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Strong Convergence Theorems for General Equilibrium Problems and Fixed Point Problems in Banach Spaces[1]

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Abstract : In this paper, we introduce an iterative scheme for finding a common element of the set of solutions of the general equilibrium problem and the set of fixed point of nonexpansive mappings in Banach space. Under suitable conditions, some strong convergence theorem for approximating a common element of the above two sets are obtained. Results obtained in this paper improve the previously known results in this area.

Keywords : Banach spaces; fixed point; nonexpansive mapping.
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1 Introduction

Let E be a real Banach space and let E^* be the dual of E. Let C be a closed convex subset of E. We denote by J the normalized duality mapping from E to 2^{E^*}. defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

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where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We denote the strong convergence and the weak convergence of a sequence $\{x_n\} \to x$ in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. First, we recall that a mapping $A : C \to E^*$ is said to be:

1. monotone if $\langle Ax - Ay, x - y \rangle \geq 0, \ \ \forall x, y \in C$.
2. α-inverse-strongly monotone if there exists a positive real number α such that
$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. A point $x \in C$ is said to be a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Let $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ denotes the sets of real numbers. In this paper we consider the following generalized equilibrium problem of finding $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle \geq 0, \qquad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP, i.e.,

$$EP = \{u \in C : f(u, y) + \langle Au, y - u \rangle \geq 0, \qquad \forall y \in C\}.$$

When $E = H$ is a Hilbert space, problem (1.1) was introduced and studied by Takahashi and Takahashi [1]. We remark that problem (1.1) and related problems were extensively studied recently. See, e.g., [2–31].

In the case of $A \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that $f(u, y) \geq 0, \ \ \forall y \in C$, which is called equilibrium problem. The set of its solutions is denoted by $EP(f)$. In the case of $f \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that $\langle Au, y - u \rangle \geq 0, \ \ \forall y \in C$, which is called the variational inequality of Browder type. The set of its solutions is denoted by $VI(A, C)$.

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [32] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{1.2}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (1.2) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.3}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.

(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,
$$\phi(x,y) = 0 \text{ if and only if } x = y.$$

Recently, Yao, Liou and Shahzad [33] introduced the following iterative scheme. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iterative by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C[(1 - \beta_n)Tx_n], n \geq 0.$$

Where $Q_C : E \to C$ is sunny nonexpansive retraction and $T : C \to C$ is nonexpansive mapping. They proved strong convergence theorem for the iterative algorithm under some mild conditions.

Very recently, Cai and Bu [34] introduced the new iterative algorithm (1.4) for finding a common element of the set of solutions of the general equilibrium problem and the set of solutions of the variational inequality for an inverse-strongly monotone operator and the set of common fixed points of two infinite families of relatively nonexpansive mappings or the set of common fixed points of an infinite family of relatively quasi-nonexpansive mappings in Banach spaces. More precisely, they proved that the sequence $\{x_n\}$ generated by $u_1 \in C$,

$$\begin{cases} x_n \in C \text{ such that } f(x_n, y) + \langle Bx_n, y - x_n\rangle + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \forall y \in C, \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + \beta_n JT_n z_n + \gamma_n JS_n z_n), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $A : C \to E^*$ is α-inverse strongly monotone operator and $B : C \to E^*$ is β-inverse strongly monotone operator. They proved some weak convergence theorems for the iterative algorithm under some mild conditions.

In this paper, motivated and inspired by Yao, Liou and Shahzad [33], Cai and Bu [34], we prove strong convergence theorem for finding a common element of the set of solutions of the general equilibrium problem and the set of fixed point of nonexpansive mappings in Banach space. Our results extend and improve the corresponding results of Yao, Liou and Shahzad [33].

2 Preliminaries

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n -$

$y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [36–39].

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(P)$, where $R(P)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.1 ([39]). *(Demiclosedness Principle) Let C be a nonempty closed convex subset of a uniformly convex Banach E. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then T is demiclosed on C, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to y$ strongly, then $(I - T)x = y$.*

Lemma 2.2 ([40]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. [41] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \qquad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y)$;

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.4 ([21])**.** *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping, let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then there hold the following*

(1) For $x \in E$, there exists $u \in C$ such that

$$f(u, y) + \langle Au, y-u\rangle + \frac{1}{r}\langle y-u, Ju - Jx\rangle \ge 0, \forall y \in C.$$

(2) If E is additionally uniformly smooth and $K_r : E \to C$ is defined as

$$K_r(x) = \left\{u \in C : f(u, y) + \langle Au, y-u\rangle + \frac{1}{r}\langle y-u, Ju - Jx\rangle \ge 0, \forall y \in C\right\}, \\ \forall y \in E, \tag{2.1}$$

then the mapping K_r has the following properties:

(i) K_r is single-valued;

(ii) K_r is a firmly nonexpansive-type mapping, i.e.,

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle, \forall x, y \in E;$$

(iii) $F(K_r) = EP$;

(iv) EP is closed convex subset of C;

(v) $\phi(p, K_r x) + \phi(K_r x, x) \le \phi(p, x), \qquad \forall p \in F(K_r)$.

Lemma 2.5 ([42])**.** *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.6 ([42])**.** *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \le \phi(x, y)$ for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \le r\}$.*

Lemma 2.7 ([43])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|tx - (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r(0) := \{x \in E : \|x\| \le r\}$ and for any $t \in [0, 1]$.

Let E be a real uniformly convex and uniformly smooth Banach. Let C be a nonempty closed convex and sunny nonexpansive retract of E with $Q_C : E \to C$ as the sunny nonexpansive retraction. Let $T : C \to C$ be a nonexpansive mapping. Given a real number $t \in (0,1)$. Define a mapping $T_t : C \to C$ by

$$T_t x = Q_C[(1-t)Tx], x \in C.$$

It is easy to see that T_t is a contraction on C. Let $x, y \in C$, we have

$$\begin{aligned}\|T_t x - T_t y\| &= \|Q_C[(1-t)Tx] - Q_C[(1-t)Ty]\| \\ &\leq (1-t)\|Tx - Ty\| \\ &\leq (1-t)\|x - y\|.\end{aligned}$$

Let $x_t \in C$ be the unique fixed point of T_t, that is, x_t satisfies the following fixed point equation

$$x_t = Q_C[(1-t)Tx_t], \quad t \in (0,1). \tag{2.2}$$

Lemma 2.8 ([33]). *Suppose that $Fix(T) \neq \emptyset$. For $t \in (0,1)$, let the net $\{x_t\}$ be defined by (2.2). Then as $t \to 0+$, the net $\{x_t\}$ converges strongly to $x \in Fix(T)$.*

3 Main Results

In this section, we will introduce our methods and prove the strong convergence theorem.

Theorem 3.1. *Let E be a real uniformly convex and uniformly smooth Banach. Let C be a nonempty closed convex subset and sunny nonexpansive retract of E with $Q_C : E \to C$ as the sunny nonexpansive retraction. Let $T : C \to C$ be a nonexpansive mapping. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) with $F := Fix(T) \cap EP \neq \emptyset$. Let $B : C \to E^*$ be a β-inverse strongly monotone operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$,*

$$\begin{cases} u_n \in C \text{ such that } f(u_n, y) + \langle Bu_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0, \forall y \in C \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n], \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

Assume that the following conditions are satisfied

1. $\lim_{n\to\infty} \beta_n = 0$;
2. $\sum_{n=0}^{\infty} \beta_n = \infty$;
3. $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

where J is the normalized duality mapping on E. Then $\{x_n\}$ converges strongly to a fixed point $z \in F$.

Proof. First we show that $\{x_n\}$ is bounded. Take a point $p \in Fix(T)$ and notice that $u_n = K_{r_n}x_n$

$$\|u_n - p\| = \|K_{r_n}x_n - p\| \leq \|x_n - p\|$$

and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(x_n - p) + (1-\alpha_n)(Q_C[(1-\beta_n)Tu_n] - p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|Q_C[(1-\beta_n)Tu_n] - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|(1-\beta_n)Tu_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)[(1-\beta_n)\|u_n - p\| + \beta_n\|p\|] \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)[(1-\beta_n)\|x_n - p\| + \beta_n\|p\|] \\
&= [1 - (1-\alpha_n)\beta_n]\|x_n - p\| + (1-\alpha_n)\beta_n\|p\|.
\end{aligned}$$

By induction,

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}, \;\; n \geq 0,$$

and $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{Tx_n\}$. Next, we show that

$$\|x_{n+1} - x_n\| \to 0.$$

We can rewritten (3.1) as $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$ where $y_n = Q_C[(1-\beta_n)Tu_n]$ for all $n \geq 0$. It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|Q_C[(1-\beta_{n+1})Tu_{n+1}] - Q_C[(1-\beta_n)Tu_n]\| \\
&\leq \|(1-\beta_{n+1})Tu_{n+1} - (1-\beta_n)Tu_n\| \\
&\leq (1-\beta_n)\|Tu_{n+1} - Tu_n\| + |\beta_{n+1} - \beta_n|\|Tu_n\| \\
&\leq (1-\beta_n)\|u_{n+1} - u_n\| + (\beta_n - \beta_{n+1})\|Tu_n\|,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|u_{n+1} - u_n\|) \leq 0.$$

This together with Lemma 2.2 imply that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.2}$$

By the convexity of $\|\cdot\|^2$ and Lemma 2.4, we obtain

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]) \\
&= \|p\|^2 - 2\langle p, \alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]\rangle \\
&\quad + \|\alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]\|^2 \\
&= \|p\|^2 - 2\langle p, x_n\rangle - 2(1-\alpha_n)\langle p, Q_C[(1-\beta_n)Tu_n]\rangle \\
&\quad\; + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Q_C[(1-\beta_n)Tu_n]\|^2
\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, Q_C[(1-\beta_n)Tu_n]) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, (1-\beta_n)Tu_n) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)(1-\beta_n)\phi(p, Tu_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, Tu_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, u_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, K_{r_n}x_n) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, x_n) \\
&\leq \phi(p,x_n). \qquad (3.3)
\end{aligned}$$

This implies that $\lim_{n\to\infty}\phi(u,x_n)$ exists. It follows that $\{\phi(u,x_n)\}$ is bounded. Let $r_1 = \sup_{n\geq 1}\{\|x_n\|, \|y_n\|\}$. From Lemma 2.7, we have

$$\begin{aligned}
\phi(p,x_{n+1}) &= \phi(p, \alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]) \\
&= \|p\|^2 - 2\langle p, \alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]\rangle \\
&\quad + \|\alpha_n x_n + (1-\alpha_n)Q_C[(1-\beta_n)Tu_n]\|^2 \\
&= \|p\|^2 - 2\langle p, x_n\rangle - 2(1-\alpha_n)\langle p, Q_C[(1-\beta_n)Tu_n]\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Q_C[(1-\beta_n)Tu_n]\|^2 - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, Q_C[(1-\beta_n)Tu_n]) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, (1-\beta_n)Tu_n - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|)) \\
&= \alpha_n\phi(p,x_n) + (1-\alpha_n)(1-\beta_n)\phi(p, Tu_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, Tu_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, u_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, K_{r_n}x_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&\leq \alpha_n\phi(p,x_n) + (1-\alpha_n)\phi(p, x_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \\
&\leq \phi(p,x_n) - \alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|). \qquad (3.4)
\end{aligned}$$

Which implies that

$$\alpha_n(1-\alpha_n)g_1(\|x_n-y_n\|) \leq \phi(p,x_n) - \phi(p,x_{n+1}). \qquad (3.5)$$

Noticing condition (iii), by taking the limits in (3.4), we get

$$\lim_{n\to\infty} g_1(\|x_n-y_n\|) = 0. \qquad (3.6)$$

From the property of g_1, we have

$$\lim_{n\to\infty} \|x_n-y_n\| = 0. \qquad (3.7)$$

Hence,

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0.$$

Note that

$$\begin{aligned}\|x_n - Tu_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - Tu_n\| \\ &= \|x_{n+1} - x_n\| + \|\alpha_n x_n + Q_C[(1-\beta_n)Tu_n] - Q_C[Tu_n]\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - Tu_n\| + \beta_n\|Tu_n\|, \end{aligned} \tag{3.8}$$

that is,

$$\|x_n - Tu_n\| \le \frac{1}{1-\alpha_n}\{\|x_{n+1} - x_n\| + \beta_n\|Tu_n\|\}.$$

Therefore,

$$\lim_{n\to\infty} \|x_n - Tu_n\| = 0. \tag{3.9}$$

Combining (3.2),(3.7) and (3.9), we have

$$\|u_n - Tu_n\| \le \|u_n - y_n\| + \|y_n - x_n\| + \|x_n - Tu_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Since $\{u_n\}$ is bounded, we obtain that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that u_{n_i} converges weakly to $x^* \in C$. From (3.10) and Lemma 2.1, we have $x^* \in Fix(T)$.

Next we show that $x^* \in EP$. Let $r_2 = \sup_{n\ge1}\{\|x_n\|, \|u_n\|\}$. From Lemma 2.6, there exists a continuous strictly increasing and convex function g_2 with $g_2(0) = 0$ such that $g_2(\|x - y\|) \le \phi(x, y)$, $\forall x, y \in B_{r_2}(0)$. Noticing $x_n = K_{r_n}u_n$ and from Lemma 2.4 and (3.3), for $p \in F$ we have

$$g_2(\|x_n - u_n\|) \le \phi(x_n, u_n) - \phi(p, x_n) \le \phi(u, x_{n-1}) - \phi(u, x_n).$$

Since $\lim_{n\to\infty} \phi(u, x_n)$ exists, we obtain $\lim_{n\to\infty} g_2(\|x_n - u_n\|) = 0$. If follows from the property of g_2 that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

Since J is uniformly norm-to-norm continuous on bounded sets of E, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.12}$$

From condition (iii), we have

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.13}$$

By the definition of $x_n = K_{r_n}u_n$, we have

$$F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \ge, \quad \forall y \in C, \tag{3.14}$$

where $F(x_n, y) = f(x_n, y) + \langle Bx_n, y - x_n\rangle$. Replacing n by n_i, we have from (A2) that

$$\frac{1}{r_{n_i}}\langle y - x_{n_i}, Jx_{n_i} - Ju_{n_i}\rangle \geq -F(x_{n_i}, y) \geq F(y, x_{n_i}), \qquad \forall y \in C. \tag{3.15}$$

From (3.11) and $u_{n_i} \rightharpoonup x^*$, we have $x_{n_i} \rightharpoonup x^*$. Since $y \mapsto f(x, y) + \langle Bx, y - x\rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $i \to \infty$ in (3.15), from (3.13) and (A4) we have

$$F(y, x^*) \leq 0, \qquad \forall y \in C.$$

For t, with $0 < t < 1$, and $y \in C$, let $y_t = ty + (1 - t)x^*$. Since $y \in C$ and $x^* \in C$ then $y_t \in C$ and hence $F(y_t, x^*) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1 - t)F(y_t, x^*) \leq tF(y_t, y).$$

Dividing by t, we have

$$F(y_t, y) \geq 0, \qquad \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(x^*, y) \geq 0, \qquad \forall y \in C.$$

And hence

$$f(x^*, y) + \langle Bx^*, y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

So $x^* \in EP$

We next show that

$$\limsup_{n\to\infty}\langle x^*, j(x^* - Tu_n)\rangle = \limsup_{n\to\infty}\langle x^*, j(x^* - u_n)\rangle = \limsup_{n\to\infty}\langle x^*, j(x^* - x_n)\rangle \leq 0,$$

where $x^* = \lim_{t\to 0+} x_t$ and x_t is the defined by (2.2). Nothing that $x_t = Q_C[(1 - t)Tx_t]$ and $x_n \in C$, we have

$$\langle x_t - (1 - t)Tx_t, j(x_t - x_n)\rangle \leq 0.$$

Hence,

$$\begin{aligned}
\|x_t - x_n\|^2 &= \langle x_t - x_n, j(x_t - x_n)\rangle \\
&= \langle x_t - (1 - t)Tx_t, j(x_t - x_n)\rangle + \langle (1 - t)Tx_t - x_n, j(x_t - x_n)\rangle \\
&\leq \langle (1 - t)Tx_t - x_n, j(x_t - x_n)\rangle \\
&= (1 - t)\langle Tx_t - Tx_n, j(x_t - x_n)\rangle + (1 - t)\langle Tx_n - x_n, j(x_t - x_n)\rangle \\
&\qquad + t\langle x_t - x_n, j(x_t - x_n)\rangle - t\langle x_t, j(x_t - x_n)\rangle \\
&\leq \|x_t - x_n\|^2 + (1 - t)\|x_n - Tx_n\|\|x_t - x_n\| - t\langle x_t, j(x_t - x_n)\rangle.
\end{aligned}$$

It follows that

$$\langle x_t, j(x_t - x_n)\rangle \leq \frac{1 - t}{t}\|x_n - Tx_n\|\|x_t - x_n\|.$$

Therefore,

$$\limsup_{n\to\infty}\langle x_t, j(x_t - x_n)\rangle \leq 0.$$

It follows from (3.11) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle x^*, j(x^* - Tu_n)\rangle &= \limsup_{n\to\infty}\langle x^*, j(x^* - u_n)\rangle \\
&= \limsup_{n\to\infty}\langle x^*, j(x^* - x_n)\rangle \\
&\leq 0. \qquad (3.16)
\end{aligned}$$

Finally, we show that $x_n \to x^*$. As a matter of fact, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(Q_C[(1-\beta_n)Tu_n] - x^*)\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|Q_C[(1-\beta_n)Tu_n] - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|(1-\beta_n)(Tu_n - x^*) - \beta_n x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[(1-\beta_n)^2\|Tu_n - x^*\|^2 - \beta_n x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle -x^*, j(Tu_n - x^*)\rangle + \beta_n^2\|x^*\|^2] \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[(1-\beta_n)^2\|u_n - x^*\|^2 - \beta_n x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle -x^*, j(Tu_n - x^*)\rangle + \beta_n^2\|x^*\|^2] \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[(1-\beta_n)^2\|x_n - x^*\|^2 - \beta_n x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle -x^*, j(Tu_n - x^*)\rangle + \beta_n^2\|x^*\|^2] \\
&\leq [1 - 2(1-\alpha_n)\beta_n]\|x_n - x^*\|^2 + 2(1-\alpha_n)(1-\beta_n)\beta_n\langle -x^*, j(Tu_n - x^*)\rangle \\
&\quad + (1-\alpha_n)\beta_n^2(\|x_n - x^*\|^2 + \|x\|^2) \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\delta_n,
\end{aligned}$$

where $\gamma_n = 2(1-\alpha_n)\beta_n$ and $\delta_n = \{(1-\beta_n)\langle\langle -x^*, j(Tu_n - x^*)\rangle\} + \frac{\beta_n}{2}(\|x_n - x^*\|^2 + \|x^*\|^2)$. It is easily seen that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. By Lemma 2.3, we deduce that $x_n \to x^*$. This completes the proof. □

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