, \dots, w^j_d)$ *for each* $j = 1, \dots, d$. *Thus,* $S_{u_nj} \to S_{uj}$ *under the Hausdorff distance,* $w^{(j)} \in S_{uj}, w^j_i = w_i$ *for* $i \neq j$ *and* $f_j(w^{(j)}) = w_j$. *That is,* w *is a point of intersection of* $f_1, f_2, \dots, f_d$ *corresponding to* u.

(4) *In the proof of the main result to follow, we will consider a negative part and a positive part of the function* f_1 *over* S_{u1}, *for each* u. *They are defined respectively as*

$$\begin{aligned} N^0(f_1, u) &= \{(x_1, \dots, x_d) \in S_{u1} : f_1((x_1, \dots, x_d)) < x_1\}, \\ P^0(f_1, u) &= \{(x_1, \dots, x_d) \in S_{u1} : f_1((x_1, \dots, x_d)) > x_1\}. \end{aligned}$$

As subsets of the $d-1$ *dimensional Euclidean space* $\mathbb{R}^{d-1}$, *let partition* $N^0(f_1, u)$ *and* $P^0(f_1, u)$ *into (open) components, say,*

$$N^0(f_1, u) = \bigcup_\alpha N^0_\alpha(f_1, u) \ , \quad P^0(f_1, u) = \bigcup_\beta P^0_\beta(f_1, u).$$

Let $N(f_1, u)$ *and* $P(f_1, u)$ *be the closure of* $N^0(f_1, u)$ *and* $P^0(f_1, u)$ *respectively. Analogously, let* $N_\alpha(f_1, u)$ *and* $P_\beta(f_1, u)$ *be the closure of* $N^0_\alpha(f_1, u)$ *and* $P^0_\beta(f_1, u)$ *respectively. Put* $Z_u = S_{u1} \backslash (N(f_1, u) \cup P(f_1, u))$. *Thus,* $Z_u \subset \{(x_1, \dots, x_d) \in S_{u1} : f_1((x_1, \dots, x_d)) = x_1\}$. *For* $0 \leq u \leq \sqrt{d}$ *and* $\lambda \in \mathbb{R}$, *let* $f_{\lambda 1}(w_0) = f_{\lambda u1}(w_0) = \lambda g_{u1}(w_0) + (1-\lambda) w$ *for* $w = (x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) \in S_{u1}$ *where* $w_0 = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots,$ $x_d)$. *One may need to truncate the function* $f_{\lambda 1}$ *to lie within* K. *For each* α *and* β, *put* $N^\alpha_{\lambda u} = f_{\lambda u1} \cdot \chi_{N_\alpha(f_1,u)}$ *and* $P^\beta_{\lambda u} = f_{\lambda u1} \cdot \chi_{P_\beta(f_1,u)}$, *and* $O_u = f_{u1} . \chi_{Z_u}$.

By Lemma 3.1, for a given closed interval $[a, b]$ in $\mathbb{R}$, the functions

$$\begin{aligned} & N^\alpha_{\lambda u}, f_2, \dots, f_d, \quad \text{for some } \alpha, \\ \text{or} \quad & P^\beta_{\lambda u}, f_2, \dots, f_d, \quad \text{for some } \beta, \\ \text{or} \quad & O_u, f_2, \dots, f_d, \end{aligned} \tag{3.1}$$

intersect at a point, for all λ in $[a, b]$.

To see why (3.1) holds, we argue on the contrary that the functions $O_u, f_2, \dots,$ f_d, do not intersect and for each α and β, there exists respectively λ_α and λ_β such that neither $N^\alpha_{\lambda_\alpha u}, f_2, \dots, f_d$, nor $P^\beta_{\lambda_\beta u}, f_2, \dots, f_d$, intersect. We claim that the union, called h, of the functions $O_u, N^\alpha_{\lambda_\alpha u}, P^\beta_{\lambda_\beta u}$, for all α and β is continuous, and clearly it does not have a common point with the functions $f_2, \dots, f_d$ which contradicts Lemma 3.1. To verify the claim, let $\{w_n\}$ be a sequence in S_{u1} converging to w. Suppose that w belongs to $P_\beta(f_1, u)$ for some β (the proof for the case w belongs to $N_\alpha(f_1, u)$ for some α follows the same lines). Write $w = (x_1, \dots, x_d)$ and $w_n = (x_{n1}, \dots, x_{nd})$. If $f_1(w) > x_1$, then $f_1(w_n) > x_{n1}$ for all large n, i.e. $w_n \in P^0_\beta(f_1, u)$ for those n. Now for such $n, h(w_n) = f_{\lambda_\beta 1}(w_n) = \lambda_\beta g_{u1}((w_n)_0) + (1-\lambda_\beta)w_n \to \lambda_\beta P^\beta_{\lambda_\beta u}(w) + (1-\lambda_\beta)w = h(w)$ as desired. On the other hand, if $f_1(w) = x_1$, then $f_1(w_n) \to x_1$. From the estimate

$$\|h(w_n) - w_n\| \ \le\ M\|f_1(w_n) - w_n\| \to \|x_1 - x_1\| \ =\ 0,$$

and the fact that $w_{n1} \to x_1$, we get $h(w_n) \to x_1 = h(w)$ as desired. Here M is a bound for λ's. For the remaining case when $w \in Z, w_n$ belongs to a $(d-1)$-ball $B(w, n)$ for all large n which in turn $h(w_n) = w_{n1} \to x_1 = h(w)$.

We are now ready to prove the Brouwer Fixed Point Theorem:

Proof. If there is a sequence $\{w_n\}$ of points of intersection lying strictly below S_{u1}, i.e. in the direction of the first component, for all u, a convergent subsequence of $\{w_n\}$, by Remark 3.2 (3), must converge to a point of intersection which must lie in $S_{01} = \{\bar{O}\}$ and thus $\bar{O}$ is a fixed point of T.

So we suppose that for some u there holds for each given bounded interval $[a, b]$ (which we will assume in the sequent that it properly contains $[0, 1]$), there exists a β such that (3.1) holds for $P^\beta_{\lambda u}, f_2, \dots, f_d$ for all $\lambda \in [a, b]$. Now let

$$\begin{aligned} u_0 = \sup\Big\{ u \in [0, \sqrt{d}] &: \text{for each bounded interval } [a,b], \text{ there exists } \beta \\ &\text{such that (3.1) holds for } P^\beta_{\lambda u}, f_2, \dots, f_d \text{ for all } \lambda \in [a,b] \Big\} \\ := \sup A & \end{aligned}$$

We show that a point of intersection of $f_1, f_2, \dots, f_d$ lies in Δ_{u_0} and we are done. We suppose that

$$\text{all points of intersection of functions} f_1, f_2, \dots, f_d \ \text{ do not lie in } \ \Delta_{u_0}. \qquad (3.2)$$

First, we claim that $u_0 \in A$. To achieve this, we are given any bounded interval $[a, b]$ in $\mathbb{R}$ and any λ in $[a, b]$, and choose a sequence $\{u_n\}$ in A converging increasingly to u_0. Take a sequence $\{\beta_n\}$ described in A corresponding to $\{u_n\}$. Thus there exists, for each n, a point of intersection w_n of functions $P^{\beta_n}_{\lambda u_n}, f_2, \dots, f_d$. Assume without loss of generality that $w_n \to c_\lambda$.

By Remark 3.2 (3), c_λ is a point of intersection of $f_{\lambda 1}, f_2, \dots, f_d$. In particular, when $\lambda = 1$, there exists a point c_1 of intersection of $f_1, f_2, \dots, f_d$. We note from the uniform continuity of T that $P_{\beta_n}(f_1, u)$ converges to $P_{\beta_0}(f_1, u)$, for some β_0, under the Hausdorf distance. Moreover, under our assumption (3.2), c_λ is a point of intersection of $P^{\beta_0}_{\lambda u_0}, f_2, \dots, f_d$. This proof holds for all λ in $[a, b]$, and it ends the proof of the claim that $u_0 \in A$.

If $u_0 = \sqrt{d}$, then $\bar{1}$ is a fixed point of T as we reasoned for $\bar{O}$. So we consider the case $u_0 < \sqrt{d}$.

Case I [All $c_1 \in ◪_{u_0 1}$]: The proof in this case is simpler than the following case, so we omit the proof.

Case II [Some $c_1 \notin ◪_{u_0 1}$]. Choose $\varepsilon_0 > 0$ such that, under the subspace topology, $B(w, \varepsilon_0) \cap \Delta_{u_0} \subset P_{\beta_0}(f_1, u_0)$ for all points of intersection w of $P^{\beta_0}_{0u_0}, f_2, \ldots, f_d$. Otherwise, a sequence of such points w would converge to a point of intersection of $f_1, f_2, \ldots, f_d$ which lies in Δ_{u_0} contradicting to our assumption. Thus the following sets are nonempty for all small $\delta > 0$. For $\delta > 0$ to be chosen appropriately later, let $u_1 \in (u_0, u_0 + \delta)$.

Put

$$P_\delta = \{(0, x_2, \ldots, x_d) : (x_1, \ldots, x_d) \in S_{u1}, P^{\beta_1}_{1u_1}(x_1, \ldots, x_d) > x_1 + \delta\},$$

for $\delta > 0$, and let $cl(P_\delta)$ be its closure in $[0,1]^{d-1}$. Note that P_δ is open in $[0,1]^{d-1}$. For each $j = 2, \ldots, d$, let

$$Q^j_\delta = (\pi_j f_j)^{-1}(cl(P_{2\delta})) \quad \text{and} \quad R^j_\delta = [0,1]^{d-1} \setminus (\pi_j f_j)^{-1}(P_\delta).$$

Thus both Q^j_δ and R^j_δ are nonempty for small δ and they are disjoint compact sets in $[0,1]^{d-1}$. Hence the minimum distance between elements of the two sets is positive. Thus the union of two continuous real valued functions on Q^j_δ and R^j_δ is always continuous, and in turn it is extendable continuously on $[0,1]^{d-1}$. We use this fact to redefine $f_1, \ldots, f_d$. First put

$$h_j = f_j . \chi_{Q^j_\delta}.$$

If $\bar{O} \notin Q^1_\delta$, we then redefine only h_2 as

$$h_2 = f_2 . \chi_{\bar{Q}^2_\delta} + a_1 . \chi_{\{\bar{O}\}}$$

where $\bar{Q}^2_\delta = Q^2_\delta \bigcap ([0,1] \times [\eta, 1] \times [0,1]^{d-2})$ for some small η and a_1 is any fixed element chosen from Q^1_δ. Extend each h_j over $[0,1]^{d-1}$ continuously by Tietze extension theorem to obtain new set of $f_1, \ldots, f_n$. Note that new functions $f_1, \ldots, f_n$ remain unchanged on $Q^1_\delta, \ldots, Q^d_\delta$ and they do not intersect on $R^1_\delta, \ldots, R^d_\delta$.

Consequently, by Lemma 3.1, they intersect only on $P_{\beta_1}(f_1, u_1)$. Now if there are sequences $\{v_n\}, \{\delta_n\}$, and $\{\lambda_n\}$ with $\{v_n\}$ is strictly decreasing to u_0, $\{\delta_n\}$ is strictly decreasing to 0 and $\{\lambda_n\} \subset [a, b]$ such that $f_{\lambda_n 1}, f_2, \ldots, f_d$ do not intersect when f_1 is restricted to $Q^1_{\delta_n}$. This means that intersection occurs for f_1 restricted to $R^1_{\delta_n} \setminus Q^1_{\delta_n}$. But then under (3.2) and by uniform continuity of T, we obtain a contradiction. At this point we can find small $\delta > 0$ and $u_1 \in (u_0, u_0 + \delta)$ as planned so that (3.1) holds for $P^{\beta_1}_{\lambda u_1}, f_2, \ldots, f_d$ for all $\lambda \in [a, b]$.

The above argument shows that there is a number u_1 in A which is bigger than u_0 and this is not possible. Hence our claim that some point of intersection of $f_1, f_2, \ldots, f_d$ lies in Δ_{u_0} is justified, and therefore the proof is complete.

□

Remark 3.3. *It is clear from the proof that, in practice, we only need to consider the graphs of functions* $f_1, \ldots, f_d$ *restricted to each* Δ_u*. To find a fixed point, we move the vector* $\bar{u} = (u, \ldots, u) \in \mathbb{R}^d$ *along the vector* $\bar{1}$ *until* Δ_u *meets a point of intersection. That point is a fixed point of* T*. This method of finding a fixed point can be described as "catching a fish by a fishing net".*

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References

[1] S. Park, Ninety years of the Brouwer fixed point theorem, Vietnam J. Math. 27 (3) (1999) 187-222.

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