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Note on the Properties of Type Π_1 Subfactors induced from Non-Degenerate Commuting Square[1]

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Abstract : In his celebrated article [1], V. Jones introduced Index theory of subfactors , which is called Jones Index theory to his honor. In this article , he showed that to any type Π_1 subfactors, $A \subset B$ corresponds a number $[A : B]$, which is independent from the Hilbert space on which the above subfactors act upon. He proved that for values of $[A : B]$ less than 4 the values of index are given by the following set of numbers, ,$4\cos^2(\pi/n), n = 1, 2, \ldots$

For a given subfactors $B_1 \subset B_2$, Jones introduced a construction in term of extending the above inclusion into the tower of subfactors , $B_1 \subset B_2 \subset \ldots \subset B_k \subset ..B_\infty$, which is called Jones tower. One of the main tools to construct subfactors is called commuting square. A commuting square consists of four finite dimensional C^* algebras that satisfy the following geometry,

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dimensional C^* algebras that satisfy the following geometry,

$$\begin{array}{ll} B_{2,1} & \subset B_{2,2} \\ \bigcup & \bigcup \\ B_{1,1} & \subset B_{1,2} \end{array}$$

If the commuting square is non-degenerate and equipped with Markov trace then we can extend it vertically using Jones construction to get type Π_1 limiting algebras $B_1 \subset B_2$. Now using Jones construction on the subfactors $B_1 \subset B_2$, we get the tower $B_1 \subset B_2 \subset \ldots \subset B_k \subset ..B_\infty$. Next let us define the following algebras , $D_i = (B_i)' \bigcap B_\infty = (x \in B_\infty$, x commute with $B_i)$, i=1,2. then we say that the graph of the inclusion $B_1 \subset B_2$ is Ergodic if the algebras D_1 and D_2 are factors , i.e, have trivial centers. We say that the inclusion $B_1 \subset B_2$ is strongly amenable if the exist a von Neumann algebra isomorphism taking the subfactors $B_1 \subset B_2$ onto the subfactors $D_2 \subset D_1$. Also usually we represent D_1 by M^{st} and D_2 by N^{st}. Note that in any case we always have, $[D_2 : D_1]$=$[B_1 : B_2]$. Now keeping the same notations as in the above , given a non-degenerate commuting square, we are going to show that the corresponding induced Π_1 subfactors $B_2 \supset B_1$, have Ergodic principal and dual graphs. Furthermore, we will show that if the induced subfactors index fall within certain interval then their inclusion is either strongly amenable or the inclusion of corresponding derived subalgebras $D_2 \subset D_1$ is isomorphic to Jones subfactors. In the later case we show that the corresponding graphs to the higher relative commutants of the above inclusion, Γ_{B_1,B_2}, have norms equal to 2. This extends the results of U.Haagerup in[2], Scott Morrison and Noah Snyder in [3], stating that the only infinite depth principal graphs corresponding to subfactors with their indices located in the interval (4, 5) are A_∞ graphs.

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1 Introduction and Preleminaries

In [4] and [5], V.Jones used results from Index theory to find new polinomial invariant for Link and knots. Later on Edward Witten in [6] applied Jones polinomials to Quantum field theory and introduced polinomial invariant for three manifold. Further applications to physics and other branches of mathematics has been done . Since then the advances of Jones Index theory motivated the need to answer unknown questions regarding irreducible subfactors of finite index. For example finding the set $IRRH$ of values of index for hyperfinite irreducible subfactors , with index larger than 4 is still and open problem. In Corollary 4.5 [7] , S.Popa showed that $IRRH$ contains a gap between 4 to 4.026. In [8], the authors prove that $IRRH \supset [37.0037]$. escope of this field of research is immense and at

this article we try to answer few of the open problems.
One of the main tools to analyze and to construct a pair of irreducible type Π_1 subfactors is called commuting square. They were first introduced by S.Popa in [9]. In this article we are going to use commuting squares to show some important properties of the corresponding induced subfactors.
Suppose we are Given a following non-degenerate commuting square

$$\begin{array}{ll} B_{2,1} & \subset B_{2,2} \\ \bigcup & \bigcup \\ B_{1,1} & \subset B_{1,2} \end{array} \qquad (1)$$

For a given inclusion of finite dimensional C^* algebras $A \subset B$, let T_A^B, be the matrix representation of the above inclusion with $\|T_A^B\|$, the norm of the linear operator T_A^B . Then the fact that (1) is non-degenerate implies that matrices $T = T_{B_{1,1}}^{B_{2,1}}$, $S = T_{B_{1,1}}^{B_{1,2}}$, $G = T_{B_{1,2}}^{B_{2,2}}$, $L = T_{B_{2,1}}^{B_{2,2}}$, are indecomposable with, , $\|T\| = \|G\|$ and $\|S\| = \|L\|$. Throughout this article the unique normal faithful normalized trace on a type Π_1 factor M is represented by tr_M. In particular if M is a limiting algebra corresponding to periodic tower of finite dimensional algebras, $(B_i \subset B_{i+1})_{i=1}^{\infty}$ with $T_{B_1}^{B_2}$ indecomposable , then tr_M is the The Markov trace corresponding to the inclusion $B_1 \subset B_2$. For the definitions of commuting square , Markov trace , Jones tower ,. and other preliminaries see for instance [10] , [11] , [12],[13] and [14]. Now by the standard arguments in [11], we can extend (1), upward using the basic construction on the pair $B_{1,1} \subset B_{2,1}$, to get the following tower of commuting squares

$$\begin{array}{ll} B_1 & \subset B_2 \\ \bigcup & \bigcup \\ \vdots & \vdots \\ \bigcup & \bigcup \\ B_{k,1} & \subset B_{k,2} \\ \bigcup & \bigcup \\ \vdots & \vdots \\ \bigcup & \bigcup \\ B_{3,1} & \subset B_{3,2} \\ \bigcup & \bigcup \\ B_{2,1} & \subset B_{2,2} \\ \bigcup & \bigcup \\ B_{1,1} & \subset B_{1,2} \end{array} \qquad (2)$$

with $B_{3,2} = < B_{2,2} \, , e_{B_{2,2}} >$, $B_{3,1} = < B_{2,1} \, , e_{B_{2,2}} >$. Proceeding inductively, for any integer k larger that 3 , set, $B_{k,2} = < B_{k-1,2} \, , e_{B_{k-2,2}} >$, $B_{k,1} = < B_{k-1,1} \, , e_{B_{k-2,2}} >$, where $e_{B_{k-2,2}}$ is the projection corresponding to the action of $B_{k-2,2}$ on $L^2_{(B_{k-1,2},tr_{B_2})}$. Let us set new and more convenient names for the Jones projections in the above ,$f_1 = e_{B_{1,2}}, f_2 = e_{B_{2,2}}, \ldots, f_k = e_{B_{k,2}}$, then we have $B_{k,2} =< B_{k-1,2}, f_{k-2} >$ for each integer k that is equal or larger than 3. Also note that

B_1 and B_2 are the limiting algebras of the towers $\{B_{k,1}\}$ and $\{B_{k,2}\}$ respectively. Next we can extend the inclusion $B_1 \subset B_2$ right ward using the basic construction to get the tower,

$$B_1 \subset B_2 \subset B_3 =< B_2, e_{B_1} >\subset \ldots \subset< B_k, e_{B_{k-1}} >= B_{k+1} \subset \ldots \subset B^{\infty}$$

where B^{∞} is the limiting algebra of the above tower. Furthermore for each $n \geq 1$, we have the following tower of finite dimensional algebras.

$$B_{n,1} \subset B_{n,2} \subset B_{n,3} =< B_{n,2}, e_{B_1} >\subset \ldots \subset B_{n,k} =< B_{n-1,k}, e_{B_{k-2}} >\subset \ldots \subset B^{n}$$

with B^n the limiting algebra of the above tower, which is isomorphic to the Jones tower induced from the finite algebras inclusion $B_{n,1} \subset B_{n,2}$ by standard arguments. Now in order to facilitate our notations, we rename the above Jones projections as in the following ,

$$e_1 = e_{B_1}, e_2 = e_{B_2}, \ldots, e_k = e_{B_k}, \ldots$$

The above construction will provide us with the following Jones system of commuting squares, i.e, each of the horizontal and vertical towers are isomorphic to Jones tower. Therefore we get the following tower of commuting squares

$$\begin{array}{cccccccc}
B_1 \subset & B_2 & \subset & \ldots & \subset & B_k & \subset \ \ldots & B^{\infty} \\
\bigcup & \bigcup & & & & \bigcup & & \bigcup \\
\vdots & \vdots & & & & \vdots & & \vdots \\
\bigcup & \bigcup & & & & \bigcup & & \bigcup \\
B_{n,1} \subset & B_{n,2} & \subset & \ldots & \subset & B_{n,k} & \subset \ \ldots & B^{n} \\
\bigcup & \bigcup & & & & \bigcup & & \bigcup \\
\vdots & \vdots & & & & \vdots & & \vdots \\
\bigcup & \bigcup & & & & \bigcup & & \bigcup \\
B_{2,1} \subset & B_{2,2} & \subset & \ldots & \subset & B_{2,k} & \subset \ \ldots & B^{2} \\
\bigcup & \bigcup & & & & \bigcup & & \bigcup \\
B_{1,1} \subset & B_{1,2} & \subset & \ldots & \subset & B_{1,k} & \subset \ \ldots & B^{1}
\end{array} \qquad (3)$$

In particular B^1 is the limiting algebra of the tower, $B_{1,1} \subset B_{1,2} \subset \ldots B_{1,k} \subset \ldots \subset B^1$. Finally, we have , $B_{k,2} =< B_{k-1,2}, f_{k-2} >$ and $B_{n,k} =< B_{n,k-1}, e_{k-2} >$ In the process of writing this article we use perturbation technics frequently.These technics are mainly based on the results of E. Christensen [10] and A. Ocneanu [15]. In the last section of this work some open problems have been addressed, with partial solution provided. The techniques we use here are simple based on S.Popa's work.

2 On the Properties of Subfactors induced from Commuting Squares

Considering the diagram (3), we need to bring some facts from S. Popa [16] that we will need in this section. As we mentioned before let $B_1^{st} = B_1^{'} \bigcap B_{\infty}$

and $(B^1)^{st} = (B^1)^{'} \bigcap B_\infty$. S.Popa showed that B_1^{st} is a factor if and only if $(B^2)^{st} = (B^2)^{'} \bigcap B_\infty$is factor. In this case we say that the diagram(3) is Ergodic or the inclusion $B_1 \subset B_2$ is Ergodic.

Let IR be the set of all irreducible type Π_1 subfactors of finite index. Also by the results of M.Pimsner and S.Popa in [16] for the couple $M \supset N$, in IR, we have, $H(M : N) = ln(\lambda^{-1}(M, N)) = ln([M : N])$.In general subfactors inclusion $M \supset N$, of finite index , is called extremal if $H(M : N) = ln(\lambda^{-1}(M, N)) = ln([M : N])$.Let $IRP \subset IR$ be subset of IR consisting of subfactors that are limiting algebras of periodic tower of commuting squares. Then by Theorem.1[17], any inclusion $M \supset N$ that is in IRP is extremal. Suppose we are given a subfactors $M \supset N$ of finite index. let us define $M^{st} \supset N^{st}$, be the limiting algebras corresponding to the tower of higher relative commutants of the inclusion $M \supset N$. Then by Theorem.1[18] we have, $\lambda^{-1}(M^{st} : N^{st}) = [M : N]$. Furthermore, Lemma.1[18] implies that the tower of higher relative commutant corresponding to the inclusion $M \supset N$ is a system of commuting square. Thus M^{st} (respectively N^{st}) is a factor or has an infinite dimensional center. But since $\lambda^{-1}(M^{st} : N^{st}) < \infty$ this implies that M^{st} is factor if and only if N^{st} is factor. Keeping the same notations as in section-1, we proceed to state the main result of this section in Theorem 2.2 ,proving that in general the inclusions $B^1 \subset B^2$ and $B_1 \subset B_2$ have Ergodic principal and dual graphs.At this point note that for the extremal inclusion of subfactors the dual graph is Ergodic if and only if the principal graph is Ergodic. This fact allow us to just state that the inclusion graphs are Ergodic in the similar cases as in the above.This fact was first proved by the first author at [18]. Unfortunately this fact is not yet a common knowledge in the field. Suppose we are given an subfactor inclusion $N_0 = N \subset M$.Then it is well known (following result of S.Popa in [16]), there exist a projection a projection $e = e_0$ in N , such that N_1 its relative commutant inside N is a subfactor and e induces the expectation of N onto N_1. Furthermore, we have $[M : N] = [N : N_1] = 1/tr_M(e)$. continuing this process inductively, we will get an infinite sequence of projections $(e_k)_{k=1}^{\infty}$ and corresponding infinite sequence of subfactors $(N_k \subset N_{k+1})_{k=1}^{\infty}$, with e_k a projection in N_{k-1} that induces the expectation of N_k onto N_{k+1}. Furthermore N_{k+1} is equal to the relative commutant of e_k in N_k. The above decreasing sequence of subfactors is called a choice of tunnel. Let us define, $L_k = N_k^{'} \cap M, P_k = N_k^{'} \cap N, L = \bigcup_{k=1}^{\infty} L_k$ and $P = \bigcup_{k=1}^{\infty} P_k$. Following the notations in S.Popa's article [19], we have that M^{st} is equal to the von Neumann algebra generated by L (respectively, N^{st} is equal to the Von Neumann algebra generated by P).The sequence of projections $(e_k)_{k=1}^{\infty}$ acts on the inclusion $N^{st} \subset M^{st}$ in exactly the same way as it acted on the inclusion $N \subset M$ to produce the following tonnel $M^{st} \overset{e_0}{\supset} N^{st} \overset{e_1}{\supset} N_1^{st} \overset{e_2}{\supset} N_2^{st} \supset \ldots \supset N_k^{st} \supset \ldots$. In theorem 5.3.1[19], S.Popa showed that if the inclusions $N \subset M$ and $N^{st} \subset M^{st}$, are extremal, then the inclusion $N \subset M$ is strongly amenable. The following lemma is one of the consequences of the remark in Section-1[19]. It will express one of the important canonical properties of inclusions, $M \supset N$ of type Π_1 factors with ergodic graph.

Lemma 2.1. *Suppose $M \supset N$ is extremal inclusion of Π_1, factors with [M:N] $< \infty$ and Γ_N^M ergodic. Then $M^{st} \supset N^{st}$ is strongly amenable.*

Proof. Let $M \overset{e_0}{\supset} N \overset{e_1}{\supset} N_1 \overset{e_2}{\supset} N_2 \ldots \overset{e_k}{\supset} N_k \supset \ldots$ be a choice of a tunnel as in the above. The following the definition,

$$M^{st} =< \bigcup_{k=1}^{\infty} N_k' \cap M >, \; N^{st} =< \bigcup_{k=1}^{\infty} N_k' \cap N > .$$

Then we will be provided a choice of the tunnel $M^{st} \overset{e_0}{\supset} N^{st} \overset{e_1}{\supset} N_1^{st} \overset{e_2}{\supset} N_2^{st} \supset \ldots$. Now by Theorem (5.3.2)[15], it is enough to show that $(M^{st})^{st} = M^{st}$ and $(N^{st})^{st} = N^{st}$. But

$$(M^{st})^{st} = \; < \bigcup_{k=1}^{\infty} M^{st} \cap (N_k^{st})' > \; \subset M^{st}$$

On the other hand since $N_k^{st} \subset N_k$, we have ,

$$\bigcup_{k=1}^{\infty} M^{st} \cap (N_k^{st})' = \bigcup_{k=1}^{\infty} (< \bigcup_{i=1}^{\infty} N_i' \cap M > \cap (N_k^{st})') \supset \bigcup_{k=1}^{\infty} (< \bigcup_{i=1}^{\infty} N_i' \cap M > \cap N_k') =$$

$$< \bigcup_{i=1}^{\infty} N_i' \cap M > \cap (\bigcup_{k=1}^{\infty} N_k').$$

Thus

$$< \bigcup_{k=1}^{\infty} M^{st} \cap (N_k^{st})' > \supset << \bigcup_{i=1}^{\infty} N_i' \cap M > \cap (\bigcup_{k=1}^{\infty} N_k') > \supset M^{st}$$

hence $M^{st} = (M^{st})^{st}$ and similarly $N^{st} = (N^{st})^{st}$. □

As we mentioned earlier or by Corollary 4.5[19] , if $M^{st} \supset N^{st}$ is extremal then the inclusion $N \subset M$ is strongly amenable . But if the inclusion is not extremal then $N \subset M$ might not be strongly amenable. In particular Suppose $N \subset M$ is strongly amenable. Next if the relative commutant of N inside M is not trivial , let us reduce the inclusion by minimal projections in order to get subfactors with trivial relative commutant. Whether the inclusions of these induced subfactors are strongly amenable is a question that has not been yet answered and we will talk about it at the later sections. Set $Q_1 = B_1^{st}$ and $Q_2 = B_2^{st}$ respectively (set $Q^1 = (B^1)^{st}$ and $Q_2 = (B^2)^{st}$) . Note , that by theorem 2.2, the inclusion $B_1 \subset B_2$, respectively ($B^1 \subset B^2$) has Ergodic graph i.e, Q_1 and Q_2, respectively(Q^1 and Q^2) are factors. By Theorem 5.3.1[19] if inclusions of derived algebras are extremal then the inclusions of the limiting algebras $B_1 \subset B_2$ and $B^1 \subset B^2$ are strongly amenable. But generally this is not the case. We have proved that if M^{st}, N^{st} are factors then the inclusion $M^{st} \supset N^{st}$ is strongly amenable and we are done.

Hence we only have to consider the cases , where the inclusion $M^{st} \supset N^{st}$ is not extremal.
The following is well know example of the inclusion with ergodic graph and has been mentioned in [16] and [19]. Consider an infinite sequence $e_1, e_2, \ldots, e_n, \ldots$ of Jones projections and set $M =< e_1, e_2, \ldots, e_n, \ldots >$, $N =< e_2, e_3, \ldots, e_n, \ldots >$ then we get the algebras, $N_1 =< e_3, \ldots, e_n, \ldots >$, $N_2 =< e_4, \ldots, e_n, \ldots >$,...,$N_k =< e_{k+2}, \ldots >$,...
That induce the choice of a tunnel $M \overset{e_1}{\supset} N \overset{e_2}{\supset} N_1 \overset{e_3}{\supset} N_2 \ldots \supset N_k \overset{e_{k+2}}{\supset} N_{k+1} \supset \ldots$ on the pair $M \supset N$ and it is easy to see that in this case, $M^{st} = M$ and $N^{st} = N$, hence by Theorem(5.3.2)[19], the inclusion $M \supset N$ is strongly amenable.

For a given the pair $B_1 \subset B_2$ of finite subfactors using standard basic construction, we can construct increasing tower of finite subfactors $B_1 \subset B_2 \subset \ldots \subset B_k \subset \ldots \subset B_\infty$ of finite subfactors and corresponding infinite set of Jones projections, $e_1, e_2, ..., e_k$.... Now using downward construction we get another infinite set of Jones projections , $e_0, e_{-1}, ..., e_{-k}$... Finally the union of the above two sets of projections, i.e, the set (e_i) , i running from $-\infty$ to $+\infty$ is a Jones sequence of projections , hence every infinite tail of it will generates a type Π_1 factor. These facts are going to be used in proving Theorem 2.2. But before that we are going to state two well known properties of subfactors. We say that the inclusion of subfactors $A \subset B$ is irreducible if $(A)' \bigcap B$ is trivial. Note by [16] for a given subfactors $A \subset B$ of finite index the inclusion is extremal if and only if for any projection $p \in (A)' \bigcap B$, the index of reduced subfactors is given by $[A_p : B_p]$= $(tr_A(p))^2[A : B]$. In the process of proving the following theorem we will use our results in [17], that for sufactors that are limiting algebras corresponding to a non-degenerate commuting square, their inclusion is extremal. But to guarantee the limiting algebras inclusion to be irreducible there are some conditions that are expressed in [17] and [20].

Theorem 2.2. *Considering the the tower of commuting squares in diagram (3) , and let the subfactors $B^1 \subset B^2$ be corresponding limiting algebras. Then $(B^1)' \bigcap B_\infty$ and $(B^2)' \bigcap B_\infty$, are factors , hence the graph of the inclusion $B^1 \subset B^2$ is Ergodic.*

Proof. Let the sequence of Jones projections $(e_k)_{k=1}^{\infty}$ be as in the arguments in connection to diagram(3). Then using downward construction on the couple $B^1 \subset B^2$ we get the sequence of Jones projections $(e_k)_{-\infty}^{\infty}$ as in the above. In according to the results of S.Popa in [19], to complete the proof of theorem 2.2 , we only have to prove that $(B^1)' \bigcap B^\infty$ is a factor.Let Let p be a projection in the center $(B^1)' \bigcap B^\infty$. for a given small $\varepsilon > 0$, using the arguments in the Proposition 2.3[10] , we can choose , an integer n large enough depending on ε such that , $p_n = E_{B_n}(p)$, the expectation of p onto B^n be a positive operator with , $\|E_{B_n}(p) - p\|_2 < \varepsilon$. Also note that $p_n \epsilon (B^1)' \bigcap B^n$, hence using Ocneanu's compactness Theorem in [15], $p_n \epsilon B_{n,1}$ .Furthermore by Lemma 2.1[10], p_n is a positive operator, with $\|p_n\|$ close enough to identity. Now consider the projection

e= $e_n.e_{n-1} \ldots\ e_3$ and a positive operator p_n , Since p commutes with all the projections (e_i) for i larger or equal to 3 , we have , $tr(p.e) = tr(p)tr(e) = tr(p_n.e)$

By Proposition 3.1.5 [11] , there exists a positive central operator r_n in $B_{1,1}$ such that $p_n.e = r_n.e$. So $tr(p.e) = tr(r_n.e)$. Since r_n commutes with all set of projections (e_i) , for i larger or equal to 3 , we have $tr(r_n.e) = tr(r_n).tr(e)$. This implies that $tr(r_n) = tr(p)$. Note that by the arguments in the proof of Proposition 2.3[10] there exists a number $\beta > 1$, which is close enough to one such that $(\beta)p_n$ is larger than an spectral projection corresponding to p_n. This implies that $\|p_n e\|$ is close to identity. Hence $\|r_n\|$ is close to identity. Therefore without loss of generality we can assume that r_n dominate a central projection in $B_{1,1}$. This implies that the center of $(B^1)' \bigcap B^\infty$ is finite dimensional. Thus by the results of S.Popa in 1.4[19] ,$(B^1)' \bigcap B^\infty$ is a factor. □

At this point we finalize this section by discussing the nature of irreducible subfactors $B_1 \subset B_2$, that that are induced from a non-degenerate commuting squares. To faciliate notations let us set $Q_1 = B_1^{st}$ and $Q_2 = B_2^{st}$

Corollary 2.3. *Keeping the same notations as in the above ,suppose the system(3) is such that the inclusion $B_1 \subset B_2$ is irreducible and the subfactor inclusion $Q_1 \subset Q_2$ is extremal. Then the subfactor inclusion $B_1 \subset B_2$ is strongly amenable and hence the inclusion $Q_1 \subset Q_2$ is irreducible.*

Proof. The proof is the direct application of Theorem 5.3.1 [19]. □

Theorem 2.4. *Keeping the above notations , if [$B_1 : B_2$]< 5.8 , then the inclusion of $B_1 \subset B_2$ is either strongly amenable or the inclusion $Q_1 \subset Q_2$ is locally trivial. in later case case, $Q_1 \subset Q_2$ is isomorphic to the inclusion of Jones subfactors*

Proof. if the inclusion $Q_1 \subset Q_2$ is extremal then by Lemma 2.3 , inclusion $B_1 \subset B_2$ is strongly amenable and the relative commutant of Q_1 inside Q_2 is trivial. Otherwise the relative commutant $C_1 = (Q_1)' \bigcap Q_2$ is not trivial and we can use lemma(2.2.2)[12], to find the relation between traces of projection belong to any partition of unity in C_1 and the index , of Q_1 in Q_2 which is equal to $[B_1 : B_2]$. Hence by the arguments at the end of Corollary 4.6[16] , for $[B_1 : B_2]< 9$, C_1 can only have 2 orthogonal projection. But if C_1 is isomorphic to $M_2(C)$, then the inclusion of Q_1 in Q_2 is extremal and hence by Theorem 5.3.1[19] , C_1 is trivial and this is contradiction. So we can assume that,C_1 is isomorphic to sum of two copies of identity . that means C_1 is generated by two projections p and $q = 1 - p$. if $[(Q_1)_p : (Q_2)_p] = [(Q_1)_q : (Q_2)_q] = 2$,then in this case the minimum value of $[Q_1 : Q_2]$ can reach to 8. This implies that for $[Q_1 : Q_2]$ smaller than 8 , either the inclusion $Q_1 \subset Q_2$ is locally trivial , i.e, $[(Q_1)_p : (Q_2)_p] = [(Q_1)_q : (Q_2)_q] = 1$ or one of the indices is equal to 1 and the other must be equal to 2 , in the later case the minimum value of index can reach to approximately 5.8. So for index less than 5.8 , the inclusion of $Q_1 \subset Q_2$ is locally trivial. But by the results of S.Popa in Secton-5[16] , these kind of inclusions are isomorphic to Jones subfactors. This completes the proof.

□

In some cases it is very hard to examine from the structure of commuting square if the corresponding limiting algebras inclusion , $B_1 \subset B_2$ is irreducible. Given certain conditions the following lemma can help. But first let us consider the following set , $E = (8, (1 + 2cos(\pi/(n + 2))^2, n \geq 1)$.

Corollary 2.5. *Keeping the same notations as in the above. Let $Ind = [B_2 : B_1] \leq 9$, with Ind not a member of E. Then the inclusion $B_1 \subset B_2$ is irreducible.*

Proof. By results of theorem 1.3.2[19], in diagram(1), if $(\|T_{B_{1,1}}^{B_{1,2}}\|)^2 \leq 9$ and is not equal to any elements in the set $E = (8, (1 + 2cos(\pi/(n + 2))^2, n \geq 1)$, the fact that the inclusion $B_1 \subset B_2$ is extremal, implies that the inclusion $B_1 \subset B_2$ is irreducible. This completes the proof. □

In order to prove Theorem 2.8 , which is the conclusive result of this section we need to prove the following lemmas.

Lemma 2.6. *Following the above notations for each integer k , $(N_k^{st})^{'} \cap M^{st} \supset (N_k)^{'} \cap M$.*

Proof. As in Lemma 2.1, we have, $M^{st} =< \bigcup_{k=1}^{\infty} N_k' \cap M >$ and $N_k^{st} =< \bigcup_{l=1}^{\infty} N_l' \cap N_k >$.Note that $(N_l^{'} \cap N_k)^{'} \supset N_l \cup N_k^{'}$. This implies that , $(N_l^{'} \cap N_k)^{'} \cap M^{st} \supset (N_l \cup N_k^{'}) \cap M^{st}$. But $(N_l \cup N_k^{'}) \cap M^{st} = (N_l \cap M^{st}) \cup (N_k^{'} \cap M^{st})$ and $(N_k)^{'} \cap M^{st} = (N_k)^{'} \cap M$. Since this is true for each integer l , we have $(N_k^{st})^{'} \cap M^{st} \supset (N_k)^{'} \cap M$. □

Lemma 2.7. *The sequence of inclusions $(N_k^{st})^{'} \cap M^{st} \supset (N_k)^{'} \cap M, k = 1, 2, 3, ...,$ is a tower of commuting squares.*

Proof. By the definition for each integer k, e_k is an element of ,N_{k+1}^{st} and induces the expectation of N_k onto N_{k-1}, (respectively it induces the expectation of N_k^{st} onto N_{k-1}^{st}). This means that for each element y in N_k^{st} , $e_k y e_k = E_{N_{k-1}}(y) e_k = E_{N_{k-1}^{st}}(y) e_k$. Hence we have, $E_{N_{k-1}}(y) = E_{N_{k-1}^{st}}(y)$ □

Let Γ_{B_1,B_2} be as defined in Section-5[19].

Theorem 2.8. *(Classification) Keeping the same notations as earlier , suppose,$4 \leq Ind = [B_1 : B_2] \leq 5.8$. Suppose the inclusion $N = B_1 \subset M = B_2$ is not strongly amenable. Then $\|\Gamma_{N,M}\|^2 = 4$.*

Proof. By Lemma 2.7, the sequence $(N_k^{st})^{'} \cap M^{st} \supset (N_k)^{'} \cap M, k = 1, 2, 3, ...,$ is a tower of commuting squares..But by results in section-5[16] the inclusion $N^{st} \subset M^{st}$ is isomorphic to the Jones subfactors. This implies that $\Gamma_{N^{st},M^{st}} = A_\infty$. Next the properties of commuting squares resulted from the tower of higher relative commutants implies that $\|\Gamma_{N,M}\|^2 = 4$. □

Remark 2.9. *Since there are only finite numbers of connected bipartite graphs of norm equal two, namely $A_\infty, A_{n,\infty}, A_{\infty,\infty}, D_\infty, D_{n,\infty}, n = 1, 2, ...$ theorem(2.8) classifies subfactors of index less than 5.8 in term of their principal and dual graphs. This is also coherent with results of Scott Morrison and Noah Snyder in[3]. Keeping the above notations, at this point it is crucial to see if being an strongly amenable subfactor is a generic property. That means if the inclusion $B_1 \subset B_2$ is strongly amenable then the inclusion $B^1 \subset B^2$ is strongly amenable too. If the above statement is correct,then it generalizes the results of N.Sato in [21] , proving that if one of the inclusions of induced subfactors, $B_1 \subset B_2$, is of finite depth , then the inclusion of other induced subfactors $B^1 \subset B^2$,is of finite depth too. The other property to check is the following. Let $B_1 \subset B_2$ be strongly amenable and C_1 is a middle sunfactor the the inclusions $C_1 \subset B_2$ and $B_1 \subset B_2$ are strongly amenable too. We are going to show that the second property is in fact true.*

Theorem 2.10. *Suppose we are given a subfactors inclusion $B_1 \subset B_2$ of finite index , such that the inclusion is strongly amenable.Let C_1 be a middle subfactor. Then the inclusions $C \subset B_2$ and $B_1 \subset C$ are strongly amenable too.*

Proof. To prove the above theorem , we are going to use methods similar to the methods employed by D.Bisch in [22]. Now we use Jones basic construction on the couple $B_1 \subset B_2$ inductively to get the following tower.
$B_1 \subset C_1 \subset B_2 \subset C_2 \subset B_3 \subset C_3 \subset \ldots \subset B_{k-1} \subset C_{k-1} \subset B_k \subset C_k \subset B_{k+1} \subset C_{k+1} \subset \ldots$
Where the inductive process of construction can be formulated at the k_{th} stage where all the operators are acting on $L^2(B_k, tr_{B_k})$ as in the following, $B_{k+1} = J_{B_k} B^{'}_{k-1} J_{B_k}$ and $C_{k+1} = J_{B_k} C^{'}_{k-1} J_{B_k}$.
This implies that for each integer k, there exists a projection f_k in $C^{'}_k \cap C_{k+1}$ such that $(C_{k+1})_{f_k} = (C_k)_{f_k}$. Now consider the following projections. $f^*_{2k} = f_2.f_4 \ldots f_{2k}, f^+_{2k} = f_4.f_6 \ldots f_{2k}$ and the sequence $B_1 \subset B_3 \subset B_5 \subset \ldots \subset B_{2k+1}$. For any odd integer n which is less or equal than $2k + 1$, set $A_n = B_n.f^*_{2k}$, Then it is easy to show that the sequence $A_1 \subset A_3 \subset A_5 \subset .... \subset A_{2k+1}$ is isomorphic to the sequence $B_1 \subset C_1 \subset C^*_2 \subset C^*_3 \subset ...C^*_k \subset ...$ which is the result of iterating the basic construction on the couple $B_1 \subset C_1$.For an odd integer $n = 2k + 1$, let us denote $m = 2k$, furthermore consider the inclusion $B^{'}_1 \cap C^*_n \subset (C^*_2)^{'} \cap C^*_n$. By the above arguments this inclusion is isomorphic to the inclusion $(A_1)^{'} \cap A_n \subset (A_3)^{'} \cap A_n$. By the definition we have $f^*_m = f_2 f^+_m$. This implies the following equalities, $(A_1)^{'} \cap A_n = (B^{'}_1 \cap B_n)_{f_2 f^+_m} = ((B^{'}_1 \cap B_n)_{f_2})_{f^+_m}$ and $(A_3)^{'} \cap A_n = ((B^{'}_3 \cap B_n)_{f_2})_{f^+_m}$ Hence the inclusion $(A_1)^{'} \cap A_n \supset (A_3)^{'} \cap A_n$ is isomorphic to the inclusion $(B^{'}_1 \cap B_n)_{f_2} \supset (B^{'}_3 \cap B_n)_{f_2}$ Let us define D_1 to be the Von Neumann algebra generated by the algebras $B^{'}_1 \cap B_n, n = 1, 2, 3, ...$ and D_3 be the Von Neumann algebra generated by the algebras $B^{'}_3 \cap B_n, n = 1, 2, 3, ...$ By our assumption since the inclusion $B_1 \subset B_2$ is strongly amenable, for and projection q in the relative commutant of D_3 in D_1 , the inclusion $(D_3)_q \subset (D_1)_q$ is extremal. This implies that if an odd integer $n = 2k + 1$ is large enough then

the values , $ln[\lambda^{-1}((A_1)^{'} \cap A_n : (A_3)^{'} \cap A_n)]$, $H((A_1)^{'} \cap A_n : (A_3)^{'} \cap A_n)$ and $ln(\lambda^{-1}((D_3)_{f_2} : (D_1)_{f_2}))$ are close enough to each other, Hence Theorem(5.3.1)[16] implies That the inclusion $B_1 \subset C_1$ is strongly amenable. Similarly we can show that the inclusion $C_1 \subset B_2$ is strongly amenable. □

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