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Convergence Theorems of a New Three-Step Iteration for Nonself Asymptotically Nonexpansive Mappings

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Abstract : Let E be a real uniformly convex and smooth Banach space with P as a sunny nonexpansive retraction, K be a nonempty closed convex subset of E. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three of weakly inward and nonself asymptotically nonexpansive mappings with respect to P. It is proved that three step iteration converges weakly and strongly to a common fixed point of T_i $(i = 1, 2, 3)$ under certain conditions. It presents some new results in this paper.

Keywords : nonself asymptotically nonexpansive mapping; strong and weak convergence; common fixed point.
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1 Introduction

Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to K$ is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if

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there exists a sequence $\{k_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \le (1+k_n)\,\|x-y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$. A mapping $T : K \to K$ is called uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\,\|x-y\| \tag{1.2}$$

for all $x, y \in K$ and $n \ge 1$. Also T is called asymptotically quasi-nonexpansive if $F(T) = \{x \in K : Tx = x\} \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that for all $x \in K$, the following inequality holds:

$$\|T^n x - p\| \le (1+k_n)\,\|x-p\|,\ \forall p \in F(T),\ n \ge 1, \tag{1.3}$$

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive. Every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive, but the converse may be not true.

The class of asymptotically nonexpansive self-mappings was introduced by Goebel and Kirk [1] in 1972, who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.

The concept of nonself asymptotically nonexpansive mappings was introduced by Chidu me et al. [2] in 2003 as the generalization of asymptotically nonexpansive self-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.1. *[2] Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K.*

(i) A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le (1+k_n)\,\|x-y\|\,,\quad \forall x, y \in K,\ n \ge 1. \tag{1.4}$$

(ii) A nonself mapping $T : K \to E$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le L\,\|x-y\|\,,\quad \forall x, y \in K,\ n \ge 1. \tag{1.5}$$

(iii) A nonself mapping $T : K \to E$ is called asymptotically quasi-nonexpansive if $F(T) \ne \emptyset$ and there exists a sequence $\{k_n\} \in [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that for all $x \in K$,

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}p\right\| \le (1+k_n)\,\|x-p\|\,,\quad \forall p \in F,\ n \ge 1. \tag{1.6}$$

In [2], they studied the following iterative algorithm :

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad \forall n \geq 1. \tag{1.7}$$

to approximate some fixed point of T under suitable conditions.

Remark 1.2. *If T is a self-mapping, then P becomes the identity mapping so that* (1.4)*,* (1.5) *and* (1.6) *reduce to* (1.1)*,* (1.2) *and* (1.3) *respectively.*

Recently, Zhou et al. [3] introduced the following definition.

Definition 1.3. *[3] Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\} \in [0, \infty)$ with $k_n \to 0$ as $n \to \infty$ such that*

$$\|(PT)^n x - (PT)^n y\| \leq (1 + k_n)\|x - y\|, \quad \forall x, y \in K, \ n \geq 1. \tag{1.8}$$

T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L \geq 0$ such that

$$\|(PT)^n x - (PT)^n y\| \leq L\|x - y\|, \quad \forall x, y \in K, \ n \geq 1. \tag{1.9}$$

Remark 1.4. *[3] If $T : K \to E$ is an asymptotically nonexpansive in the light of* (1.4) *and $P : E \to K$ is a nonexpansive retraction, then for all $x, y \in K$, $n \geq 1$, we have*

$$\begin{aligned}
\|(PT)^n x - (PT)^n y\| &= \left\|PT(PT)^{n-1}x - PT(PT)^{n-1}y\right\| \\
&\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\
&\leq k_n \|x - y\|.
\end{aligned}$$

But, the converse may not be true. Actually they studied the iteration algorithm

$$x_1 \in K, \quad x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad n \geq 1, \tag{1.10}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[a, 1-a]$ for some $a \in (0,1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$. Zhou et al. [3] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to P in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [2] were deduced.

The main purpose of this paper is to construct an iteration scheme (2.1) below for approximating common fixed points of three nonself asymptotically nonexpansive mappings and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

2 Preliminaries

Let E be a real normed linear space, and K be a nonempty closed convex subset of E which is also a nonexpansive retraction of E with a retraction P. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be nonself asymptotically nonexpansive mappings with respect to P. Then for a given $x_1 \in K$ and $n \geq 1$, compute the iterative sequences $\{x_n\}$ defined by,

$$\begin{cases} y_n = (1-a_{n3})x_n + a_{n3}(PT_3)^n x_n, \\ y_{n+1} = (1-a_{n2}-b_{n2})x_n + a_{n2}(PT_2)^n y_n + b_{n2}(PT_3)^n x_n, \\ x_{n+1} = (1-a_{n1}-b_{n1})x_n + a_{n1}(PT_1)^n y_{n+1} + b_{n1}(PT_2)^n y_n, \end{cases} \tag{2.1}$$

where $\{a_{ni}\}$, $\{b_{ni}\}$, $\{1-a_{ni}-b_{ni}\}$ are sequences in $[0,1]$ for all $i \in \{1,2,3\}$ and $n \geq 1$.

If $b_{n2} = b_{n1} = 0$ for all $n \geq 1$, then (2.1) reduces to the iteration defined by,

$$\begin{cases} y_n = (1-a_{n3})x_n + a_{n3}(PT_3)^n x_n, \\ y_{n+1} = (1-a_{n2})x_n + a_{n2}(PT_2)^n y_n, \\ x_{n+1} = (1-a_{n1})x_n + a_{n1}(PT_1)^n y_{n+1},\ n \geq 1, \end{cases} \tag{2.2}$$

If $a_{n3} = b_{n2} = b_{n1} = 0$ for all $n \geq 1$, then (2.1) reduces to the iteration defined by,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(PT_1)^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n(PT_2)^n x_n,\ n \geq 1. \end{cases} \tag{2.3}$$

If $a_{n3} = a_{n2} = b_{n2} = 0$ for all $n \geq 1$, then (2.1) reduces to the iteration (1.10) defined by Zhou et al. [3].

If $a_{n3} = b_{n2} = b_{n1} = a_{n2} = 0$, and $T_1 = T_2 = T$ then (2.1) reduces to the iteration defined by,

$$x_n = (1-\alpha_n)x_n + \alpha_n(PT)^n x_n, \forall x_1 \in K,\ n \geq 1. \tag{2.4}$$

If $T_1 = T_2 = T_3 = T$ are self-mappings and $b_{n2} = b_{n1} = 0$ for all $n \geq 1$, then (2.1) reduces to the Noor iteration defined by Xu and Noor [10]

$$\begin{cases} y_n = (1-a_{n3})x_n + a_{n3}T^n x_n, \\ y_{n+1} = (1-a_{n2})x_n + a_{n2}T^n y_n, \\ x_{n+1} = (1-a_{n1})x_n + a_{n1}T^n y_{n+1},\ n \geq 1, \end{cases} \tag{2.5}$$

where $\{a_{ni}\}$ are sequences in $[0,1]$ for all $i \in \{1,2,3\}$.

If $T_1 = T_2 = T$ are self-mappings and $a_{n3} = b_{n2} = b_{n1} = 0$ for all $n \geq 1$, then (2.1) reduces to the modified Ishikawa iterative scheme [11]

$$\begin{cases} y_{n+1} = (1-a_{n2})x_n + a_{n2}T^n x_n, \\ x_{n+1} = (1-a_{n1})x_n + a_{n1}T^n y_{n+1},\ n \geq 1, \end{cases} \tag{2.6}$$

If $T_1 = T$ is a self-mapping and $a_{n3} = a_{n2} = b_{n2} = b_{n1} = 0$ for all $n \geq 1$, then (2.1) reduces to the modified Mann iterative scheme [12]

$$x_{n+1} = (1 - a_{n1})x_n + a_{n1}T^n x_n, \ n \geq 1$$

where $\{a_{n1}\}$ is sequence in $[0, 1]$.

Now we list the following definitions and results which are useful in the sequel.

Let E be Banach space with $dimE \geq 2$, the modulus of E is the function $\delta_E(\varepsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = 1, \ \ \|y\| = 1, \ \varepsilon = \|x - y\| \right\}.$$

The Banach space E is uniformly convex if and only if with $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. The space E said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$.

The Banach space E is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ weakly as $n \to \infty$ implying that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the inward set of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \to E$ is called weakly inward if $Tx \in cl[IK(x)]$ for all $x \in K$, where $cl[IK(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let C, D be nonempty subset of a Banach space E such that C is nonempty closed convex and $D \subset C$, A retraction $P : C \to D$ is said to be sunny [4] if $P(Px + t(x - Px)) = Px$ for all $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p, if for each sequence $\{x_n\}$ in $D(T)$, the conditions $x_n \to x_0$ weakly and $Tx_n \to p$ strongly imply $Tx_0 = p$.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element of the range T.

A mapping $T : K \to K$ is said to demi-compact if any sequence $\{x_n\}$ in K satisfying $x_n - Tx_n \to 0$ as $n \to \infty$ has a convergent subsequence.

Recall that the mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$,

$f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf \{\|x - p\| : p \in F(T)\}$. Senter and Dotson [8] pointed out that every continuous and demi-compact mapping must satisfy Condition (A). Different modifications of the condition (A) for two finite families of selfmaps have been made recently in the literature [13, 14]. Yang and Xie [9] modified these conditions for three nonself asymptotically nonexpansive mappings $T_i : K \to E$ $(i = 1, 2, 3)$ as follows:

Mappings $T_i : K \to E$ $(i = 1, 2, 3)$ with the nonempty common fixed point set $F = \cap_{i=1}^{3} F(T_i)$ in K is said to satisfy Condition (B) with respect to the sequence $\{u_n\}$ if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\max\limits_{1 \leq i \leq 3} \|u_n - PT_i u_n\| \geq f(d(u_n, F))$ for all $n \geq 1$. We know that Condition (B) is weaker than the demi-compactness of mappings T_i $(i = 1, 2, 3)$.

The following lemmas are needed to prove our main results.

Lemma 2.1. *[5] Let $p > 1$ and $D > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous, strictly increasing and convex function $g_1 : [0, \infty) \to [0, \infty)$, $g_1(0) = 0$, such that*

$$\|\lambda x + (1 - \lambda) y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda) g_1(\|x - y\|) \tag{2.7}$$

for all $x, y \in B_D$ and $0 \leq \lambda \leq 1$, where B_D is the closed ball with center zero and radius D, $wp(\lambda) = \lambda (1 - \lambda)^p + \lambda^p (1 - \lambda)$.

Lemma 2.2. *[6] Let E be a uniformly convex Banach space and $B_D = \{x \in E : \|x\| \leq D\}$, $D > 0$. Then there exists a continuous, strictly increasing and convex function $g_2 : [0, \infty) \to [0, \infty)$, $g_2(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\| \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda\beta g_2(\|x - y\|), \tag{2.8}$$

for all $x, y, z \in B_D$ and $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.3. *[7] If $\{r_n\}, \{t_n\}$ are two sequences of nonnegative real numbers such that*

$$r_{n+1} \leq (1 + t_n) r_n, \ \ n \geq 1$$

and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim\limits_{n \to \infty} r_n$ exists.

Lemma 2.4. *[15] Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

Lemma 2.5. *[3] Let E be a real smooth and uniformly convex Banach space, K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with the sequence $k_n \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3 Main Results

In this section, we will prove the strong and weak convergence of the iteration scheme (2.1) to a common fixed point for three asymptotically nonexpansive nonself mappings in a uniformly convex and smooth Banach space. We first prove the following lemmas.

Lemma 3.1. *Let E be a real normed linear space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three nonself-asymptotically nonexpansive mappings with respect to P with sequences $\left\{k_n^{(i)}\right\}$ such that $\sum\limits_{n=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap\limits_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for $q \in F$.*

Proof. Let $q \in F = \bigcap\limits_{i=1}^{3} F(T_i)$. Setting $k_n = \max\left\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}\right\}$. Since $\sum\limits_{n=1}^{\infty} k_n^{(i)} < \infty$, then $\sum\limits_{n=1}^{\infty} k_n < \infty$. From (2.1), we have

$$\begin{aligned}
\|y_n - q\| &= \|(1-a_{n3})x_n + a_{n3}(PT_3)^n x_n - q\| \\
&\le (1-a_{n3})\|x_n - q\| + a_{n3}\|(PT_3)^n x_n - q\| \\
&\le (1-a_{n3})\|x_n - q\| + a_{n3}(1+k_n)\|x_n - q\| \\
&= (1+a_{n3}k_n)\|x_n - q\| \\
&\le (1+k_n)\|x_n - q\|. \qquad (3.1)
\end{aligned}$$

By (2.1) and (3.1), we obtain

$$\begin{aligned}
\|y_{n+1} - q\| &= \|(1-a_{n2}-b_{n2})x_n + a_{n2}(PT_2)^n y_n + b_{n2}(PT_3)^n x_n - q\| \\
&\le (1-a_{n2}-b_{n2})\|x_n - q\| + a_{n2}\|(PT_2)^n y_n - q\| \\
&\quad + b_{n2}\|(PT_3)^n x_n - q\| \\
&\le (1-a_{n2}-b_{n2})\|x_n - q\| + a_{n2}(1+k_n)\|y_n - q\| \\
&\quad + b_{n2}(1+k_n)\|x_n - q\| \\
&\le (1-a_{n2}-b_{n2})\|x_n - q\| + a_{n2}(1+k_n)^2\|x_n - q\| \\
&\quad + b_{n2}(1+k_n)\|x_n - q\| \\
&= \|x_n - q\| + a_{n2}(2k_n + k_n^2)\|x_n - q\| + b_{n2}k_n\|x_n - q\| \\
&\le \|x_n - q\| + (3k_n + k_n^2)\|x_n - q\| \\
&= (1+3k_n+k_n^2)\|x_n - q\|. \qquad (3.2)
\end{aligned}$$

Using (2.1), (3.1) and (3.2), we have

$$\begin{aligned}
\|x_{n+1}-q\| &= \|(1-a_{n1}-b_{n1})x_n + a_{n1}(PT_1)^n y_{n+1} + b_{n1}(PT_2)^n y_n - q\| \\
&\le (1-a_{n1}-b_{n1})\|x_n-q\| + a_{n1}\|(PT_1)^n y_{n+1}-q\| \\
&\quad + b_{n1}\|(PT_2)^n y_n - q\| \\
&\le (1-a_{n1}-b_{n1})\|x_n-q\| + a_{n1}(1+k_n)\|y_{n+1}-q\| \\
&\quad + b_{n1}(1+k_n)\|y_n-q\| \\
&\le (1-a_{n1}-b_{n1})\|x_n-q\| + a_{n1}(1+k_n)(1+3k_n+k_n^2)\|x_n-q\| \\
&\quad + b_{n1}(1+k_n)(1+k_n)\|x_n-q\| \\
&\le \|x_n-q\| + a_{n1}(4k_n+4k_n^2+k_n^3)\|x_n-q\| + b_{n1}(2k_n+k_n^2)\|x_n-q\| \\
&\le \|x_n-q\| + (4k_n+4k_n^2+k_n^3)\|x_n-q\| + (2k_n+k_n^2)\|x_n-q\| \\
&= (1+6k_n+5k_n^2+k_n^3)\|x_n-q\|. \qquad (3.3)
\end{aligned}$$

Defining

$$r_n = \|x_n-q\|, \;\; t_n = 6k_n+5k_n^2+k_n^3$$

in (3.3) we get $r_{n+1} \le (1+t_n)\,r_n$. Since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, Lemma 2.3 implies the existence of the limit $\lim_{n\to\infty}\|x_n-q\|$. This yields the assertion. □

Now we are ready to formulate and prove a criterion on strong convergence of $\{x_n\}$ given by (2.1).

Theorem 3.2. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i=1,2,3)$ be three nonself-asymptotically nonexpansive mappings with respect to P with sequences $\left\{k_n^{(i)}\right\}$ such that $\sum_{i=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1). Then the sequence $\{x_n\}$, defined by (2.1), converges strongly to a common fixed point of T_i $(i=1,2,3)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \qquad (3.4)$$

where $d(x_n,F) = \inf\{\|x-p\| : p \in F\}$.

Proof. The necessity of condition (3.4) is obvious. Let us prove the sufficiency part of the theorem.

For any given $q \in F$, we have (see (3.3))

$$\|x_{n+1}-q\| \le (1+t_n)\|x_n-q\|, \qquad (3.5)$$

and hence, we get

$$d(x_{n+1},F) \le (1+t_n)d(x_n,F). \qquad (3.6)$$

Now applying Lemma 2.3 to (3.6) we obtain the existence of the limit $\lim_{n\to\infty} d(x_n, F)$. By condition (3.4), we get

$$\lim_{n\to\infty} d(x_n, F) = \liminf_{n\to\infty} d(x_n, F) = 0 \tag{3.7}$$

Let us show that the sequence $\{x_n\}$ converges to a common fixed point of T_i $(i = 1, 2, 3)$. We first show that $\{x_n\}$ is a Cauchy sequence in E. It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the (3.5), we have

$$\|x_{n+1} - q\| \leq \exp(t_n) \|x_n - q\| \tag{3.8}$$

Thus, for any $m, n \geq n_0$ iterating (3.8) and noting $\sum\limits_{n=1}^{\infty} t_n < \infty$, we get

$$\begin{aligned} \|x_{n+m} - q\| &\leq \exp(t_{n+m-1}) \|x_{n+m-1} - q\| \\ &\leq \exp(t_{n+m-1}) \exp(t_{n+m-2}) \|x_{n+m-2} - q\| \\ &\leq \ldots \\ &\leq \exp\left(\sum_{i=n}^{n+m-1} t_n\right) \|x_n - q\|. \end{aligned}$$

Let $M = \exp\left(\sum\limits_{i=n}^{n+m-1} t_n\right)$, then $0 < M < \infty$ and

$$\|x_{n+m} - q\| \leq M \|x_n - q\|, \forall n, m \geq n_0. \tag{3.9}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, given any $\varepsilon > 0$, there exists a positive integer $n \geq n_1$ such that $d(x_n, F) < \varepsilon/(1 + M)$ for all $n \geq n_1$. So we have $q^* \in F$ such that $\|x_{n_1} - q^*\| \leq \varepsilon/(1 + M)$. It follows from (3.9) that for all $n \geq n_1$ and $m \geq 1$

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\ &\leq (1 + M) \|x_n - q^*\| < \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, thus $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = q$. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since the set of fixed points of asymptotically nonexpansive mappings is closed, we have $q \in F$. This completes the proof of the theorem. □

Lemma 3.3. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three nonself-asymptotically nonexpansive mappings with respect to P with sequences $\left\{k_n^{(i)}\right\}$ such that $\sum\limits_{i=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap\limits_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1) satisfying the following conditions:*

i) $0 < \lim\inf\limits_{n\to\infty} a_{n3} < \lim\sup\limits_{n\to\infty} a_{n3} < 1$

ii) $0 < \lim\inf\limits_{n\to\infty} a_{nj} < \lim\sup\limits_{n\to\infty} (a_{nj} + b_{nj}) < 1$ *for* $j = 2, 3$.

Then $\lim_{n\to\infty} \|x_n - (PT_i)\, x_n\| = 0$ *for* $i = 1, 2, 3$.

Proof. From Lemma 3.1, $\{x_n - q\}$ is bounded for $q \in F$. Thus, we get boundedness of $\{y_n - q\}$, $\{y_{n+1} - q\}$ from (3.1) and (3.2). Since T_i is nonself asymptotically nonexpansive mappings, we can prove the sequences $\{(PT_i)^n\, y_n - q\}_{i=1}^{2}$, $\{(PT_i)^n\, x_n - q\}_{i=2}^{3}$, $\{(PT_i)^n\, y_{n+1} - q\}$ are all bounded. By using (2.1) and Lemma 2.1 we have, for some constant $D_1 > 0$,

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1-a_{n3})x_n + a_{n3}(PT_3)^n x_n - q\|^2 \\
&= \|(1-a_{n3})\,(x_n - q) + a_{n3}\,((PT_3)^n x_n - q)\|^2 \\
&\le (1-a_{n3})\,\|x_n - q\|^2 + a_{n3}\,\|(PT_3)^n x_n - q\|^2 \\
&\quad\;\; - a_{n3}(1-a_{n3})g_1\,(\|x_n - (PT_3)^n x_n\|) \\
&\le (1-a_{n3})\,\|x_n - q\|^2 + a_{n3}\,(1+k_n)^2\,\|x_n - q\|^2 \\
&\quad\;\; - a_{n3}(1-a_{n3})g_1\,(\|x_n - (PT_3)^n x_n\|) \\
&\le \|x_n - q\|^2 + D_1 k_n - a_{n3}(1-a_{n3})g_1\,(\|x_n - (PT_3)^n x_n\|)\,. \qquad (3.10)
\end{aligned}$$

It follows from (2.1), Lemma 2.2, and (3.10) that for $D_2 > 0$,

$$\begin{aligned}
\|y_{n+1} - q\|^2 &= \|(1-a_{n2}-b_{n2})x_n + a_{n2}(PT_2)^n y_n + b_{n2}(PT_3)^n x_n - q\|^2 \\
&= \|(1-a_{n2}-b_{n2})\,(x_n - q) + a_{n2}\,((PT_2)^n y_n - q) + b_{n2}\,((PT_3)^n x_n - q)\|^2 \\
&\le (1-a_{n2}-b_{n2})\,\|x_n - q\|^2 + a_{n2}\,\|(PT_2)^n y_n - q\|^2 \\
&\quad\;\; + b_{n2}\,\|(PT_3)^n x_n - q\|^2 - a_{n2}(1-a_{n2}-b_{n2})g_2\,(\|x_n - (PT_2)^n y_n\|) \\
&\le (1-a_{n2}-b_{n2})\,\|x_n - q\|^2 + a_{n2}\,(1+k_n)^2\,\|y_n - q\|^2 \\
&\quad\;\; + b_{n2}\,(1+k_n)^2\,\|x_n - q\|^2 - a_{n2}(1-a_{n2}-b_{n2})g_2\,(\|x_n - (PT_2)^n y_n\|) \\
&\le \|x_n - q\|^2 + 2D_2 k_n - a_{n2}a_{n3}(1-a_{n3})g_1\,(\|x_n - (PT_3)^n x_n\|) \\
&\quad\;\; - a_{n2}(1-a_{n2}-b_{n2})g_2\,(\|x_n - (PT_2)^n y_n\|)\,. \qquad (3.11)
\end{aligned}$$

Similarly, using (2.1), Lemma 2.2, (3.10) and (3.11) we have, for some constant $D_3 > 0$,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-a_{n1}-b_{n1})x_n + a_{n1}(PT_1)^n y_{n+1} + b_{n1}(PT_2)^n y_n - q\|^2 \\
&= \|(1-a_{n1}-b_{n1})\,(x_n - q) + a_{n1}\,((PT_1)^n y_{n+1} - q) + b_{n1}\,((PT_2)^n y_n - q)\|^2 \\
&\le (1-a_{n1}-b_{n1})\,\|x_n - q\|^2 + a_{n1}\,\|(PT_1)^n y_{n+1} - q\|^2 \\
&\quad\;\; + b_{n1}\,\|(PT_2)^n y_n - q\|^2 - a_{n1}(1-a_{n1}-b_{n1})g_2\,(\|x_n - (PT_1)^n y_{n+1}\|) \\
&\le (1-a_{n1}-b_{n1})\,\|x_n - q\|^2 + a_{n1}\,(1+k_n)^2\,\|y_{n+1} - q\|^2 \\
&\quad\;\; + b_{n1}\,(1+k_n)^2\,\|y_n - q\|^2 - a_{n1}(1-a_{n1}-b_{n1})g_2\,(\|x_n - (PT_1)^n y_{n+1}\|) \\
&\le \|x_n - q\|^2 + 3D_2 k_n - a_{n1}(1-a_{n1}-b_{n1})g_2\,(\|x_n - (PT_1)^n y_{n+1}\|) \\
&\quad\;\; - a_{n1}a_{n2}(1-a_{n2}-b_{n2})g_2\,(\|x_n - (PT_2)^n y_n\|) \\
&\quad\;\; - a_{n1}a_{n2}a_{n3}(1-a_{n3})g_1\,(\|x_n - (PT_3)^n x_n\|)\,. \qquad (3.12)
\end{aligned}$$

By the inequality (3.12), we get

$$a_{n1}a_{n2}a_{n3}(1-a_{n3})g_1\left(\|x_n-(PT_3)^n x_n\|\right) \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+3D_2k_n, \tag{3.13}$$

$$a_{n1}a_{n2}(1-a_{n2}-b_{n2})g_2\left(\|x_n-(PT_2)^n y_n\|\right) \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+3D_2k_n \tag{3.14}$$

and

$$a_{n1}(1-a_{n1}-b_{n1})g_2\left(\|x_n-(PT_1)^n y_{n+1}\|\right) \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+3D_2k_n. \tag{3.15}$$

If $\liminf_{n\to\infty} a_{ni} > 0$ for $i=1,2$ and $0<\liminf a_{n3}<\limsup a_{n3}<1$, there exist a positive integer n_0 and $t,t'\in(0,1)$ such that

$$0<t<a_{n3},(i=1,2,3), \qquad a_{n3}<t'<1.$$

Using above inequalities, we get from (3.13) that

$$t^3(1-t')g_1\left(\|x_n-(PT_3)^n x_n\|\right) \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+3D_2k_n. \tag{3.16}$$

By (3.16), we derive

$$\begin{aligned}\sum_{n=n_0}^{m} g_1\left(\|x_n-(PT_3)^n x_n\|\right) \\ &\le \frac{1}{t^3(1-t')}\left(\sum_{n=n_0}^{m}\left(\|x_n-q\|^2-\|x_{n+1}-q\|^2\right)+3D_2\sum_{n=n_0}^{m}k_n\right) \\ &\le \frac{1}{t^3(1-t')}\left(\|x_{n_0}-q\|^2+3D_2\sum_{n=n_0}^{m}k_n\right).\end{aligned}$$

Because the right side of the above inequality is finite sum, we obtain

$$\sum_{n=n_0}^{m} g_1\left(\|x_n-(PT_3)^n x_n\|\right)<\infty,$$

and so $\lim_{n\to\infty} g_1\left(\|x_n-(PT_3)^n x_n\|\right)=0$. Then properties of g_1 imply

$$\lim_{n\to\infty}\|x_n-(PT_3)^n x_n\|=0. \tag{3.17}$$

By the same argument, from (3.14), (3.15) we obtain that

$$\lim_{n\to\infty}\|x_n-(PT_2)^n y_n\|=0, \tag{3.18}$$

$$\lim_{n\to\infty}\|x_n-(PT_1)^n y_{n+1}\|=0. \tag{3.19}$$

On the other hand note that

$$\begin{aligned}\|y_n - x_n\| &= \|(1-a_{n3})x_n + a_{n3}(PT_3)^n x_n - x_n\| \\ &\leq \|x_n - (PT_3)^n x_n\| \\ &\to 0 \ \ (n\to\infty).\end{aligned} \tag{3.20}$$

Since T_2 nonself-asymptotically nonexpansive mapping, we get from (3.18) and (3.20) that

$$\begin{aligned}\|x_n - (PT_2)^n x_n\| &\leq \|x_n - (PT_2)^n y_n\| + \|(PT_2)^n y_n - (PT_2)^n x_n\| \\ &\leq \|x_n - (PT_2)^n y_n\| + (1+k_n)\|y_n - x_n\| \\ &\to 0 \ \ (n\to\infty).\end{aligned} \tag{3.21}$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}\|y_{n+1} - x_n\| &= \|(1-a_{n2}-b_{n2})x_n + a_{n2}(PT_2)^n y_n + b_{n2}(PT_3)^n x_n - x_n\| \\ &\leq a_{n2}\|x_n - (PT_2)^n y_n\| + b_{n2}\|x_n - (PT_3)^n x_n\| \\ &\to 0 \ \ as\ n\to\infty.\end{aligned} \tag{3.22}$$

Thus we get from (3.19) and (3.22)

$$\begin{aligned}\|x_n - (PT_1)^n x_n\| &\leq \|x_n - (PT_1)^n y_{n+1}\| + \|(PT_1)^n y_{n+1} - (PT_1)^n x_n\| \\ &\leq \|x_n - (PT_1)^n y_{n+1}\| + (1+k_n)\|y_{n+1} - x_n\| \\ &\to 0 \ \ as\ n\to\infty.\end{aligned} \tag{3.23}$$

It follows from (2.1), (3.18) and (3.19) that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-a_{n1}-b_{n1})x_n + a_{n1}(PT_1)^n y_{n+1} + b_{n1}(PT_2)^n y_n - x_n\| \\ &\leq a_{n1}\|x_n - (PT_1)^n y_{n+1}\| + b_{n1}\|x_n - (PT_2)^n y_n\| \\ &\to 0 \ \ as\ n\to\infty.\end{aligned} \tag{3.24}$$

Since an asymptotically nonexpansive mapping with respect to P must be uniformly Lipschitzian with respect to P, we have

$$\begin{aligned}\|x_{n+1} - (PT_i)x_{n+1}\| &\leq \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + \left\|(PT_i)^{n+1}x_{n+1} - (PT_i)x_{n+1}\right\| \\ &\leq \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + L\|x_{n+1} - (PT_i)^n x_{n+1}\| \\ &\leq \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + L\|(x_{n+1} - x_n) \\ &\qquad + (x_n - (PT_i)^n x_n) + ((PT_i)^n x_n - (PT_i)^n x_{n+1})\| \\ &\leq \left\|x_{n+1} - (PT_i)^{n+1}x_{n+1}\right\| + L\|x_n - (PT_i)^n x_n\| \\ &\qquad + L(L+1)\|x_{n+1} - x_n\|.\end{aligned}$$

This with (3.17), (3.21), (3.23) and (3.24) implies that $\lim_{n\to\infty}\|x_n - (PT_i)\,x_n\| = 0$ for $i = 1,2,3$.

This completes the proof of the theorem. □

Theorem 3.4. *Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with sequences $\left\{k_n^{(i)}\right\}$ such that $\sum_{i=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1) satisfying the following conditions:*

$$i)\ 0 < \lim_{n\to\infty}\inf a_{n3} < \lim_{n\to\infty}\sup a_{n3} < 1$$

$$ii)\ 0 < \lim_{n\to\infty}\inf a_{nj} < \lim_{n\to\infty}\sup (a_{nj} + b_{nj}) < 1 \text{ for } j = 2, 3.$$

If $\{T_1, T_2, T_3\}$ satisfies Condition (B) with respect to sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3\}$.

Proof. By Lemma 3.3 we have $\lim_{n\to\infty} \|x_n - (PT_i)x_n\| = 0$ for $i = 1, 2, 3$. Since $\{T_1, T_2, T_3\}$ satisfy Condition (B) with respect to sequence $\{x_n\}$, we have

$$\max_{1\le i\le 3} \|x_n - PT_i x_n\| \ge f(d(x_n, F)).$$

Thus $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now applying the Theorem 3.2, we obtain the result. This completes the proof. □

Remark 3.5. *[9] Since $x_n = Px_n$ for all $n \ge 1$, we have $\|x_n - PTx_n\| \le \|x_n - Tx_n\|$ for all $n \ge 1$. Therefore, the Condition (B) is weaker than the Condition (A).*

From Theorem 3.4, we have following corollary.

Corollary 3.6. *Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with common sequences $\left\{k_n^{(i)}\right\}$ such that $\sum_{i=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1) satisfying the following conditions:*

$$i)\ 0 < \lim_{n\to\infty}\inf a_{n3} < \lim_{n\to\infty}\sup a_{n3} < 1$$

$$ii)\ 0 < \lim_{n\to\infty}\inf a_{nj} < \lim_{n\to\infty}\sup (a_{nj} + b_{nj}) < 1 \text{ for } j = 2, 3.$$

If one of $\{T_1, T_2, T_3\}$ is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3\}$.

Proof. Suppose that one of $\{T_1, T_2, T_3\}$ is demi-compact, so is one of $\{PT_1, PT_2, PT_3\}$. By continuity of T_1, T_2, T_3 and P, we get continuity of PT_1, PT_2, PT_3. It is well known that every continuous and demi-compact mapping must satisfy Condition (B). Thus, the rest of the proof follows from Theorem 3.4. This completes the proof. □

Finally, we prove the weak convergence of the iterative scheme (2.1) for three asymptotically nonexpansive nonself-mappings in a real smooth and uniformly convex Banach space satisfying Opial's condition.

Theorem 3.7. *Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E satisfying Opial's condition with P as a sunny nonexpansive retraction. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with common sequences $\left\{k_n^{(i)}\right\}$ such that $\sum\limits_{i=1}^{\infty} k_n^{(i)} < \infty$ and $F = \bigcap\limits_{i=1}^{3} F(T_i) \neq \varnothing$. For an arbitrary $x_0 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.1) satisfying the following conditions:*

$$i)\ 0 < \lim\inf_{n\to\infty} a_{n3} < \lim\sup_{n\to\infty} a_{n3} < 1$$

$$ii)\ 0 < \lim\inf_{n\to\infty} a_{nj} < \lim\sup_{n\to\infty} (a_{nj} + b_{nj}) < 1 \text{ for } j = 2, 3.$$

Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, T_3\}$.

Proof. Let $q \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists, and $\{x_n\}$ is bounded. We now show that $\{x_n\}$ has a unique weak subsequential limit in F. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to q_1 and q_2, respectively. By Lemma 3.3, we have $\lim_{n\to\infty} \|x_n - (PT_i)\, x_n\| = 0$ for $i = 1, 2, 3$. Lemma 2.5 implies that $(I - PT_i)\, q_1 = 0$, that is, $(PT_i)\, q_1 = q_1$. Similiarly, we obtain that $(PT_i)\, q_2 = q_2$. Also Lemma 2.4 guarantees that $q_1,\ q_2 \in F$. Next, we prove the uniqueness. For this, suppose that $q_1 \neq q_2$. Then, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{k\to\infty} \|x_{n_k} - q_1\| \\
&< \lim_{k\to\infty} \|x_{n_k} - q_2\| \\
&= \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{j\to\infty} \left\|x_{n_j} - q_2\right\| \\
&< \lim_{j\to\infty} \left\|x_{n_j} - q_1\right\| \\
&= \lim_{n\to\infty} \|x_n - q_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point of F. This completes the proof. □

Remark 3.8. *All the above theorems, the iterative sequence* (2.2)*,* (2.3) *and* (2.4) *can be replaced by the three step iterative process* (2.1)*. If the error terms are added in* (2.1) *and assumed to be bounded, then the results of this paper still hold.*

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