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Equivalence Problem for the Canonical Form of Linear Second Order Parabolic Equations[1]

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Abstract : The article is devoted to the equivalence problem for the class of linear second order parabolic equations, the first

$$u_t = u_{xx} + a(x)u, \tag{0.1}$$

and

$$u_t = u_{xx} + \frac{k}{x^2}u \tag{0.2}$$

k a nonzero constant. Conditions which the parabolic equation

$$a_1(t,x)\,u_t + a_2(t,x)\,u_x + a_3(t,x)\,u + u_{xx} = 0 \tag{0.3}$$

to be equivalent to (0.1)-(0.2) are obtained.

Keywords : Parabolic equations, equivalence transformation, equivalence problem, semi-invariants, invariants
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1 Introduction

1.1 Equivalence transformations

The equivalence problem of a linear second-order parabolic partial differential equations in two independent variables

$$a_1(t,x)u_t + a_2(t,x)u_x + a_3(t,x)u + u_{xx} = 0 \tag{1.1}$$

is considered in the paper. Recall that the well-known group of equivalence transformations for Eq. (1.1) (given in [1]), i.e. the changes of the independent variables t, x and the dependent variable u that do not change the form of Eq. (1.1), is composed of the linear transformation of the dependent variable

$$v = u/V(t,x) \tag{1.2}$$

and the following change of the independent variables:

$$\tau = H(t), \quad y = Y(t,x), \tag{1.3}$$

where $V(t,x), H(t)$ and $Y(t,x)$ are arbitrary functions obeying the invertibility conditions, $V(t,x) \neq 0$, $H'(t) \neq 0$ and $Y_x(t,x) \neq 0$. The form invariance of Equation (1.1) means that the transformations (1.2)–(1.3) map Eq. (1.1) into an equation of the same form:

$$b_1(\tau,y)v_\tau + b_2(\tau,y)v_y + b_3(\tau,y)v - v_{yy} = 0 \tag{1.4}$$

Equations (1.1) and (1.4), related by an equivalence transformation, are called equivalent equations. Lie [1] also obtained the classification of a linear second-order partial differential equations (1.1). Ovsiannikov [2] studied the group classification of a nonlinear parabolic equation. It was shown that Ibragimov [3] found first and second order semi-invariants of a parabolic partial differential equation (1.1). In [4] Johnpillai and Mahomed showed that there are no first, second, third and fourth order invariants other than constant and they obtained one relative invariant. Sixth and seventh-order differential invariants of linear second-order parabolic partial differential equation (1.1) under an action of the equivalence group of point transformations (1.2)–(1.3) were found in [5]. The paper [6] gives an extension of Euler's method to linear parabolic equations with two independent variables. The new method allowed deriving an explicit formula for the general solution of a wide class of parabolic equations. Morozov [7] studied invariants of contact transformations for linear parabolic equations.

1.2 Canonical parabolic equations

According to Lie's classification [1], the canonical forms of linear second-order parabolic partial differential equations (1.4) are the heat equation

$$u_t = u_{xx}, \tag{1.5}$$

the equation

$$u_t = u_{xx} + a(x)u, \tag{1.6}$$

with arbitrary function $a(x) \neq \frac{k}{x^2}$ and the equation

$$u_t = u_{xx} + \frac{k}{x^2}u, \tag{1.7}$$

where k a nonzero constant. Conditions which the parabolic equation

$$b_1(\tau, y)v_\tau + b_2(\tau, y)v_y + b_3(\tau, y)v - v_{yy} = 0 \tag{1.8}$$

to be equivalent to (1.6)-(1.7) are obtained.

Equivalence problem for the heat equation was studied by Johnpillai and Mahomed [4]. Criterion for an equation (1.1) to be equivalent to the heat equation (1.5) was obtained by Johnpillai and Mahomed

$$\lambda = 0.$$

The quantity λ is defined by

$$\begin{aligned}\lambda = \ & (-8b_{1\tau yyy}b_1^5 + 36b_{1\tau yy}b_{1y}b_1^4 - 4b_{1\tau y}b_{1\tau}b_1^5 + 28b_{1\tau y}b_{1yy}b_1^4 \\ & -80b_{1\tau y}b_{1y}^2b_1^3 + 4b_{1\tau\tau y}b_1^6 - 4b_{1\tau\tau}b_{1y}b_1^5 + 4b_{1\tau}^2b_{1y}b_1^4 + 8b_{1\tau}b_{1yyy}b_1^4 \\ & -64b_{1\tau}b_{1yy}b_{1y}b_1^3 + 80b_{1\tau}b_{1y}^3b_1^2 + 4b_{1yyyyy}b_1^4 - 40b_{1yyyy}b_{1y}b_1^3 \\ & -64b_{1yyy}b_{1yy}b_1^3 + 220b_{1yyy}b_{1y}^2b_1^2 + 288b_{1yy}^2b_{1y}b_1^2 - 810b_{1yy}b_{1y}^3b_1 \\ & +12b_{1yy}b_1^7k + 405b_{1y}^5 + 20b_{1y}K_yb_1^7 + 8K_{yy}b_1^8)/b_1^{10},\end{aligned} \tag{1.9}$$

where

$$K = (2b_{1y}b_{2y} - b_{1y}b_2^2 - 4b_{1y}b_3 + 2b_{2\tau}b_1^2 - 2b_{2yy}b_1 + 2b_{2y}b_1b_2 + 4b_{3y}b_1)/(2b_1^4).$$

The present paper is devoted to obtain conditions for equation (1.1) to be equivalent to (1.6) and (1.7).

2 Preliminaries

For this we suppose that $u_0(t, x)$ is a given function. Substituting $u_0(t, x)$ into (1.2), one obtains

$$v_0(t, x) = u_0(t, x)/V(t, x). \tag{2.1}$$

By virtue of the inverse function theorem, there exist functions $T(\tau, y)$, $X(\tau, y)$ such that

$$t = T(\tau, y), \ x = X(\tau, y). \tag{2.2}$$

After substituting (2.2) into equation (1.2), one obtains the transformation of the function $u_0(t, x)$:

$$v_0(\tau, y) = u_0(t, x)/V(t, x),$$

where t and x are defined by (2.2). Notice that the function $v_0(\tau, y)$ satisfies the relation

$$v_0(H(t), Y(t,x)) = u_0(t,x)/V(t,x). \tag{2.3}$$

Differentiating equation (2.3) with respect to t and x, one gets

$$v_{0\tau}H' + v_{0y}Y_t = (u_0/V)_t, \quad v_{0y}Y_x = (u_0/V)_x. \tag{2.4}$$

Solving linear system (2.4) with respect to the derivatives $v_{0\tau}$ and v_{0y}, one has

$$\begin{aligned} v_{0\tau}(\tau, y) &= \triangle^{-1}(Y_t(u_0/V)_x - Y_x(u_0/V)_t), \\ v_{0y}(\tau, y) &= -\triangle^{-1}H'(u_0/V)_x, \end{aligned} \tag{2.5}$$

where it assumed that $\triangle = -H'Y_x \neq 0$. Differentiating second equation (2.5) with respect to t and x, one obtains

$$v_{0y\tau}H' + v_{0yy}Y_t = A_1, \quad v_{0yy}Y_x = A_2, \tag{2.6}$$

where

$$A_1 = ((H'(u_0/V)_x)\triangle_t - \triangle(H'(u_0/V)_x)_t)/\triangle^2,$$
$$A_2 = ((H'(u_0/V)_x)\triangle_x - \triangle(H'(u_0/V)_x)_x)/\triangle^2.$$

Hence, the derivative v_{yy} is

$$v_{0yy} = \triangle^{-1}A_2H'. \tag{2.7}$$

That is the equation (1.8) become (1.1), where

$$\begin{aligned} a_1 = &\ \Delta^{-1}Y_x^3b_1, \\ a_2 = &\ -\Delta^{-1}V^{-1}(2H'V_xY_x - H'Y_{xx}V - H'Y_x^2Vb_2 + Y_tY_x^2Vb_1), \\ a_3 = &\ -\Delta^{-1}V^{-1}(H'V_{xx}Y_x - H'V_xY_{xx} - H'V_xY_x^2b_2 - H'Y_x^3b_3V \\ &\ -V_tY_x^3b_1 + V_xY_tY_x^2b_1). \end{aligned} \tag{2.8}$$

3 Main Results

3.1 Equivalence problem for equation $u_t = u_{xx} + a(x)u$

This section studies equations (1.8) which are equivalent to equation (1.6). Since for equation (1.6)

$$a_1 = -1, \quad a_2 = 0, \quad a_3 = a\,(x)\,. \tag{3.1}$$

Hence equation (2.8) become

$$\begin{aligned} 1 = &\ -\Delta^{-1}Y_x^3b_1, \\ 0 = &\ 2H'V_xY_x - H'Y_{xx}V - H'Y_x^2Vb_2 + Y_tY_x^2Vb_1, \\ a = &\ -\Delta^{-1}V^{-1}(H'V_{xx}Y_x - H'V_xY_{xx} - H'V_xY_x^2b_2 - H'Y_x^3b_3V \\ &\ -V_tY_x^3b_1 + V_xY_tY_x^2b_1). \end{aligned} \tag{3.2}$$

The problem is to find conditions for the coefficients $b_1(\tau, y)$, $b_2(\tau, y)$, $b_3(\tau, y)$ which guarantee existence of the functions $H(t)$, $Y(t,x)$, $V(t,x)$ transforming the coefficients of (1.4) into (3.1). Solution of this problem consists of the analysis of compatibility of (3.2).

From the first equation of equation (3.2), one has

$$H' = b_1 Y_x^2. \tag{3.3}$$

Then

$$Y_{xx} = -b_{1y} Y_x^2/(2b_1). \tag{3.4}$$

The second equation and the third equation of equation (3.2) can be solved with respect to Y_t and V_{xx}:

$$Y_t = (-4V_x Y_x b_1 + Y_x^2 V(-b_{1y} + 2b_1 b_2))/(2b_1 V), \tag{3.5}$$

$$\begin{aligned} V_{xx} = \quad & (-b_{1y} V_x Y_x^2 + 2V_t Y_x b_1 - 2V_x Y_t b_1 + 2V_x Y_x^2 b_1 b_2 \\ & +2Y_x^3 b_1 b_3 V + 2Y_x a b_1 V)/(2Y_x b_1). \end{aligned} \tag{3.6}$$

Comparing the mixed derivatives $(Y_t)_{xx} - (Y_{xx})_t = 0$, one finds

$$\begin{aligned} V_{tx} = \quad & (4V_t Y_t Y_x b_1^3 + 2V_t Y_x^3 b_1^2 (b_{1y} - 2b_1 b_2) - 4V_x Y_t^2 b_1^3 + 4V_x Y_t Y_x^2 b_1^2 (-b_{1y} \\ & +2b_1 b_2) + V_x Y_x^4 b_1 (-4b_{1yy} b_1 + 7b_{1y}^2 + 4b_{1y} b_1 b_2 - 4b_1^2 b_2^2 - 8b_1^2 b_3) \\ & -8V_x Y_x^2 a b_1^3 + 2Y_t Y_x^3 b_1 V(-b_{1yy} b_1 + 2b_{1y}^2 + 2b_1^2 b_3) + 4Y_t Y_x a b_1^3 V \\ & +Y_x^5 V(-2b_{1yyy} b_1^2 + 9b_{1yy} b_{1y} b_1 - 7b_{1y}^3 - 6b_{1y} b_{2y} b_1^2 + 10b_{1y} b_1^2 b_3 \\ & +4b_{2yy} b_1^3 - 8b_{3y} b_1^3 - 4b_1^3 b_2 b_3) + 2Y_x^3 b_1 V(b_{1\tau y} H' b_1 - b_{1\tau} b_{1y} H' \\ & +b_{1y} a b_1 - 2a b_1^2 b_2) - 8Y_x^2 a_x b_1^3 V)/(8Y_x^2 b_1^3). \end{aligned} \tag{3.7}$$

The equation $(V_{xx})_t - (V_{tx})_x = 0$ gives

$$\begin{aligned} V_{tt} = \; & (-32V_t^2Y_x^3b_1^4V + 128V_tV_x^2Y_x^3b_1^4 + 80V_tV_xY_tY_x^2b_1^4V + 8V_tV_xY_x^4b_1^3V(3b_{1y} \\ & -10b_1b_2) + 8V_tY_t^2Y_xb_1^4V^2 - 16V_tY_tY_x^3b_1^4b_2V^2 + 2V_tY_x^5b_1^2V^2(-b_{1y}^2 \\ & +4b_1^2b_2^2) - 128V_tY_x^3ab_1^4V^2 - 64V_x^3Y_tY_x^2b_1^4 + 32V_x^3Y_x^4b_1^3(b_{1y} + 2b_1b_2) \\ & -32V_x^2Y_t^2Y_xb_1^4V + 32V_x^2Y_tY_x^3b_1^3V(b_{1y} + 2b_1b_2) + 8V_x^2Y_x^5b_1^2V(8b_{1yy}b_1 \\ & -9b_{1y}^2 - 4b_{1y}b_1b_2 - 8b_{2y}b_1^2 - 4b_1^2b_2^2 + 24b_1^2b_3) + 64V_x^2Y_x^3b_1^3V(-b_{1\tau}H' \\ & +3ab_1) - 8V_xY_t^3b_1^4V^2 + 4V_xY_t^2Y_x^2b_1^3V^2(b_{1y} + 6b_1b_2) + 2V_xY_tY_x^4b_1^2V^2 \\ & (4b_{1yy}b_1 + 5b_{1y}^2 - 4b_{1y}b_1b_2 - 16b_{2y}b_1^2 - 12b_1^2b_2^2 + 24b_1^2b_3) + 16V_xY_tY_x^2b_1^3V^2 \\ & (-2b_{1\tau}H' + 7ab_1) + V_xY_x^6b_1V^2(-8b_{1yyy}b_1^2 + 68b_{1yy}b_{1y}b_1 - 73b_{1y}^3 \\ & -26b_{1y}^2b_1b_2 + 8b_{1y}b_{2y}b_1^2 + 4b_{1y}b_1^2b_2^2 + 72b_{1y}b_1^2b_3 - 16b_{2yy}b_1^3 + 32b_{2y}b_1^3b_2 \\ & +8b_1^3b_2^3 - 48b_1^3b_2b_3) + 8V_xY_x^4b_1^2V^2(-b_{1\tau y}H'b_1 - b_{1\tau}b_{1y}H' + 4b_{1\tau}H'b_1b_2 \\ & +5b_{1y}ab_1 - 14ab_1^2b_2) + 4Y_t^2Y_x^3b_1^2V^3(-3b_{1yy}b_1 + 6b_{1y}^2 + 2b_1^2b_3) \\ & +8Y_t^2Y_xab_1^4V^3 + 2Y_tY_x^5b_1V^3(-10b_{1yyy}b_1^2 + 50b_{1yy}b_{1y}b_1 + 6b_{1yy}b_1^2b_2 \\ & -43b_{1y}^3 - 12b_{1y}^2b_1b_2 - 18b_{1y}b_{2y}b_1^2 + 48b_{1y}b_1^2b_3 + 12b_{2yy}b_1^3 - 32b_{3y}b_1^3 \\ & -8b_1^3b_2b_3) + 4Y_tY_x^3b_1^2V^3(3b_{1\tau y}H'b_1 - 3b_{1\tau}b_{1y}H' - 4ab_1^2b_2) - 32Y_tY_x^2a_xb_1^4V^3 \\ & +Y_x^7V^3(-8b_{1yyyy}b_1^3 + 50b_{1yyy}b_{1y}b_1^2 + 12b_{1yyy}b_1^3b_2 + 40b_{1yy}^2b_1^2 - 195b_{1yy}b_{1y}^2b_1 \\ & -54b_{1yy}b_{1y}b_1^2b_2 - 32b_{1yy}b_{2y}b_1^3 + 64b_{1yy}b_1^3b_3 + 113b_{1y}^4 + 42b_{1y}^3b_1b_2 \\ & +58b_{1y}^2b_{2y}b_1^2 - 98b_{1y}^2b_1^2b_3 - 36b_{1y}b_{2yy}b_1^3 + 36b_{1y}b_{2y}b_1^3b_2 + 64b_{1y}b_{3y}b_1^3 \\ & -32b_{1y}b_1^3b_2b_3 + 16b_{2yyy}b_1^4 - 24b_{2yy}b_1^4b_2 - 64b_{2y}b_1^4b_3 - 32b_{3yy}b_1^4 \\ & +32b_{3y}b_1^4b_2 + 8b_1^4b_2^2b_3 + 64b_1^4b_3^2) + 2Y_x^5b_1V^3(4b_{1\tau yy}H'b_1^2 - 7b_{1\tau y}b_{1y}H'b_1 \\ & -6b_{1\tau y}H'b_1^2b_2 - 4b_{1\tau}b_{1yy}H'b_1 + 7b_{1\tau}b_{1y}^2H' + 6b_{1\tau}b_{1y}H'b_1b_2 - b_{1y}^2ab_1 \\ & -16b_{3\tau}H'b_1^3 + 4ab_1^3b_2^2) + 16Y_x^4a_xb_1^3V^3(-b_{1y} + 2b_1b_2) \\ & +32Y_x^3b_1^4V^3(-a_{xx} - 2a^2))/(32Y_x^3b_1^4V^2). \end{aligned} \tag{3.8}$$

Equating $(V_{tt})_x = (V_{tx})_t$, one obtains

$$a_{xxx} = (Y_x^5\lambda_3)/(16b_1^5), \tag{3.9}$$

where $\lambda_3 = -b_1^{10}\lambda$. Notice that by virtue of $\lambda \neq 0$, one has $\lambda_3 \neq 0$. Because a does not depend on t, differentiating (3.9) with respect to t, one has

$$\begin{aligned} V_t = \; & (-20V_x^2b_1^2\lambda_3 + 2V_xY_xb_1V(15b_{1y}\lambda_3 - 2\lambda_{3y}b_1) + Y_x^2\lambda_4V^2 \\ & -20ab_1^2\lambda_3V^2)/(20b_1^2\lambda_3V), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \lambda_2 = \; & -b_{1yy}b_1 + 2b_{1y}^2 - 2b_{1y}b_1b_2 + 2b_{2y}b_1^2 - 4b_1^2b_3, \\ \lambda_4 = \; & -10b_{1\tau}b_1^2\lambda_3 + 5b_{1y}^2\lambda_3 - b_{1y}\lambda_{3y}b_1 - 10b_{1y}b_1b_2\lambda_3 + 2\lambda_{3\tau}b_1^3 \\ & +2\lambda_{3y}b_1^2b_2 + 5\lambda_2\lambda_3. \end{aligned}$$

Substitution of V_t into (3.7) and (3.8) gives

$$\begin{aligned} a_{xx} = \; & (-16V_x^2Y_x^2b_1^2\lambda_5 - 8V_xY_x^3b_1\lambda_6V + Y_x^4\lambda_7V^2 + 40Y_xa_xb_1^3\lambda_3V^2 \\ & (15b_{1y}\lambda_3 - 2\lambda_{3y}b_1))/(400b_1^4\lambda_3^2V^2), \end{aligned} \tag{3.11}$$

$$4V_xb_1\lambda_5 + Y_x\lambda_6V = 0, \tag{3.12}$$

where

$$\begin{aligned}\lambda_1 = \;& 2b_{1\tau y}b_1^3 - 2b_{1\tau}b_{1y}b_1^2 - 2b_{1yyy}b_1^2 + 10b_{1yy}b_{1y}b_1 - 2b_{1yy}b_1^2b_2 - 9b_{1y}^3\\ &+4b_{1y}^2b_1b_2 - 6b_{1y}b_{2y}b_1^2 + 8b_{1y}b_1^2b_3 + 4b_{2yy}b_1^3 - 8b_{3y}b_1^3,\\ \lambda_5 = \;& -150b_{1y}^2\lambda_3^2 + 25b_{1y}\lambda_{3y}b_1\lambda_3 + 150b_{1y}b_1b_2\lambda_3^2 - 150b_{2y}b_1^2\lambda_3^2\\ &+10\lambda_{3yy}b_1^2\lambda_3 - 12\lambda_{3y}^2b_1^2 + 300b_1^2b_3\lambda_3^2 + 75\lambda_2\lambda_3^2,\end{aligned}$$

$$\begin{aligned}\lambda_6 = \;& -300b_{1y}b_1^2b_3\lambda_3^2 + 15b_{1y}\lambda_3\lambda_4 + 40\lambda_{3y}b_1^3b_3\lambda_3 + 12\lambda_{3y}b_1\lambda_4,\\ &-10\lambda_{4y}b_1\lambda_3 + 25\lambda_1\lambda_3^2,\\ \lambda_7 = \;& 140b_{1\tau}b_1^2\lambda_3\lambda_4 - 300b_{1y}^2b_1^2b_3\lambda_3^2 - 55b_{1y}^2\lambda_3\lambda_4 + 200b_{1y}b_{3y}b_1^3\lambda_3^2,\\ &+40b_{1y}\lambda_{3y}b_1^3b_3\lambda_3 + 12b_{1y}\lambda_{3y}b_1\lambda_4 + 600b_{1y}b_1^3b_2b_3\lambda_3^2\\ &+110b_{1y}b_1b_2\lambda_3\lambda_4 - 275b_{1y}\lambda_1\lambda_3^2 - b_{1y}\lambda_6 - 400b_{3\tau}b_1^5\lambda_3^2\\ &-400b_{3y}b_1^4b_2\lambda_3^2 + 50\lambda_{1y}b_1\lambda_3^2 - 80\lambda_{3y}b_1^4b_2b_3\lambda_3 - 24\lambda_{3y}b_1^2b_2\lambda_4\\ &+10\lambda_{3y}b_1\lambda_1\lambda_3 - 20\lambda_{4\tau}b_1^3\lambda_3 - 800b_1^4b_3^2\lambda_3^2 - 400b_1^2b_3\lambda_2\lambda_3^2\\ &-50b_1b_2\lambda_1\lambda_3^2 + 2b_1b_2\lambda_6 - 70\lambda_2\lambda_3\lambda_4 + 12\lambda_4^2.\end{aligned}$$

Differentiating (3.11) with respect to t, one has

$$\begin{aligned}&64V_x^3Y_x^3b_1^3(15b_{1y}\lambda_3\lambda_5 - 14\lambda_{3y}b_1\lambda_5 + 5\lambda_{5y}b_1\lambda_3)\\ &+16V_x^2Y_x^4b_1^2V(120b_{1\tau}b_1^2\lambda_3\lambda_5 - 60b_{1y}^2\lambda_3\lambda_5 + 5b_{1y}\lambda_{5y}b_1\lambda_3\\ &+120b_{1y}b_1b_2\lambda_3\lambda_5 + 15b_{1y}\lambda_3\lambda_6 - 28\lambda_{3y}b_1\lambda_6 - 10\lambda_{5\tau}b_1^3\lambda_3\\ &-10\lambda_{5y}b_1^2b_2\lambda_3 + 10\lambda_{6y}b_1\lambda_3 + 40b_1^2b3\lambda_3\lambda_5 - 60\lambda_2\lambda_3\lambda_5 + 14\lambda_4\lambda_5)\\ &+4V_xY_x^5b_1V^2(260b_{1\tau}b_1^2\lambda_3\lambda_6 - 130b_{1y}^2\lambda_3\lambda_6 + 10b_{1y}\lambda_{6y}b_1\lambda_3\\ &+260b_{1y}b_1b_2\lambda_3\lambda_6 + 14\lambda_{3y}b_1\lambda_7 - 20\lambda_{6\tau}b_1^3\lambda_3 - 20\lambda_{6y}b_1^2b_2\lambda_3 - 5\lambda_{7y}b_1\lambda_3\\ &+40b_1^2b3\lambda_3\lambda_6 - 10\lambda_1\lambda_3\lambda_5 - 130\lambda_2\lambda_3\lambda_6 + 28\lambda_4\lambda_6) + 480V_xY_x^2a_xb_1^4\lambda_3\lambda_5V^2\\ &+Y_x^6V^3(-140b_{1\tau}b_1^2\lambda_3\lambda_7 + 70b_{1y}^2\lambda_3\lambda_7 - 5b_{1y}\lambda_{7y}b_1\lambda_3 - 140b_{1y}b_1b_2\lambda_3\lambda_7\\ &+10\lambda_{7\tau}b_1^3\lambda_3 + 10\lambda_{7y}b_1^2b_2\lambda_3 - 10\lambda_1\lambda_3\lambda_6 + 70\lambda_2\lambda_3\lambda_7 - 14\lambda_4\lambda_7)\\ &+120Y_x^3a_xb_1^3\lambda_3\lambda_6V^3 = 0.\end{aligned} \tag{3.13}$$

Substituting a_{xx} into (3.9), one finds

$$\begin{aligned}&16V_x^2Y_xb_1^2(15b_{1y}\lambda_3\lambda_5 - 26\lambda_{3y}b_1\lambda_5 + 10\lambda_{5y}b_1\lambda_3) + 8V_xY_x^2b_1V\\ &(-15b_{1y}\lambda_3\lambda_6 - 24\lambda_{3y}b_1\lambda_6 + 10\lambda_{6y}b_1\lambda_3 + 40b_1^2b_3\lambda_3\lambda_5 + 2\lambda_4\lambda_5)\\ &+Y_x^3V^2(45b_{1y}\lambda_3\lambda_7 + 22\lambda_{3y}b_1\lambda_7 - 10\lambda_{7y}b_1\lambda_3 + 80b_1^2b_3\lambda_3\lambda_6\\ &+250\lambda_3^4 + 4\lambda_4\lambda_6) + 80a_xb_1^3\lambda_3\lambda_5V^2 = 0.\end{aligned} \tag{3.14}$$

Further study depends on the value of λ_5.

Case 1 : $\lambda_5 \neq 0$.
From equations (3.12) and (3.14), one finds

$$V_x = -Y_x\lambda_6V/(4b_1\lambda_5) \tag{3.15}$$

$$a_x = Y_x^3\lambda_8/(80b_1^3\lambda_3\lambda_5^3), \tag{3.16}$$

where

$$\begin{aligned}\lambda_8 = \;& 10\lambda_{7y}b_1\lambda_3\lambda_5^2 - 45b_{1y}\lambda_3\lambda_5\lambda_6^2 - 22\lambda_{3y}b_1\lambda_5^2\lambda_7 - 22\lambda_{3y}b_1\lambda_5\lambda_6^2\\ &-10\lambda_{5y}b_1\lambda_3\lambda_6^2 - 45b_{1y}\lambda_3\lambda_5^2\lambda_7 + 20\lambda_{6y}b_1\lambda_3\lambda_5\lambda_6 - 250\lambda_3^4\lambda_5^2.\end{aligned}$$

Equation (3.13) becomes

$$\begin{array}{l}95b_{1y}^2\lambda_3\lambda_5^3\lambda_7 - 280b_{1\tau}b_1^2\lambda_3\lambda_5^3\lambda_7 - 280b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_6^2 + 95b_{1y}^2\lambda_3\lambda_5^2\lambda_6^2 \\ -22b_{1y}\lambda_{3y}b_1\lambda_5^3\lambda_7 - 22b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_6^2 - 190b_{1y}b_1b_2\lambda_3\lambda_5^3\lambda_7 \\ -190b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_6^2 - 250b_{1y}\lambda_3^4\lambda_5^3 + 45b_{1y}\lambda_3\lambda_5^2\lambda_6\lambda_7 + 45b_{1y}\lambda_3\lambda_5\lambda_6^3 \\ -b_{1y}\lambda_5\lambda_8 + 44\lambda_{3y}b_1^2b_2\lambda_5^3\lambda_7 + 44\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_6^2 - 6\lambda_{3y}b_1\lambda_5^2\lambda_6\lambda_7 \\ -6\lambda_{3y}b_1\lambda_5\lambda_6^3 - 20\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_6^2 + 40\lambda_{6\tau}b_1^3\lambda_3\lambda_5^2\lambda_6 + 20\lambda_{7\tau}b_1^3\lambda_3\lambda_5^3 \\ +500b_1b_2\lambda_3^4\lambda_5^3 + 2b_1b_2\lambda_5\lambda_8 + 140\lambda_2\lambda_3\lambda_5^3\lambda_7 + 140\lambda_2\lambda_3\lambda_5^2\lambda_6^2 \\ +250\lambda_3^4\lambda_5^2\lambda_6 - 28\lambda_4\lambda_5^3\lambda_7 - 28\lambda_4\lambda_5^2\lambda_6^2 + \lambda_6\lambda_8 = 0.\end{array} \tag{3.17}$$

Substituting V_x into (3.6), one has

$$\begin{array}{rl}\lambda_{6y} = & (15b_{1y}\lambda_3\lambda_5\lambda_6 - \lambda_{3y}b_1\lambda_5\lambda_6 + 5\lambda_{5y}b_1\lambda_3\lambda_6 - 20b_1^2b_3\lambda_3\lambda_5^2 \\ & -\lambda_4\lambda_5^2)/(5b_1\lambda_3\lambda_5).\end{array} \tag{3.18}$$

Comparing the mixed derivatives $(V_t)_x - (V_x)_t = 0$, one gets

$$\begin{array}{l}40b_{1y}b_1^2b_3\lambda_3\lambda_5^3 - 20b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_6 - 20b_{1y}^2\lambda_3\lambda_5^2\lambda_6 + 2b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_6 \\ +40b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_6 + 15b_{1y}\lambda_3\lambda_5\lambda_6^2 + 2b_{1y}\lambda_4\lambda_5^3 - 4\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_6 \\ -2\lambda_{3y}b_1\lambda_5\lambda_6^2 - 20\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_6 + 20\lambda_{6\tau}b_1^3\lambda_3\lambda_5^2 - 80b_1^3b_2b_3\lambda_3\lambda_5^3 \\ -4b_1b_2\lambda_4\lambda_5^3 + 10\lambda_1\lambda_3\lambda_5^3 + 10\lambda_2\lambda_3\lambda_5^2\lambda_6 - 2\lambda_4\lambda_5^2\lambda_6 - \lambda_8 = 0.\end{array} \tag{3.19}$$

Substituting a_x into (3.11), one obtains

$$15b_{1y}\lambda_3\lambda_5\lambda_8 - 2\lambda_{3y}b_1\lambda_5\lambda_8 + 5\lambda_{12} + 2\lambda_5^4\lambda_7 + 2\lambda_5^3\lambda_6^2 = 0, \tag{3.20}$$

where

$$\lambda_{12} = 9b_{1y}\lambda_3\lambda_5\lambda_8 + 2\lambda_{3y}b_1\lambda_5\lambda_8 + 6\lambda_{5y}b_1\lambda_3\lambda_8 - 2\lambda_{8y}b_1\lambda_3\lambda_5.$$

Since a does not depend on t, the equation $(a_x)_t = 0$ gives

$$\begin{array}{l}35b_{1y}^2\lambda_3\lambda_5^2\lambda_8 - 10b_{1y}\lambda_{3y}b_1\lambda_5^2\lambda_8 - 160b_{1\tau}b_1^2\lambda_3\lambda_5^2\lambda_8 \\ -70b_{1y}b_1b_2\lambda_3\lambda_5^2\lambda_8 + 5b_{1y}\lambda_{12}\lambda_5 + 45b_{1y}\lambda_3\lambda_5\lambda_6\lambda_8 + 20\lambda_{3y}b_1^2b_2\lambda_5^2\lambda_8 \\ -6\lambda_{3y}b_1\lambda_5\lambda_6\lambda_8 - 60\lambda_{5\tau}b_1^3\lambda_3\lambda_5\lambda_8 + 20\lambda_{8\tau}b_1^3\lambda_3\lambda_5^2 - 10b_1b_2\lambda_{12}\lambda_5 \\ -5\lambda_{12}\lambda_6 + 80\lambda_2\lambda_3\lambda_5^2\lambda_8 - 16\lambda_4\lambda_5^2\lambda_8 = 0.\end{array} \tag{3.21}$$

If conditions (3.17)-(3.21) are satisfied, then the system of equation (3.2) is compatible. Thus, we have obtained that conditions (3.17)-(3.21) guarantee that the parabolic equation (1.4) is equivalent to (1.6).

Case 2 : $\lambda_5 = 0$.
From (3.12), one has that, $\lambda_6 = 0$, and equation (3.14) becomes

$$\lambda_{7y} = (45b_{1y}\lambda_3\lambda_7 + 22\lambda_{3y}b_1\lambda_7 + 250\lambda_3^4)/(10b_1\lambda_3). \tag{3.22}$$

Notice that the condition $\lambda_3 \neq 0$ implies $\lambda_7 \neq 0$. Differentiating a_{xx} in (3.11) with respect to t, one gets

$$12V_xb_1\lambda_9 + Y_x\lambda_{10}V = 0, \tag{3.23}$$

where

$$\begin{aligned}\lambda_9 = {} & 6\lambda_{3y}b_1\lambda_7 - 45b_{1y}\lambda_3\lambda_7 - 250\lambda_3^4,\\ \lambda_{10} = {} & 420b_{1y}b_1b_2\lambda_3\lambda_7 - 840b_{1\tau}b_1^2\lambda_3\lambda_7 - 210b_{1y}^2\lambda_3\lambda_7 - 3500b_{1y}\lambda_3^4\\ & -11b_{1y}\lambda_9 + 60\lambda_{7\tau}b_1^3\lambda_3 + 7000b_1b_2\lambda_3^4 + 22b_1b_2\lambda_9 + 420\lambda_2\lambda_3\lambda_7\\ & -84\lambda_4\lambda_7.\end{aligned}$$

From definition of λ_9, one finds λ_{3y}. Then (3.22) becomes

$$\lambda_{7y} = (630b_{1y}\lambda_3\lambda_7 + 3500\lambda_3^4 + 11\lambda_9)/(30b_1\lambda_3). \tag{3.24}$$

Case 2.1: $\lambda_9 \neq 0$.
From (3.23), one finds

$$V_x = -Y_x\lambda_{10}V/(12b_1\lambda_9). \tag{3.25}$$

Substituting V_x into (3.6) and (3.7), one has

$$\begin{aligned}\lambda_{10y} = {} & (45b_{1y}\lambda_{10}\lambda_3\lambda_7\lambda_9 + 30\lambda_{9y}b_1\lambda_{10}\lambda_3\lambda_7 - 360b_1^2b_3\lambda_3\lambda_7\lambda_9^2\\ & -250\lambda_{10}\lambda_3^4\lambda_9 - \lambda_{10}\lambda_9^2 - 18\lambda_4\lambda_7\lambda_9^2)/(30b_1\lambda_3\lambda_7\lambda_9),\end{aligned} \tag{3.26}$$

$$a_x = Y_x^3\lambda_{11}, \tag{3.27}$$

where

$$\begin{aligned}\lambda_{11} = {} & (-12b_{1\tau}b_1^2\lambda_{10}\lambda_9^2 + 6b_{1y}^2\lambda_{10}\lambda_9^2 - 6b_{1y}\lambda_{10y}b_1\lambda_9^2 + 6b_{1y}\lambda_{9y}b_1\lambda_{10}\lambda_9\\ & -12b_{1y}b_1b_2\lambda_{10}\lambda_9^2 - 3b_{1y}\lambda_{10}^2\lambda_9 + 12\lambda_{10\tau}b_1^3\lambda_9^2 + 12\lambda_{10y}b_1^2b_2\lambda_9^2\\ & +2\lambda_{10y}b_1\lambda_{10}\lambda_9 - 12\lambda_{9\tau}b_1^3\lambda_{10}\lambda_9 - 12\lambda_{9y}b_1^2b_2\lambda_{10}\lambda_9 - 2\lambda_{9y}b_1\lambda_{10}^2\\ & +24b_1^2b_3\lambda_{10}\lambda_9^2 + 18\lambda_1\lambda_9^3 + 6\lambda_{10}\lambda_2\lambda_9^2)/(144b_1^3\lambda_9^3).\end{aligned}$$

Differentiating (3.27) with respect to t, and substituting a_x into (3.11), one gets

$$\begin{aligned}
&5400b_{1\tau}b_{1y}b_1^2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 - 10800b_{1\tau}b_1^3b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^3\\
&-1800b_{1\tau}b_1^2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2 - 2700b_{1y}^3\lambda_{10}\lambda_3\lambda_7\lambda_9^3\\
&+135b_{1y}^2\lambda_{9y}b_1\lambda_{10}\lambda_3\lambda_7\lambda_9^2 - 145800b_{1y}^2b_1^3\lambda_{11}\lambda_3^2\lambda_7^2\lambda_9^3\\
&+10800b_{1y}^2b_1b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^3 + 2250b_{1y}^2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2\\
&-270b_{1y}\lambda_{9\tau}b_1^3\lambda_{10}\lambda_3\lambda_7\lambda_9^2 - 540b_{1y}\lambda_{9y}b_1^2b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^2\\
&-90b_{1y}\lambda_{9y}b_1\lambda_{10}^2\lambda_3\lambda_7\lambda_9 + 291600b_{1y}b_1^4b_2\lambda_{11}\lambda_3^2\lambda_7^2\lambda_9^3\\
&-810000b_{1y}b_1^3\lambda_{11}\lambda_3^5\lambda_7\lambda_9^3 - 3240b_{1y}b_1^3\lambda_{11}\lambda_3\lambda_7\lambda_9^4\\
&+97200b_{1y}b_1^3\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^3 - 10800b_{1y}b_1^2b_2^2\lambda_{10}\lambda_3\lambda_7\lambda_9^3\\
&-4500b_{1y}b_1b_2\lambda_{10}^2\lambda_3\lambda_7\lambda_9^2 - 450b_{1y}\lambda_{10}^3\lambda_3\lambda_7\lambda_9\\
&+7500b_{1y}\lambda_{10}^2\lambda_3^4\lambda_9^2 + 30b_{1y}\lambda_{10}^2\lambda_9^3 - 2700b_{1y}\lambda_{10}\lambda_2\lambda_3\lambda_7\lambda_9^3\\
&+540b_{1y}\lambda_{10}\lambda_4\lambda_7\lambda_9^3 + 243b_{1y}\lambda_7^3\lambda_9^3\\
&+194400\lambda_{11\tau}b_1^6\lambda_3^2\lambda_7^2\lambda_9^3 + 540\lambda_{9\tau}b_1^4b_2\lambda_{10}\lambda_3\lambda_7\lambda_9^2\\
&+90\lambda_{9\tau}b_1^3\lambda_{10}^2\lambda_3\lambda_7\lambda_9 + 540\lambda_{9y}b_1^3b_2^2\lambda_{10}\lambda_3\lambda_7\lambda_9^2\\
&+180\lambda_{9y}b_1^2b_2\lambda_{10}^2\lambda_3\lambda_7\lambda_9 + 15\lambda_{9y}b_1\lambda_{10}^3\lambda_3\lambda_7\\
&+1620000b_1^4b_2\lambda_{11}\lambda_3^5\lambda_7\lambda_9^3 + 6480b_1^4b_2\lambda_{11}\lambda_3\lambda_7\lambda_9^4\\
&-194400b_1^4b_2\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^3 - 540000b_1^3\lambda_{10}\lambda_{11}\lambda_3^5\lambda_7\lambda_9^2\\
&-2160b_1^3\lambda_{10}\lambda_{11}\lambda_3\lambda_7\lambda_9^3 - 32400b_1^3\lambda_{10}\lambda_{13}\lambda_3^2\lambda_7^2\lambda_9^2\\
&+291600b_1^3\lambda_{11}\lambda_2\lambda_3^2\lambda_7^2\lambda_9^3 - 58320b_1^3\lambda_{11}\lambda_3\lambda_4\lambda_7^2\lambda_9^3\\
&-15000b_1b_2\lambda_{10}^2\lambda_3^4\lambda_9^2 - 60b_1b_2\lambda_{10}^2\lambda_9^3\\
&+5400b_1b_2\lambda_{10}\lambda_2\lambda_3\lambda_7\lambda_9^3 - 1080b_1b_2\lambda_{10}\lambda_4\lambda_7\lambda_9^3\\
&-486b_1b_2\lambda_7^3\lambda_9^3 - 2500\lambda_{10}^3\lambda_3^4\lambda_9 - 10\lambda_{10}^3\lambda_9^2\\
&+900\lambda_{10}^2\lambda_2\lambda_3\lambda_7\lambda_9^2 - 180\lambda_{10}^2\lambda_4\lambda_7\lambda_9^2 - 81\lambda_{10}\lambda_7^3\lambda_9^2 = 0,
\end{aligned} \tag{3.28}$$

$$600b_1^3\lambda_3^2\lambda_7\lambda_{13} - 10000b_1^3\lambda_{11}\lambda_3^5 - 40b_1^3\lambda_{11}\lambda_3\lambda_9 + 3\lambda_7^2 = 0, \tag{3.29}$$

where

$$\lambda_{13} = 3b_{1y}\lambda_{11} - 2\lambda_{11y}b_1.$$

Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, (3.24), (3.26), (3.28) and (3.29) guarantee that the parabolic equation (1.4) equivalent to equation (1.6).

Case 2.2 : $\lambda_9 = 0$.
From (3.23) and (3.24), it follow that

$$\lambda_{10} = 0,$$

$$\lambda_{7y} = 7(9b_{1y}\lambda_7 + 50\lambda_3^3)/(3b_1). \tag{3.30}$$

Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$ and (3.30) guarantee that the parabolic equation (1.4) equivalent to equation (1.6).

We can summarize the results by following theorem.

Theorem 3.1. *The parabolic equation (1.4) is equivalent to equation (1.6) if and only if the coefficients of (1.4) obey one of the following conditions:*
(A) equations (3.17)-(3.21), in this case the functions $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (3.3)-(3.5), (3.10),*

(3.15), (3.16);
(B) equations $\lambda_5 = 0$, $\lambda_6 = 0$, *(3.24), (3.26), (3.28), (3.29), in this case the functions* $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (3.3)-(3.5), (3.10), (3.25), (3.27);*
(C) equations $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$, *(3.30), in this case the functions* $H(t), Y(t,x), V(t,x)$ *and* $a(x)$ *are obtained by solving involutive system of equations (3.3)-(3.6), (3.10), (3.11).*

We present an example to illustrate Theorem (3.1).

Example 3.2. *Consider the linear second-order parabolic partial differential equation*

$$\tau v_\tau + y v_y - (e^{y/\tau}\tau + 1)v - \tau^3 v_{yy} = 0. \quad (3.31)$$

It has the form of equation (1.4) with the following coefficients

$$b_1 = 1/\tau^2,\ b_2 = y/\tau^3,\ b_3 = -(e^{y/\tau}\tau + 1)/\tau^3. \quad (3.32)$$

Coefficients of equation (3.32) satisfy conditions (A) *in Theorem (3.1). Hence, the parabolic equation (3.31) is equivalent to equation (1.6). For finding transformation* $H(t), Y(t,x), V(t,x)$ *which mapping equation (3.31) into equation (1.6), and* $a(x)$ *one needs to solve equations (3.3)-(3.5), (3.10), (3.15) and (3.16).*

Substituting b_1, b_2, b_3 *into* λ_6, *one has* $\lambda_6 = 0$. *Thus equation (3.15) becomes* $V_x = 0$, *i.e.,*

$$V = V(t).$$

Substituting b_1 *into equation (3.4), one has* $Y_{xx} = 0$. *Hence,*

$$Y = H(t)(\alpha(t)x + \beta(t)),$$

where $\alpha(t)$ *and* $\beta(t)$ *are arbitrary functions. From equation (3.16), one finds*

$$a = \alpha^2 e^{\alpha x + \beta} + C,$$

where C *is arbitrary constant, which can chosen, for example* $C = 0$. *Substituting* V, Y, *and* a *into (3.3), (3.10), (3.5), one obtains*

$$H' = \alpha^2,\ V' = \alpha^2 V/H,\ \alpha' = 0,\ \beta' = 0 \quad (3.33)$$

Since any particular solution for equations (3.33) can be used, we set $\alpha = 1$, $\beta = 0$. *Hence,* $H = t$, $Y = tx$, $V = t$ *and* $a(x) = e^x$. *Therefore, one obtains the following transformations*

$$\tau = t,\ y = tx,\ v = tu,$$

mapping equation (3.31) into the equation

$$u_t = u_{xx} + e^x u.$$

3.2 Equivalence problem for equation $u_t = u_{xx} + \frac{k}{x^2}u$

This section studies equations (1.4) which are equivalent to equation (1.7). Since for equation (1.7), the coefficient is

$$a(x) = k/x^2, (k \neq 0), \tag{3.34}$$

we continue studying the various cases from the previous section.
Case 1 : $\lambda_5 \neq 0$.
Substituting a in (3.34) into (3.16), one has

$$k = -Y_x^3\lambda_8 x^3/(160 b_1^3\lambda_3\lambda_5^3). \tag{3.35}$$

Since k is constant, differentiating (3.35) with respect to x, one obtains

$$Y_x\lambda_{12}x - 6b_1\lambda_3\lambda_5\lambda_8 = 0. \tag{3.36}$$

Case 1.1 : $\lambda_{12} \neq 0$.
In this case, one can find

$$Y_x = 6b_1\lambda_3\lambda_5\lambda_8/(\lambda_{12}x). \tag{3.37}$$

Substituting Y_x into (3.4), one has

$$\begin{array}{l} 3\lambda_{12y}b_1\lambda_3\lambda_5\lambda_8 - 18b_{1y}\lambda_{12}\lambda_3\lambda_5\lambda_8 - 6\lambda_{3y}b_1\lambda_{12}\lambda_5\lambda_8 \\ -12\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_8 + 2\lambda_{12}^2 = 0. \end{array} \tag{3.38}$$

Comparing the mixed derivatives $(Y_t)_x - (Y_x)_t = 0$, one obtains

$$\begin{array}{l} 60\lambda_{12}\lambda_2\lambda_3\lambda_5^2\lambda_8 - 120b_{1\tau}b_1^2\lambda_{12}\lambda_3\lambda_5^2\lambda_8 + 105b_{1y}^2\lambda_{12}\lambda_3\lambda_5^2\lambda_8 \\ -10b_{1y}\lambda_{12y}b_1\lambda_3\lambda_5^2\lambda_8 + 10b_{1y}\lambda_{3y}b_1\lambda_{12}\lambda_5^2\lambda_8 + 40b_{1y}\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_5\lambda_8 \\ -210b_{1y}b_1b_2\lambda_{12}\lambda_3\lambda_5^2\lambda_8 - 5b_{1y}\lambda_{12}^2\lambda_5 - 45b_{1y}\lambda_{12}\lambda_3\lambda_5\lambda_6\lambda_8 \\ +20\lambda_{12\tau}b_1^3\lambda_3\lambda_5^2\lambda_8 + 20\lambda_{12y}b_1^2b_2\lambda_3\lambda_5^2\lambda_8 + 10\lambda_{12y}b_1\lambda_3\lambda_5\lambda_6\lambda_8 \\ -20\lambda_{3y}b_1^2b_2\lambda_{12}\lambda_5^2\lambda_8 - 22\lambda_{3y}b_1\lambda_{12}\lambda_5\lambda_6\lambda_8 - 20\lambda_{5\tau}b_1^3\lambda_{12}\lambda_3\lambda_5\lambda_8 \\ -80\lambda_{5y}b_1^2b_2\lambda_{12}\lambda_3\lambda_5\lambda_8 - 40\lambda_{5y}b_1\lambda_{12}\lambda_3\lambda_6\lambda_8 - 20\lambda_{8\tau}b_1^3\lambda_{12}\lambda_3\lambda_5^2 \\ +10b_1b_2\lambda_{12}^2\lambda_5 + 5\lambda_{12}^2\lambda_6 - 12\lambda_{12}\lambda_4\lambda_5^2\lambda_8 = 0. \end{array} \tag{3.39}$$

Therefore the condition (3.17)-(3.21) and (3.38)-(3.39) guarantee that the parabolic equation (1.4) equivalent to equation (1.7).

Case 1.2 : $\lambda_{12} = 0$.
Equation (3.36) implies $\lambda_8 = 0$. Then (3.21) is the identity and equation (3.20) and (3.17) become

$$\lambda_7 = -\lambda_6^2/\lambda_5, \tag{3.40}$$

$$\lambda_6 = \lambda_5(b_{1y} - 2b_1b_2). \tag{3.41}$$

Substituting λ_6 into (3.18) and (3.19), one gets

$$\begin{aligned}\lambda_{6y} = \; & (15b_{1y}^2\lambda_3\lambda_5 - b_{1y}\lambda_{3y}b_1\lambda_5 + 5b_{1y}\lambda_{5y}b_1\lambda_3 - 30b_{1y}b_1b_2\lambda_3\lambda_5 \\ & +2\lambda_{3y}b_1^2b_2\lambda_5 - 10\lambda_{5y}b_1^2b_2\lambda_3 - 20b_1^2b_3\lambda_3\lambda_5 - \lambda_4\lambda_5)/(5b_1\lambda_3),\end{aligned} \tag{3.42}$$

$$\begin{aligned}& b_{1y}^3 - 4b_{1y}^2b_1b_2 + 4b_{1y}b_{2y}b_1^2 + 4b_{1y}b_1^2b_2^2 - 8b_{1y}b_1^2b_3 + 10b_{1y}\lambda_2 - 8b_{2\tau}b_1^4 \\ & -8b_{2y}b_1^3b_2 - 4\lambda_{2y}b_1 + 4\lambda_1 = 0.\end{aligned} \tag{3.43}$$

Therefore the conditions $\lambda_8 = 0$, $\lambda_{12} = 0$, (3.40)-(3.42) and (3.43) guarantee that the parabolic equation (1.4) equivalent to equation (1.7).

Case 2 : $\lambda_5 = 0$.
Case 2.1 : $\lambda_9 \neq 0$.
Substituting a from (3.34) into (3.27), one has

$$k = -Y_x^3\lambda_{11}x^3/2. \tag{3.44}$$

Differentiating k with respect to t and x, one gets

$$\begin{aligned}& 45b_{1y}^2\lambda_{11}\lambda_3\lambda_7\lambda_9 - 90b_{1y}b_1b_2\lambda_{11}\lambda_3\lambda_7\lambda_9 - 15b_{1y}\lambda_{13}\lambda_3\lambda_7\lambda_9 \\ & -60\lambda_{11\tau}b_1^3\lambda_3\lambda_7\lambda_9 + 30b_1b_2\lambda_{13}\lambda_3\lambda_7\lambda_9 + 250\lambda_{10}\lambda_{11}\lambda_3^4 \\ & +\lambda_{10}\lambda_{11}\lambda_9 + 5\lambda_{10}\lambda_{13}\lambda_3\lambda_7 - 90\lambda_{11}\lambda_2\lambda_3\lambda_7\lambda_9 + 18\lambda_{11}\lambda_4\lambda_7\lambda_9 = 0,\end{aligned} \tag{3.45}$$

$$xY_x\lambda_{13} - 6b_1\lambda_{11} = 0. \tag{3.46}$$

Case 2.1.1 : $\lambda_{13} \neq 0$.
Solving (3.46) with respect to Y_x, one obtains

$$Y_x = 6b_1\lambda_{11}/(\lambda_{13}x). \tag{3.47}$$

Substituting Y_x into (3.4), one has

$$0 = 3\lambda_{13y}b_1\lambda_{11} - 9b_{1y}\lambda_{11}\lambda_{13} + 2\lambda_{13}^2. \tag{3.48}$$

The requirement $(Y_t)_x - (Y_x)_t = 0$, leads to condition

$$\begin{aligned}& 225b_{1y}^2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9 - 180b_{1\tau}b_1^2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9 \\ & -90b_{1y}\lambda_{13y}b_1\lambda_{11}\lambda_3\lambda_7\lambda_9 - 450b_{1y}b_1b_2\lambda_{11}\lambda_{13}\lambda_3\lambda_7\lambda_9 \\ & -90b_{1y}\lambda_{10}\lambda_{11}\lambda_{13}\lambda_3\lambda_7 - 45b_{1y}\lambda_{13}^2\lambda_3\lambda_7\lambda_9 - 180\lambda_{11\tau}b_1^3\lambda_{13}\lambda_3\lambda_7\lambda_9 \\ & +180\lambda_{13\tau}b_1^3\lambda_{11}\lambda_3\lambda_7\lambda_9 + 180\lambda_{13y}b_1^2b_2\lambda_{11}\lambda_3\lambda_7\lambda_9 \\ & +30\lambda_{13y}b_1\lambda_{10}\lambda_{11}\lambda_3\lambda_7 + 90b_1b_2\lambda_{13}^2\lambda_3\lambda_7\lambda_9 - 250\lambda_{10}\lambda_{11}\lambda_{13}\lambda_3^4 \\ & -\lambda_{10}\lambda_{11}\lambda_{13}\lambda_9 + 15\lambda_{10}\lambda_{13}^2\lambda_3\lambda_7 + 90\lambda_{11}\lambda_{13}\lambda_2\lambda_3\lambda_7\lambda_9 \\ & -18\lambda_{11}\lambda_{13}\lambda_4\lambda_7\lambda_9 = 0.\end{aligned} \tag{3.49}$$

Therefore the conditions (3.24), (3.26), (3.28), (3.29), (3.45), (3.48) and (3.49) guarantee that the parabolic equation (1.4) equivalent to equation (1.7).

Case 2.1.2 : $\lambda_{13} = 0$.

Equation (3.46) and (3.29) imply $\lambda_{11} = 0$ and $\lambda_7 = 0$. This contradicts $\lambda_7 \neq 0$.

Case 2.2 : $\lambda_9 = 0$.
Equation (3.11) is

$$3Y_x^4\lambda_7^2x^4 + 800b_1^3\lambda_3^2k(25Y_x\lambda_3^3x - 9b_1\lambda_7) = 0. \tag{3.50}$$

Analyzing equation (3.50) one obtains that $25Y_x\lambda_3^3x - 9b_1\lambda_7 \neq 0$. Hence,

$$k = -3Y_x^4\lambda_7^2x^4/(800b_1^3\lambda_3^2(25Y_x\lambda_3^3x - 9b_1\lambda_7)). \tag{3.51}$$

Differentiating k with respect to x, one has

$$675Y_xb_1\lambda_3^3\lambda_7^2x - 1875Y_x^2\lambda_3^6\lambda_7x^2 + 108b_1^2\lambda_7^3 = 0. \tag{3.52}$$

Differentiating (3.52) with respect to t and x, one gets

$$\begin{aligned}&93750V_xY_x^3b_1\lambda_3^7x^2(13b_{1y}\lambda_7 + 70\lambda_3^3) + 33750V_xY_x^2b_1^2\lambda_3^4\lambda_7x\\&(-13b_{1y}\lambda_7 - 70\lambda_3^3) + 5400V_xY_xb_1^3\lambda_3\lambda_7^2(-13b_{1y}\lambda_7 - 70\lambda_3^3)\\&+1875Y_x^4\lambda_3^6\lambda_7Vx^2(-220b_{1\tau}b_1^2\lambda_3 + 110b_{1y}^2\lambda_3 - 220b_{1y}b_1b_2\lambda_3\\&+105\lambda_2\lambda_3 - 21\lambda_4) + 675Y_x^3b_1\lambda_3^3\lambda_7^2Vx(220b_{1\tau}b_1^2\lambda_3 - 110b_{1y}^2\lambda_3\\&+220b_{1y}b_1b_2\lambda_3 - 105\lambda_2\lambda_3 + 21\lambda_4) + 108Y_x^2b_1^2\lambda_7^3V(220b_{1\tau}b_1^2\lambda_3\\&-110b_{1y}^2\lambda_3 + 220b_{1y}b_1b_2\lambda_3 - 105\lambda_2\lambda_3 + 21\lambda_4) = 0,\end{aligned} \tag{3.53}$$

$$\begin{aligned}&3125Y_x^2\lambda_3^6x^2(-39b_{1y}\lambda_7 - 220\lambda_3^3) + 375Y_xb_1\lambda_3^3\lambda_7x(117b_{1y}\lambda_7\\&+635\lambda_3^3) + 135b_1^2\lambda_7^2(52b_{1y}\lambda_7 + 285\lambda_3^3) = 0.\end{aligned} \tag{3.54}$$

Using (3.52) and (3.54), one finds

$$Y_x = -3b_1\lambda_7/(25\lambda_3^3x). \tag{3.55}$$

Substituting Y_x into (3.53) and equations $Y_{xx} = (Y_x)_x$, $Y_{tx} = (Y_x)_t$, $(Y_t)_x = (Y_x)_t$, one obtains identities. Therefore the conditions $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$ and (3.30) guarantee that the parabolic equation (1.4) equivalent to equation (1.7).

Therefore the results can be summarized by the following theorem.

Theorem 3.3. *The parabolic equation (1.4) is equivalent to equation (1.7) if and only if the coefficients of (1.4) obey one of the following conditions:*
(A) equations (3.17)-(3.21), (3.38),(3.39), in this case the functions $H(t), Y(t,x)$, $V(t,x)$ *and* k *are obtained by solving involutive system of equations (3.3), (3.5), (3.10), (3.15), (3.35), (3.37);*
(B) $\lambda_8 = 0$, $\lambda_{12} = 0$, *(3.40)-(3.42), (3.43), in this case the functions* $H(t), Y(t,x)$, $V(t,x)$ *and* k *are obtained by solving involutive system of equations (3.3)-(3.5), (3.10), (3.15), (3.35);*
(C) (3.24), (3.26), (3.28), (3.29), (3.45), (3.48), (3.49), in this case the functions $H(t)$, $Y(t,x), V(t,x)$ *and* k *are obtained by solving involutive system of equations (3.3), (3.5), (3.10), (3.25), (3.44), (3.47);*
(D) $\lambda_5 = 0$, $\lambda_6 = 0$, $\lambda_9 = 0$, $\lambda_{10} = 0$, *(3.30), in this case the functions* $H(t)$, $Y(t,x), V(t,x)$ *and* k *are obtained by solving involutive system of equations (3.3), (3.5), (3.6), (3.10), (3.51), (3.55).*

Remark. Equation (1.4) equivalent to (1.7) if and only if it satisfies following relation

$$\begin{array}{rl} 3125/(3k) = & 4(506250b_{1y}^4\lambda_{3y}b_1\lambda_3^4 - 759375b_{1y}^5\lambda_3^5 - 135000b_{1y}^3\lambda_{3y}^2b_1^2\lambda_3^3 \\ & +18000b_{1y}^2\lambda_{3y}^3b_1^3\lambda_3^2 - 1200b_{1y}\lambda_{3y}^4b_1^4\lambda_3 + 32\lambda_{3y}^5b_1^5)/\lambda_3^6. \end{array} \quad (3.56)$$

We consider the following example.

Example 3.4. *Consider the linear second-order parabolic partial differential equation*

$$\tau y^2 v_\tau + y(2\tau^3 + y^2)v_y - 3\tau^3 v - \tau^3 y^2 v_{yy} = 0. \quad (3.57)$$

It is an equation of the form (1.4) with the coefficients

$$b_1 = 1/\tau^2,\ b_2 = (2\tau^3 + y^2)/(\tau^3 y),\ b_3 = -3/y^2. \quad (3.58)$$

One can check that the coefficients (3.58) obey the conditions (D) *in Theorem (3.3). Furthermore, they also satisfy condition (3.56). Thus, the parabolic equation (3.57) is equivalent to equation (1.7). For finding a transformation mapping equation (3.57) into equation (1.7), one need to solve equations (3.3), (3.5), (3.6), (3.10), (3.51), (3.55) for* $H(t), Y(t,x)$, $V(t,x)$ *and* k.

Substituting b_1, b_2, b_3 *into equation (3.55), one has*

$$xY_x - Y = 0.$$

Substituting the general solution of this equation

$$Y = c_1(t)H(t)x \quad (3.59)$$

into equation (3.51), one has $k = 1$, *where* $c_1(t)$ *is an arbitrary function. From equations (3.3) and (3.5), one obtains*

$$H' = c_1^2, \quad (3.60)$$

$$2c_1xV_x + (c_1'x^2 - 2c_1)V = 0.$$

The general solution of the last equation is

$$V = c_2(t)xe^{-(c_1'/4c_1)x^2}.$$

Substituting the function V *into (3.6), one gets*

$$(c_1''c_1c_2 - 2c_1'^2c_2)x^2 + (2c_1'c_1c_2 - 4c_2'c_1^2) = 0.$$

Splitting this equation with respect to x, *we have*

$$\begin{array}{l} c_1c_1'' = 2c_1'^2, \\ 2c_1c_2' = c_1'c_2. \end{array} \quad (3.61)$$

The general solution of the system of ordinary differential equations (3.61) is

$$c_1 = 1/(k_1 t + k_2),$$
$$c_2^2 = (1/(k_3(k_1 t + k_2))).$$

Substituting c_1 into (3.60), one has

$$H' = 1/(k_1 t + k_2)^2. \tag{3.62}$$

Further study depends on quantity of k_1.

Case $k_1 \neq 0$

The general solution of (3.62) is

$$H = k_4 - 1/(k_1(k_1 t + k_2)).$$

Because of $k_1 \neq 0$, $k_3 \neq 0$ and k_2, k_4 are arbitrary constant. Setting $k_1 = 1$, $k_2 = 0$, $k_3 = 1$, $k_4 = 0$. Therefore, one obtains the following transformations

$$\tau = -\frac{1}{t},\ y = -\frac{x}{t^2},\ v = \frac{x^2}{\sqrt{t}} e^{4x^2/t^3},$$

mapping equation (3.57) into the equation

$$u_t = u_{xx} + \frac{1}{x^2} u.$$

Case $k_1 = 0$

For this case we have

$$c_1 = 1/k_2,\ c_2^2 = 1/k_2,$$

where $k_2 \neq 0$. The general solution of (3.62) is

$$H = \frac{t}{k_2^2} + k_5$$

where k_5 is arbitrary constant. Setting $k_2 = 1$, $k_5 = 0$. Therefore, one obtains the following transformations

$$\tau = t,\ y = tx,\ v = xu,$$

mapping equation (3.57) into the equation

$$u_t = u_{xx} + \frac{1}{x^2} u.$$

4 Conclusion

This paper is devoted to finding conditions which the parabolic equation (0.3) to be equivalent to (0.1)-(0.2). Conditions which guarantee that the second-order parabolic differential equations is equivalent to one of the canonical forms are found in theorem (3.1) and (3.3), respectively.

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