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Rate of Convergence of P-Iteration and S-Iteration for Continuous Functions on Closed Intervals[1]

Prayong Sainuan

Department of Mathematics, Faculty of Sciences and Agricultural Technology
Rajamangala University of Technology Lanna, Chiang Mai 50300, Thailand
e-mail : psainuan@hotmail.com

Abstract : In this paper, we first give a necessary and sufficient condition for convergence of P-iteration to a fixed point of continuous functions on an arbitrary interval and prove equivalence of P-iteration and S-iteration. We also compare the rate of convergence between P-iteration and S-iteration. Some numerical examples for comparing the rate of convergence of those two methods are also given.

Keywords : rate of convergence; P-iteration; S-iteration; continuous function; closed interval.
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1 Introduction

Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. A point $p \in E$ is a *fixed point* of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f. It is known that if E also bounded, then $F(f)$ is nonempty. The *Mann iteration* (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n) u_n + \alpha_n f(u_n) \tag{1.1}$$

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for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0,1]$, and will be denoted by $M(u_1, \alpha_n, f)$. The *Ishikawa iteration* (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1-\beta_n)\, s_n + \beta_n f\,(s_n) \\ s_{n+1} = (1-\alpha_n)\, s_n + \alpha_n f\,(t_n) \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$, and will be denoted by $I(s_1, \alpha_n, \beta_n, f)$.The *S-iteration* (see [3]) is defined by $q_1 \in E$ and

$$\begin{cases} r_n = (1-\beta_n)\, q_n + \beta_n f\,(q_n) \\ q_{n+1} = (1-\alpha_n)\, f(q_n) + \alpha_n f\,(r_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$, and will be denoted by $S(q_1, \alpha_n, \beta_n, f)$.

It was shown in [4] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [5] showed that the Mann iteration converges faster than the Ishikawa iteration for the class of operators. Two years later, Qing and Rhoades [6] provided an example to show that the claim of Babu and Prasad is false. In 2013, Kosol [3] showed that the S-iteration converges faster than the Ishikawa iteration on an arbitrary interval. In 2011, Phuengrattana and Suantai [7] introduced a new three -step iteration, called SP-iteration, and showed that it converges faster than Mann, Ishikawa, Noor -iterations.

Motivated by the above results, we modify S and SP- iterations for construction a new iteration as follows: The *P-iteration* is defined by $x_1 \in E$ and

$$\begin{cases} z_n = (1-\gamma_n)\, x_n + \gamma_n f\,(x_n) \\ y_n = (1-\beta_n)\, z_n + \beta_n f\,(z_n) \\ x_{n+1} = (1-\alpha_n)\, f(z_n) + \alpha_n f\,(y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$, and will be denoted by $P(x_1, \alpha_n, \beta_n, \gamma_n, f)$.

In this paper, we give a necessary and sufficient condition for the convergence of the P-iteration of continuous non-decreasing functions on an arbitrary interval. We also prove that if the S-iteration converges, then the P-iteration converges and converges faster than the S-iteration for the class of continuous and non-decreasing functions. Moreover, we present the numerical examples for the P-iteration to compare with the Ishikawa and the S-iterations.

2 Preliminaries

In this section we recall some lemmas , definition , theorems and known results in the existing literature on this concept.

Lemma 2.1 ([3],Lemma 2.1). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $q_1 \in E$, let $\{q_n\}_{n=1}^{\infty}$ be the sequence defined by (1.3). Then the following hold:*

(i) If $f(q_1) < q_1$, then $f(q_n) \leq q_n$ for all $n \geq 1$ and $\{q_n\}_{n=1}^{\infty}$ is non-increasing.

(ii) If $f(q_1) > q_1$, then $f(q_n) \geq q_n$ for all $n \geq 1$ and $\{q_n\}_{n=1}^{\infty}$ is non-decreasing.

Proposition 2.2 ([3],Proposition2.5). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $q_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. If $f(q_1) > q_1$, then the sequence $\{q_n\}$ defined by S-iteration does not converge to a fixed point of f.*

Proposition 2.3 ([3],Proposition 2.6). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $q_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. If $f(q_1) < q_1$, then the sequence $\{q_n\}$ defined by S-iteration does not converge to a fixed point of f.*

Definition 2.4 ([7],Definition 3.1). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{y_n\}_{n=1}^{\infty}$ if $|x_n - p| \leq |y_n - p|$ for all $n \geq 1$.*

Theorem 2.5 ([3],Theorem 2.7). *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded. For $s_1 = q_1 \in E$, let $\{s_n\}$ and $\{q_n\}$ be the sequences defined by (1.2) and (1.3), respectively. If the Ishikawa iteration $\{S_n\}$ converges to $p \in F(f)$, then the S-iteration $\{q_n\}$ converges to p. Moreover, the S-iteration converges faster than the Ishikawa iteration.*

3 Main Results

We first give some useful facts for our main results.

Lemma 3.1. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by P-iteration. Then the following hold:*

(i) If $f(x_1) < x_1$, then $f(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-increasing.

(ii) If $f(x_1) > x_1$, then $f(x_n) \geq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.

Proof. (1) Let $f(x_1) < x_1$. Assume that $f(x_k) \leq x_k$ for $k > 1$. Then $f(x_k) \leq z_k \leq x_k$. Since f is non-decreasing, we have $f(z_k) \leq f(x_k) \leq z_k \leq x_k$. By (1.4), we get $f(z_k) \leq y_k \leq z_k$. Since f is non-decreasing, we have $f(y_k) \leq f(z_k) \leq y_k \leq z_k$. It follows from (1.4), that $f(y_k) \leq x_{k+1} \leq f(z_k)$. This implies $x_{k+1} \leq f(z_k) \leq y_k$. Since f is non-decreasing, we have $f(x_{k+1}) \leq f(y_k)$. Thus $f(x_{k+1}) \leq x_{k+1}$. By induction, we can conclude that $f(x_n) \leq x_n$ for all $n \geq 1$. This together with (1.4), we have $y_n \leq z_n \leq x_n$ for all $n \geq 1$. Since f is non-decreasing, we have $f(y_n) \leq f(z_n) \leq f(x_n)$ for all $n \leq 1$. It follows that, $x_{n+1} = (1-\alpha_n)f(z_n) + \alpha_n f(y_n) \leq f(z_n) \leq f(x_n) \leq x_n$ for all $n \geq 1$. Thus $\{x_n\}$ is non-increasing.
(ii) By using the same argument as in (i), We obtain the desired result. □

Theorem 3.2. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ and $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$. Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.*

Proof. If $\{x_n\}$ is convergent, then it is bounded. Now, assume that $\{x_n\}$ is bounded. we will show that $\{x_n\}$ is convergent. If $f(x_1) = x_1$, by (1.4) we have

$$z_1 = (1-\gamma_1)\, x_1 + \gamma_1 f(x_1) = x_1$$
$$y_1 = (1-\beta_1)\, z_1 + \beta_1 f(z_1) = x_1$$
$$x_2 = (1-\alpha_1)\, f(z_1) + \alpha_1 f(y_1) = x_1.$$

We can show by induction that $x_n = x_1$ for all $n \geq 1$. Thus $\{x_n\}$ is convergent. Suppose that $f(x_1) \neq x_1$,$f(x_1) < x_1$ or $f(x_1) > x_1$. By Lemma 3.1, we obtain that $\{x_n\}$ is non-increasing or non-decreasing. Since $\{x_n\}$ is bounded, it implies that $\{x_n\}$ is convergent. Next, we prove that $\{x_n\}$ converges to a fixed point of f. Let $\lim_{n\to\infty} x_n = p$ for some $p \in E$. By continuity of f and $\{x_n\}$ is bounded, we have $\{f(x_n)\}$ is bounded. By (1.4), we obtain $z_n = (1-\gamma_n)x_n + \gamma_n f(x_n) = x_n + \gamma_n(f(x_n) - x_n)$. Since $\lim_{n\to\infty} \gamma_n = 0$, we have $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = p$. By continuity of f and $\{x_n\}$ is bounded, we have $\{z_n\}$ and $f(z_n)$ are bounded.

By (1.4) , we get $y_n = (1-\beta_n)z_n + \beta_n f(z_n) = z_n + \beta_n(f(z_n) - z_n)$.
Since $\lim_{n\to\infty} \beta_n = 0$, we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = p$.
By continuity of f , we have $\lim_{n\to\infty}(f(y_n) - f(z_n)) = f(p) - f(p) = 0$.
From $x_{n+1} = f(z_n) + \alpha_n(f(y_n) - f(z_n))$ and continuity of f , we have

$$\begin{aligned} p &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} f(z_n) + \lim_{n\to\infty} \alpha_n(f(y_n) - f(z_n)) \\ &= \lim_{n\to\infty} f(z_n) \\ &= f(p). \end{aligned}$$

Hence p is a fixed point of f and $\{x_n\}$ converge to p. □

Lemma 3.3. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the P-iteration defined by (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in* $[0,1]$. *Then we have the following :*

(i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.

(ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Proof. (i) Suppose that $p \in F(f)$ and $x_1 > p$. Since f is non-decreasing, we have $f(x_1) \geq f(p) = p$. By (1.4), we get

$$z_1 = (1-\gamma_1)x_1 + \gamma_1 f(x_1) \geq (1-\gamma_1)p + (\gamma_1)p = p.$$

Thus $f(z_1) \geq f(p) = p$. From (1.4), we have

$$y_1 = (1-\beta_1)z_1 + \beta_1 f(z_1) \geq (1-\beta_1)p + (\beta_1)p = p.$$

Thus $f(y_1) \geq f(p) = p$. Again (1.4), implies that

$$x_2 = (1-\alpha_1)f(z_1) + \alpha_1 f(y_1) \geq (1-\alpha_1)p + (\alpha_1)p = p.$$

Assume that $x_k \geq p$ for $k > 2$. Thus $f(x_k) \geq f(p) = p$.
By (1.4) , we have $z_k = (1-\gamma_k)x_k + \gamma_k f(x_k) \geq (1-\gamma_k)p + (\gamma_k)p = p$.
Thus $f(z_k) \geq f(p) = p$. This implies $y_k = (1-\beta_k)z_k + \beta_k f(z_k) \geq (1-\beta_k)p + \beta_k p = p$.

Hence $f(y_k) \geq f(p) = p$. It follows that

$$x_{k+1} = (1-\alpha_k)f(z_k) + \alpha_k f(y_k) \geq (1-\alpha_k)p + \alpha_k p = p.$$

By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$.
(ii) By using the same argument as in (i), we can show that $x_n \leq p$ for all $n \geq 1$. □

Lemma 3.4. *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. For $x_1 \in E$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 = q_1 \in E$, let $\{q_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be sequences defined by (1.3) and (1.4) respectively. Then we have the following :*

(i) If $f(q_1) < q_1$, then $x_n \leq q_n$ for all $n \geq 1$.

(ii) If $f(q_1) > q_1$, then $x_n \geq q_n$ for all $n \geq 1$.

Proof. (i) Let $f(q_1) < q_1$. Since $x_1 = q_1$, we get $f(x_1) < x_1$. First, we show that $x_n \leq q_n$ for all $n \geq 1$.
From (1.4), we get $f(x_1) \leq z_1 \leq x_1$. Since f is non-decreasing, we have
$f(z_1) \leq f(x_1) \leq z_1 \leq x_1$.
By (1.4), we have $f(z_1) \leq y_1 \leq z_1$. Since f is non-decreasing, we obtain
$f(y_1) \leq f(z_1) \leq y_1 \leq z_1 \leq x_1$.

From (1.3) and (1.4), we get $z_1-q_1=(1-\gamma_1)x_1+\gamma_1 f(x_1)-q_1=\gamma_1(f(x_1)-x_1)\leq 0$. Thus $z_1\leq q_1$. Since f is non-decreasing, we have $f(z_1)\leq f(q_1)$.
By (1.3) and (1.4), we get $y_1-r_1=(1-\beta_1)(z_1-q_1)+\beta_1(f(z_1)-f(q_1))\leq 0$. Thus $y_1\leq r_1$. Since f is non-decreasing, we have $f(y_1)\leq f(r_1)$. By (1.3) and (1.4), it follows that

$$x_2-q_2=(1-\alpha_1)[f(z_1)-f(q_1)]+\alpha_1[f(y_1)-f(r_1)]\leq 0.$$

Thus $x_2\leq q_2$. Assume that $x_k\leq q_k$. Thus $f(x_k)\leq f(q_k)$. By Lemma 2.1 $f(q_k)\leq q_k$ and by (1.4), Lemma 3.1 $f(x_k)\leq x_k$. This implies $f(x_k)\leq z_k\leq x_k\leq q_k$. Since f is non-decreasing, we have $f(z_k)\leq f(q_k)$. By (1.3) and (1.4), it follows that

$$y_k-r_k=(1-\beta_k)(z_k-q_k)+\beta_k(f(z_k)-f(q_k))\leq 0.$$

Thus $y_k\leq r_k$. Since f is non-decreasing, we have $f(y_k)\leq f(r_k)$ it follows that $x_{k+1}-q_{k+1}=(1-\alpha_k)[f(z_k)-f(q_k)]+\alpha_k[f(y_k)-f(r_k)]\leq 0$. By Mathematical induction, we obtain $x_n\leq q_n$ for all $n\geq 1$.
(ii) By using the same argument as in (i), we obtain the desired result. □

The next two propositions show that convergence of P-iteration depends on how far the initial point from the fixed point set.

Proposition 3.5. *Let E be a closed interval on the real line and $f:E\to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1<\inf\{p\in E:p=f(p)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1)<x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of f.*

Proof. By Lemma 3.1(i), we have that $\{x_n\}$ is non-increasing. Since the initial point $x_1<\inf\{p\in E:p=f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Proposition 3.6. *Let E be a closed interval on the real line and $f:E\to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1>\sup\{p\in E:p=f(p)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. If $f(x_1)>x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of f.*

Proof. By Lemma 3.1(ii), we have that $\{x_n\}$ is non-decreasing. Since the initial point $x_1>\sup\{p\in E:p=f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Theorem 3.7. *Let E be a closed interval on the real line and $f:E\to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded. For $q_1=x_1\in E$, let $\{q_n\}$ and $\{x_n\}$ be the sequences defined by (1.3) and (1.4), respectively. If the S-iteration $\{q_n\}$ converges to $p\in F(f)$, then the P-iteration $\{x_n\}$ converges to p. Moreover, the P-iteration converges faster than the S- iteration.*

Proof. Suppose the S-iteration $\{q_n\}$ converges to $p\in F(f)$. Put $l=\inf\{x\in E: x=f(x)\}$ and $u=\sup\{x\in E:x=f(x)\}$. We devide our proof into the following three cases:

Case 1: $q_1 = x_1 > u$. By Proposition 2.2 and Proposition 3.6, we get $f(q_1) < q_1$ and $f(x_1) < x_1$. By Lemma 3.4 (i), we have $x_n \leq q_n$ for all $n \geq 1$. By continuity of f, we have $f(u) = u$, so $u = f(u) \leq f(x_1) < x_1$. This implies by (1.4) that $f(x_1) \leq z_1 \leq x_1$, so $u \leq z_1 \leq x_1$. Since f is non-decreasing, we have $u = f(u) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1$. It follows by (1.4), that $y_1 = (1-\beta_1)z_1 + \beta_1 f(z_1) \leq z_1$. Since f is non-decreasing, we have $u \leq f(y_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1$ and $u \leq f(y_1) \leq x_2 \leq f(z_1)$. By mathematical induction, we can show that $u \leq x_n$ for all $n \geq 1$. Hence , we have $p \leq x_n \leq q_n$ for all $n \geq 1$, which implies $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Thus $x_n \to p$ and the P-iteration converges to p faster than the S- iteration.

Case 2: $q_1 = x_1 < l$. By Proposition 2.3 and Proposition 3.5 , we get $f(q_1) > q_1$ and $f(x_1) > x_1$. By Lemma 3.4 (ii), we have $x_n \geq q_n$ for all $n \geq 1$. We note that $x_1 < l$, by (1.4) and mathematical induction, we can show that $x_n < l$ for all $n \geq 1$. So $q_n \leq x_n \leq p$ for all $n \geq 1$. Hence $|x_n - p| \leq |q_n - p|$. It follows that $x_n \to p$ and the P-iteration converges to p faster than the S-iteration.

Case 3: $l < q_1 = x_1 < u$. Suppose that $f(x_1) \neq x_1$. If $f(x_1) < x_1$, by Lemma 2.1 (i) , we have that $\{q_n\}$ is non-increasing. It follows that $p \leq q_n$ for all $n \geq 1$. By Lemma 3.3 (i) and Lemma 3.4 (i) , we get $p \leq x_n \leq q_n$ for all $n \geq 1$, This implies $|x_n - p| \leq |q_n - p|$. It follows that $x_n \to p$ and the P-iteration converges to p faster than the S-iteration.

If $f(x_1) > x_1$, by Lemma 2.1 (ii), we have that $\{q_n\}$ is non-decreasing. This implies $q_n \leq p$ for all $n \geq 1$. By Lemma 3.3 (ii) and Lemma 3.4 (ii), we get $q_n \leq x_n \leq p$ for all $n \geq 1$. It follows that $|x_n - p| \leq |q_n - p|$ for all $n \geq 1$. Hence $x_n \to p$ and the P-iteration converges to p faster than the S-iteration. □

Example 3.8. *Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = \frac{x^2+3}{4}$. Then f is a continuous and non-decreasing function. The comparisons of the convergence of the Ishikawa iteration, S-iteration and the P-iteration to the exact fixed point $p = 1$ are given in Table 1, with the initial point $x_1 = q_1 = s_1 = 2$ and $\alpha_n = \frac{1}{n}$, $\beta_n = \gamma_n = \frac{1}{2n}$.*

	Ishikawa	S-iteration	P-iteration	
n	s_n	q_n	x_n	$\lvert f(x_n) - x_n\rvert$
3	1.540872070	1.228210401	1.150245305	0.129225591
⋮	⋮	⋮	⋮	⋮
26	1.364330695	1.000000032	1.000000002	2.56567E-09
27	1.361571178	1.000000016	1.000000001	1.23348E-09
28	1.358927101	1.000000008	1.000000001	5.93889E-09

Table 1

Comparison of rate of convergence of the Ishikawa iteration, S-iteration and P-iteration for the given function in Example 3.8. From Table 1, we see that the P-iteration converges to $p = 1$ faster than the Ishikawa and S-iterations.

Example 3.9. *Let* $f : [0,5] \to [0,5]$ *be defined by* $f(x) = \sqrt[3]{x^2+4}$. *Then* f *is a continuous and non-decreasing function. The comparisons of the convergence of the Ishikawa iteration, S-iteration and the P-iteration to the exact fixed point* $p = 2$ *are given in Table 2, with the initial point* $x_1 = q_1 = s_1 = 3$ *and* $\alpha_n = \beta_n = \gamma_n = \frac{1}{n}$.

	Ishikawa	S-iteration	P-iteration	
n	s_n	q_n	x_n	$\lvert f(x_n) - x_n\rvert$
3	2.055372105	2.010415225	2.001802129	0.005000393
⋮	⋮	⋮	⋮	⋮
12	2.021489268	2.000000462	2.000000033	6.9311E-08
13	2.020359903	2.000000153	2.000000010	2.17191E-09
14	2.019368059	2.000000051	2.000000003	6.84134E-09
15	2.018488772	2.000000017	2.000000001	2.16447E-09

Table 2

Comparison of rate of convergence of the Ishikawa iteration, S-iteration and P-iteration for the given function in Example 3.9. From Table 2, we see that the P-iteration converges to $p = 2$ faster than the Ishikawa and S-iterations.

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