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Some Inequalities Concerning the Polar Derivative of a Polynomial[1]

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Abstract : In this paper, we consider the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n$, having all zeros in $|z| \leq k,\ k \leq 1$ and thereby establish several interesting estimates pertaining to the maximum modulus of the polar derivative of a polynomial $P(z)$. Our results not only generalize and refine some known polynomial inequalities, but also a variety of interesting results can be deduced from these by a fairly uniform procedure.

Keywords : polar derivative; polynomials; zeros; maximum modulus; inequalities in the complex domain.
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1 Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then according to the well-known Bernstien's inequality [4] on the derivative of a polynomial, we have

$$Max_{|z|=1}|P'(z)| \leq nMax_{|z|=1}|P(z)|. \tag{1.1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin.
For the class of polynomials $P(z)$ having all zeros in $|z| \leq 1$, Turan [11] proved

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that

$$Max_{|Z|=1}|P'(z)| \geq \frac{n}{2} Max_{|z|=1}|P(z)|. \tag{1.2}$$

Inequality (2) was refined by Aziz and Dawood [1] and they proved under the same hypothesis that

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{2}\{Max_{|z|=1}|P(z)| + Min_{|z|=1}|P(z)|\}. \tag{1.3}$$

Both the inequalities (2) and (3) are best possible and become equality for polynomials $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. As an extension of (2), it was shown by Malik [10], that if $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k} Max_{|z|=1}|P(z)|, \tag{1.4}$$

where as the corresponding extension of (3) and a refinement of (4) was given by Govil [8] who under the same hypothesis proved that

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\left\{Max_{|z|=1}|P(z)| + \frac{1}{k^{n-1}} Min_{|z|=k}|P(z)|\right\}. \tag{1.5}$$

In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Aziz and Shah [3], Dewan, Mir and Yadav [7], Govil, Rahman and Schemeisser [9], Dewan, Singh and Lal [5], etc.

By considering the class of polynomials $P(z) = a_n z^n + \sum\limits_{v=\mu}^{n} a_{n-v} z^{n-v}, 1 \leq \mu \leq n$, of degree n having all zeros in $|z| \leq k, k \leq 1$, Aziz and Shah [3] (see also Dewan, Mir and Yadav [7]) proved

$$Max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^\mu}\left\{Max_{|z|=1}|P(z)| + \frac{1}{k^{n-\mu}} Min_{|z|=k}|P(z)|\right\}. \tag{1.6}$$

For $\mu = 1$, inequality (6) reduces to inequality (5).
Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$Lim_{\alpha \to \infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} = P'(z).$$

Dewan, Singh and Lal [5] extended the inequality (6) to the polar derivative of a polynomial $P(z)$ by showing that if $P(z) = a_n z^n + \sum\limits_{v=\mu}^{n} a_{n-v} z^{n-v}, 1 \leq \mu \leq n$, has all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number α with $|\alpha| \geq k^{\mu}$,

$$Max_{|z|=1}|D_{\alpha}P(z)| \geq \frac{n(|\alpha| - k^{\mu})}{1+k^{\mu}} Max_{|z|=1}|P(z)| + \frac{n(|\alpha|+1)}{k^{n-\mu}(1+k^{\mu})} Min_{|z|=k}|P(z)| \tag{1.7}$$

If we divide both sides of (7) by $|\alpha|$ and let $|\alpha| \to \infty$, we get (6).

Here, we shall prove the following more general result which includes not only inequalities (6) and (7) as special cases, but also leads to a standard development of interesting generalizations of some well-known results.

Theorem 1. If $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number α with $|\alpha| \geq A_{\mu}$, we have

$$Max_{|z|=1}|D_{\alpha}P(z)| \geq \frac{n(|\alpha| - A_{\mu})}{1+A_{\mu}} Max_{|z|=1}|P(z)| + \frac{nA_{\mu}(|\alpha|+1)}{k^{n}(1+A_{\mu})} m \tag{1.8}$$

where

$$A_{\mu} = \frac{n(|a_n| - \frac{m}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m}{k^n})k^{\mu-1} + \mu|a_{n-\mu}|} \tag{1.9}$$

and $m = Min_{|z|=k}|P(z)|$.

Remark 1. Since by Lemma 5 (stated in section 2)we have $A_{\mu} \leq k^{\mu}$; $1 \leq \mu \leq n$, Theorem 1 in particular holds for $|\alpha| \geq k^{\mu}$ also.

Also when $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, it is easy to verify, for example by derivative test and Lemma 6, that for every α with $|\alpha| \geq x$, the function

$$\left(\frac{|\alpha| - x}{1+x}\right) Max_{|z|=1}|P(z)| + \frac{x(|\alpha|+1)m}{k^n(1+x)},$$

is a non-increasing function of x. If we combine this fact with $A_{\mu} \leq k^{\mu}$, we get inequality (7) from Theorem 1.

If we do not have the knowledge of $Min_{|z|=k}|P(z)|$, we can use the following result, whose proof is similar to that of Theorem 1.

Theorem 2. If $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number α with $|\alpha| \geq s_{\mu}$, we have

$$Max_{|z|=1}|D_{\alpha}P(z)| \geq n\left(\frac{|\alpha| - s_{\mu}}{1+s_{\mu}}\right) Max_{|z|=1}|P(z)|, \tag{1.10}$$

where

$$s_{\mu} = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}. \tag{1.11}$$

Remark 2. If we divide both sides of (10) by $|\alpha|$ and let $|\alpha| \to \infty$, we get an interesting generalisation of a result due to Govil, Rahman and Schemeisser [9].

Several other interesting results easily follow from Theorem 1. Here, we mention few of these. If we divide both sides of (8) by $|\alpha|$ and let $|\alpha| \to \infty$, we immediately get the following result.

Corollary 1. If $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then

$$Max_{|z|=1}|P'(z)| \geq \left(\frac{n}{1+A_\mu}\right) Max_{|z|=1}|P(z)| + \frac{nmA_\mu}{k^n(1+A_\mu)} \tag{1.12}$$

where $m = Min_{|z|=k}|P(z)|$ and A_μ is defined in (9).
The result is sharp and equality in (12) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Remark 3. Again by the same reasoning as in remark 1, it is easy to verify that the function

$$\left(\frac{n}{1+x} Max_{|z|=1}|P(z)| + \frac{nmx}{k^n(1+x)}\right)$$

is a non-increasing function of x. If we combine this fact with Lemma 5 according to which $A_\mu \leq k^\mu$ for $1 \leq \mu \leq n$, we get inequality (6).

2 Lemmas

For the proof of Theorem 1, we need the following lemmas.

Lemma 1. If $P(z) = a_n z^n + \sum_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, and $q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ then

$$\left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}}\right\} |P'(z)| \geq |q'(z)|, \quad for \quad |z| = 1 \tag{2.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \leq k^\mu \tag{2.2}$$

The above lemma is due to Aziz and Rather [2].

Lemma 2. If $P(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having all its zeros in $|z| \leq k, k > 0$, then

$$|q(z)| \geq \frac{m}{k^n} \qquad for \quad |z| \leq \frac{1}{k}$$

and in particular

$$|a_n| > \frac{m}{k^n}, \tag{2.3}$$

where $m = Min_{|z|=k}|P(z)|$ and $q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.
The above lemma is due to Dewan, Singh and Mir [6].
Lemma 3. The function

$$s_\mu(x) = \frac{nxk^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{nxk^{\mu-1} + \mu|a_{n-\mu}|}, \tag{2.4}$$

where $k \leq 1$ and $\mu \geq 1$, is a non-increasing function of x.
Proof of Lemma 3. The proof follows by considering the first derivative test for $s_\mu(x)$.
Lemma 4. If $P(z) = a_nz^n + \sum\limits_{v=\mu}^{n} a_{n-v}z^{n-v}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, and $q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$,

$$|q'(z)| \leq A_\mu|P'(z)| - \frac{mnA_\mu}{k^n}, \tag{2.5}$$

where

$$A_\mu = \frac{n\left(|a_n| - \frac{m}{k^n}\right)k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n\left(|a_n| - \frac{m}{k^n}\right)k^{\mu-1} + \mu|a_{n-\mu}|} \tag{2.6}$$

and

$$\frac{\mu|a_{n-\mu}|}{n(|a_n| - \frac{m}{k^n})} \leq k^\mu \tag{2.7}$$

with $m = Min_{|z|=k}|P(z)|$.

Proof of Lemma 4. By hypothesis, the polynomial $P(z) = a_nz^n + \sum\limits_{v=\mu}^{n} a_{n-v}z^{n-v}, 1 \leq \mu \leq n$, has all its zeros in $|z| \leq k, k \leq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Lemma 1. Henceforth we assume that all the zeros of $P(z)$ lie in $|z| < k, k \leq 1$, so that $m > 0$. Since $m \leq |P(z)|$ for $|z| = k$, therefore if λ is any real or complex number with $|\lambda| < 1$, then

$$\left|\frac{m\lambda z^n}{k^n}\right| < |P(z)| \qquad for \quad |z| = k.$$

Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche's theorem that all the zeros of $P(z) - \frac{m\lambda z^n}{k^n}$ also lie in $|z| < k, k \leq 1$. Hence by Guass-Lucas theorem, the polynomial

$$P'(z) - \frac{mn\lambda z^{n-1}}{k^n} \tag{2.8}$$

also has all its zeros in $|z| < k, k \leq 1$, for every λ with $|\lambda| < 1$. This implies

$$|P'(z)| \geq \frac{mn|z|^{n-1}}{k^n} \qquad for \quad |z| \geq k, k \leq 1. \tag{2.9}$$

Because if (21) is not true, then there is a point $z = z_0$ with $|z_0| \geq k$ such that

$$|P'(z_0)| < \frac{mn|z_0|^{n-1}}{k^n}.$$

We choose $\lambda = \frac{k^n P'(z_0)}{mnz_0^{n-1}}$, so that $|\lambda| < 1$ and with this choice of λ, from (20), we have

$$P'(z_0) - \frac{mn\lambda z_0^{n-1}}{k^n} = 0,$$

where $|z_0| \geq k$, which contradicts the fact that all the zeros of

$$P'(z) - \frac{mn\lambda z^{n-1}}{k^n}$$

lie in $|z| < k, k \leq 1$. Now, we can apply inequality (13) of Lemma 1 to the polynomial

$$P(z) - \frac{m\lambda z^n}{k^n}$$

and get,

$$s_{\mu}^{'}|P'(z) - \frac{mn\lambda z^{n-1}}{k^n}| \geq |q'(z)|, \qquad for\ |z| = 1, \tag{2.10}$$

where

$$s_{\mu}^{'} = \frac{n|a_n - \frac{m\lambda}{k^n}|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n - \frac{m\lambda}{k^n}|k^{\mu-1} + \mu|a_{n-\mu}|} \tag{2.11}$$

Since for every λ with $|\lambda| < 1$, we have

$$|a_n - \frac{m\lambda}{k^n}| \geq |a_n| - \frac{m|\lambda|}{k^n} \geq |a_n| - \frac{m}{k^n} \tag{2.12}$$

and $|a_n| > \frac{m}{k^n}$ by Lemma 2. Now combining (23), (24) and Lemma 3, we get for every λ with $|\lambda| < 1$,

$$s_{\mu}^{'} = \frac{n|a_n - \frac{m\lambda}{k^n}|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n - \frac{m\lambda}{k^n}|k^{\mu-1} + \mu|a_{n-\mu}|} \leq \frac{n(|a_n| - \frac{m}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m}{k^n})k^{\mu-1} + \mu|a_{n-\mu}|} = A_{\mu}. \tag{2.13}$$

Therefore using (25) and (22), we get

$$A_{\mu}|P'(z) - \frac{mn\lambda z^{n-1}}{k^n}| \geq |q'(z)| \qquad , for \quad |z| = 1. \tag{2.14}$$

If in (26), we choose the argument of λ such that

$$|P'(z) - \frac{mn\lambda z^{n-1}}{k^n}| = |P'(z)| - \frac{mn|\lambda|}{k^n}$$

which easily follows from (21), we get

$$A_{\mu}|P'(z)| - \frac{mn|\lambda|A_{\mu}}{k^n} \geq |q'(z)| \ , \quad for|z| = 1. \tag{2.15}$$

Finally letting $|\lambda| \to 1$ in (27), we get

$$A_{\mu}|P'(z)| \geq |q'(z)| + \frac{mnA_{\mu}}{k^n}, for|z| = 1,$$

which proves (17).

To prove (19), we apply inequality (14) of Lemma 1 to the polynomial $P(z) - \frac{m\lambda z^n}{k^n}$, and get

$$\frac{\mu|a_{n-\mu}|}{n|a_n - \frac{m\lambda}{k^n}|} \leq k^\mu \tag{2.16}$$

for every real or complex number λ with $|\lambda| < 1$.
Since by Lemma 2, we have $|a_n| > \frac{m}{k^n}$, we can choose argument of λ in (28) such that

$$|a_n - \frac{m\lambda}{k^n}| = |a_n| - \frac{m|\lambda|}{k^n}$$

and with this choice of the argument of λ, we get from (28) that

$$\frac{\mu|a_{n-\mu}|}{n(|a_n| - \frac{m|\lambda|}{k^n})} \leq k^\mu \tag{2.17}$$

Inequality (19) now follows by making $|\lambda| \to 1$ in (29).

Lemma 5. If $P(z) = a_n z^n + \sum\limits_{\upsilon=\mu}^{n} a_{n-\upsilon} z^{n-\upsilon}, 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then

$$A_\mu \leq k^\mu, \tag{2.18}$$

where A_μ is defined as in Theorem 1.

Proof of Lemma 5. We have from inequality (19) of Lemma 4,

$$\mu|a_{n-\mu}| \leq n(|a_n| - \frac{m}{k^n})k^\mu,$$

which implies,

$$\left\{\mu|a_{n-\mu}| - n(|a_n| - \frac{m}{k^n})k^\mu\right\} \leq 0,$$

which is equivalent to

$$(k^{\mu-1} - k^\mu)\left\{\mu|a_{n-\mu}| - n(|a_n| - \frac{m}{k^n})k^\mu\right\} \leq 0,$$

that is,

$$n(|a_n| - \frac{m}{k^n})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1} \leq \left(\mu|a_{n-\mu}| + n(|a_n| - \frac{m}{k^n})k^{\mu-1}\right)k^\mu,$$

from which inequality (30) follows.

Lemma 6. If $P(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$, is a polynomial of degree n having all zeros in $|z| \leq k,\ k \leq 1$ and $m = Min_{|z|=k}|P(z)|$, then

$$\frac{m}{k^n} \leq Max_{|z|=1}|P(z)|. \tag{2.19}$$

Proof of Lemma 6. Since $P(z)$ has all its zeros in $|z| \leq k,\ k \leq 1$, we have from inequality (17) of Lemma 4,

$$|q'(z)| \leq A_\mu |P'(z)| - \frac{mnA_\mu}{k^n}, \quad for\ |z| = 1. \tag{2.20}$$

On using (1) in (32), we get for $|z| = 1$,

$$\begin{aligned} |q'(z)| &\leq A_\mu n Max_{|z|=1}|P(z)| - \frac{mnA_\mu}{k^n} \\ &= nA_\mu \Big\{ Max_{|z|=1}|P(z)| - \frac{m}{k^n} \Big\}, \end{aligned}$$

which is true and this proves (31).

3 Proof of the Theorems

Proof of Theorem 1. If $q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then it can be easily verified that

$$|q'(z)| = |nP(z) - zP'(z)| \quad , for\ |z| = 1.$$

Also for $|z| = 1$, we have

$$\begin{aligned} n|P(z)| &= |nP(z) - zP'(z) + zP'(z)| \\ &\leq |nP(z) - zP'(z)| + |P'(z)| \\ &= |q'(z)| + |P'(z)| \end{aligned} \tag{3.1}$$

The above inequality (33) when combined with inequality (17) of Lemma 4, gives for $|z| = 1$,

$$n|P(z)| \leq (1 + A_\mu)|P'(z)| - \frac{mnA_\mu}{k^n},$$

which implies

$$|P'(z)| \geq \left(\frac{n}{1 + A_\mu}\right) Max_{|z|=1}|P(z)| + \frac{mnA_\mu}{k^n(1 + A_\mu)} \quad , for \ \ |z| = 1. \tag{3.2}$$

Now for every real or complex number α with $|\alpha| \geq A_\mu$, the polar derivative of $P(z)$ with respect to α is

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

This implies for $|z| = 1$,

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \\ &= |\alpha||P'(z)| - |q'(z)| \end{aligned} \tag{3.3}$$

Combining inequalities (35) and (17), we get

$$|D_\alpha P(z)| \geq (|\alpha| - A_\mu)|P'(z)| + \frac{mnA_\mu}{k^n}, for|z| = 1. \tag{3.4}$$

Inequality (36) in conjunction with inequality (34) gives for $|z| = 1$,

$$|D_\alpha P(z)| \geq (|\alpha| - A_\mu)\left\{(\frac{n}{1+A_\mu})Max_{|z|=1}|P(z)| + \frac{mnA_\mu}{k^n(1+A_\mu)}\right\} + \frac{mnA_\mu}{k^n},$$

from which we can obtain Theorem 1.

Proof of Theorem 2. The proof of this theorem follows on the lines of the proof of Theorem 1, but on applying Lemma 1 instead of Lemma 4. We omit the details.

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