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Location of the Zeros of Polynomials[1]

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Abstract : In this paper, we obtain an annulus that contains all the zeros of the polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where a_ν's are complex coefficients and z is a complex variable. Our result generalizes one of the recently obtained result in this direction.

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1 Introduction and Statements

The study of the zeros of polynomials have a very rich history. In addition to having numerous applications in many areas including, but not limited to, signal processing, communication theory, Cryptography, Control Theory, Combinatorics, and Bio-Mathematics. This study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra).

Gauss and Cauchy were the earliest contributors in the theory of the location of zeros of a polynomial, since then this subject has been studied by many people (for example, see [1, 2]).

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A classical result due to Cauchy (see [1, p. 122]) on the distribution of zeros of a polynomial may be stated as follows:

Theorem A. *If $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$ is a polynomial with complex coefficients of degree n, then all zeros of $P(z)$ lie in the disk $|z| \le r$ where r is the unique positive root of the real-coefficient polynomial*

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \cdots - |a_1|x - |a_0|.$$

Recently Díaz-Barrero [3] improved this estimate by identifying an annulus containing all the zeros of a polynomial, where the inner and outer radii are expressed in terms of binomial coefficients and Fibonacci numbers. In fact he has proved the following result.

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a non-constant complex polynomial of degree n. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ where*

$$r_1 = \frac{3}{2} \min_{1\le k\le n} \left\{ \frac{2^n F_k \binom{n}{k}}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}, \quad r_2 = \frac{2}{3} \max_{1\le k\le n} \left\{ \frac{F_{4n}}{2^n F_k \binom{n}{k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

Here F_j are Fibonacci's numbers, that is, $F_0 = 0$, $F_1 = 1$ and for $j \ge 2$, $F_j = F_{j-1} + F_{j-2}$.

Seon-Hong Kim [4] obtained an annulus containing all the zeros of a polynomial, where the inner and outer radii are expressed in terms of binomial coefficients and proved the following result.

Theorem C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a non-constant complex polynomial of degree n. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ where*

$$r_1 = \min_{1\le j\le n} \left\{ \frac{C(n,j)}{2^n - 1} \left| \frac{a_0}{a_j} \right| \right\}^{1/j}, \quad r_2 = \max_{1\le j\le n} \left\{ \frac{2^n - 1}{C(n,j)} \left| \frac{a_{n-j}}{a_n} \right| \right\}^{1/j}.$$

where $C(n,j) := \frac{n!}{(n-j)!j!}$ is a Binomial coefficient.

In litrature there exists several other extenstions of above results (see [5]).

The subject in this note is to study further the annular bound for the zeros of complex-coefficient polynomials, and obtain a better annular bound which include Theorem C as a special case. In addition, several numerical examples will be given to demonstrate the improvement over above mentioned results. More precisely, we prove:

Theorem 1.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a non-constant complex polynomial of degree n and α, β any non-negative real numbers such that $(\alpha, \beta) \ne (0, 0)$. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ where*

$$r_1 = \min_{1\le k\le n} \left\{ \frac{(\alpha + k\beta)\binom{n}{k}}{(2\alpha + n\beta)2^{n-1} - \alpha} \left| \frac{a_0}{a_k} \right| \right\}^{1/k},$$

$$r_2 = \max_{1\le k\le n}\left\{\frac{(2\alpha+n\beta)2^{n-1}-\alpha}{(\alpha+k\beta)\binom{n}{k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Remark 1.2. For $\beta = 0$, Theorem 1.1 reduces to Theorem C.

If we take $\alpha = 0$, we obtain the following result.

Corollary 1.3. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a non-constant complex polynomial of degree n. Then all its zeros lie in the annulus $C = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{k\binom{n}{k}}{n2^{n-1}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}, \qquad r_2 = \max_{1\le k\le n}\left\{\frac{n2^{n-1}}{k\binom{n}{k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Here, we construct examples of polynomials for which the annuli containing all the zeros of the polynomials obtained by Corollary 1.3 are considerably sharper than the annuli obtainable from the above mentioned results (Theorems B, C.)

Example 1.4. Let $P(z) = z^5+2z^2+4z+3i$, for which, as we shall see, Corollary 1.3 gives the best bound.
By using Theorem B, we find that the annulus containing all the zeros of polynomial $P(z)$ comes out to be $0.026608 \le |z| \le 3.065262$, while by Corollary 1.3 it is $0.046875 \le |z| \le 1.74716$. We also calculated the actual zeros and found that they all lie in the annulus $0.69055058 \le |z| \le 1.73312317$. Thus for this polynomial, Corollary 1.3 gives the sharpest bound. We can improve the lower bound by choosing α, β suitably in Theorem 1.1.

Example 1.5. Our next example is the polynomial $P(z) = z^3+0.1z^2+0.2z+0.7$, by Theorem C, we obtain an annulus $0.4641 \le |z| \le 1.6984$ which contains all the zeros of $P(z)$. While as by Corollary 1.3 it comes out to be $0.5593 \le |z| \le 1.409$, which clearly improves the bound obtained by Theorem C.

Example 1.6. Our last example is the polynomial $P(z) = z^5 + 3z^4 + 2z + 2$, by Theorem B, we find that the annulus obtained by Theorem B is $0.326236 \le |z| \le 84.562500$, while as, if we use Corollary 1.3, it comes out to be $0.5 \le |z| \le 48$, which considerably improves Theorem B in this case.

Remark 1.7. We can similarly improve the bounds containing all the zeros of a polynomial by choosing α and β suitably in Theorem 1.1 .

2 Lemma

To prove the above theorem, we need the following lemma.

Lemma 2.1. *Let $a, d \in \mathbb{R}$, then $\sum_{k=1}^{n}(a+kd)C(n,k) = (2a+nd)2^{n-1} - a$.*

Proof. Proof follows by using the well-known Identity $\sum_{k=1}^{n} kC(n,k) = n2^{n-1}$. □

3 proof of Theorem

Proof of Theorem 1.1. We first show that all the zeros of $P(z)$ lie in

$$|z| \leq r_2 = \max_{1\leq j\leq n} \left\{ \frac{(2\alpha+n\beta)2^{n-1}-\alpha}{(\alpha+k\beta)C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k} \tag{3.1}$$

where α, β are any non-negative real numbers such that $(\alpha,\beta) \neq (0,0)$. From (3.1), it follows that

$$\left| \frac{a_{n-k}}{a_n} \right| \leq r_2^k \frac{(\alpha+k\beta)C(n,k)}{(2\alpha+n\beta)2^{n-1}-\alpha}, \quad k = 1,2,3\cdots,n$$

or

$$\sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_2^k} \leq \sum_{k=1}^{n} \frac{(\alpha+k\beta)C(n,k)}{(2\alpha+n\beta)2^{n-1}-\alpha} = 1 \quad \text{by lemma 2.1.} \tag{3.2}$$

Now, for $|z| > r_2$, we have

$$\begin{aligned} |P(z)| =& |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\ \geq& |a_n||z|^n \left\{ 1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{|z|^k} \right\} \\ >& |a_n||z|^n \left\{ 1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_2^k} \right\} \end{aligned}$$

using (3.2), we have for $|z| > r_2$, $|P(z)| > 0$. Consequently all the zeros of $P(z)$ lie in $|z| \leq r_2$ and this proves the second part of theorem.
To prove the first part of the theorem, we use second part. If $a_0 = 0$, then $r_1 = 0$ and there is nothing to prove. Let $a_0 \neq 0$, consider the polynomial

$$Q(z) = z^n P(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n.$$

By second part of the theorem, for any non-negative real numbers α, β such that $(\alpha,\beta) \neq (0,0)$, all the zeros of the polynomial $Q(z)$ lie in

$$\begin{aligned} |z| \leq& \max_{1\leq k\leq n} \left\{ \frac{(2\alpha+n\beta)2^{n-1}-\alpha}{(\alpha+k\beta)C(n,k)} \left| \frac{a_k}{a_0} \right| \right\}^{1/k} \\ =& \frac{1}{\min\limits_{1\leq k\leq n} \left\{ \frac{(\alpha+k\beta)C(n,k)}{(2\alpha+n\beta)2^{n-1}-\alpha} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}} \\ =& \frac{1}{r_1}. \end{aligned}$$

Now replacing z by $1/z$ and observing that all the zeros of $P(z)$ lie in

$$|z| \geq r_1 = \min_{1\leq k\leq n} \left\{ \frac{(\alpha+k\beta)C(n,k)}{(2\alpha+n\beta)2^{n-1}-\alpha} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}.$$

This completes the proof of theorem 1.1. □

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