Thai Journal of Mathematics
Volume 4(2006) Number 2 : 321–328
www.math.science.cmu.ac.th/thaijournal

The Generalized Hyers-Ulam-Rassias Stability of a Quadratic Functional Equation

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Abstract : We give the general solution of the functional equation

$$f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z)=7f(x)+7f(y)+7f(z),$$

and investigate its generalized Hyers-Ulam-Rassias stability.

Keywords : Quadratic Functional Equation; Hyers-Ulam-Rassias Stability.
2000 Mathematics Subject Classification : 39B22, 39B52, 39B82.

1 Introduction

In 1940, S.M. Ulam [11] posed a problem on the stability of the linear functional equation before the Mathematics Club of the University of Wisconsin. The problem can be stated as follows:

> Let X and Y be Banach spaces. For every $\varepsilon > 0$, does there exist $\delta > 0$ such that for a function $f : X \to Y$ satisfying a δ-linear condition, $\|f(x+y) - f(x) - f(y)\| \le \delta$, for all $x, y \in X$, there exists a function $T : X \to Y$ such that $\|f(x) - T(x)\| \le \epsilon$ for all $x \in X$?

This problem was answered by D. H. Hyers [6] and was generalized by Th. M. Rassias [10]. Since then, the stability of various functional equations, including linear functional equations [5, 4, 7, 8] and quadratic functional equations [1, 2, 3, 9], was thoroughly studied,

The *classical* quadratic functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

admits a solution in the form $f(x) = ax^2$ on the set of real numbers and every solution of this equation is said to be a *quadratic function* [3]. J. M. Rassias [9] derived the stability of the generalized version of the above quadratic functional equation :

$$Q(a_1x_1+a_2x_2)+Q(a_2x_1-a_1x_2)=(a_1^2+a_2^2)[Q(x_1)+Q(x_2)],$$

which covers a wide range of quadratic functional equations in two variables.

In this paper, we propose a different quadratic functional equation in three variables:

$$f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z)=7f(x)+7f(y)+7f(z).$$

We will give the general solution of this functional equation and will investigate its generalized Hyers-Ulam-Rassias stability.

2 The General Solution

The following theorem provide the general solution of the proposed functional equation by establishing a connection with the classical quadratic functional equation.

Theorem 2.1 *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z)=7f(x)+7f(y)+7f(z) \quad (2.1)$$

for all $x, y, z \in X$ if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \quad (2.2)$$

for all $x, y \in X$.

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Setting $(x, y, z) = (x, x, x)$ in (2.1), we simply have $f(3x) = 9f(x)$ for all $x \in X$, which in turn implies that $f(0) = 0$ by putting $x = 0$. Setting $(x, y, z) = (-x, x, x)$ in (2.1), we have

$$f(-3x)+f(x)+f(3x)+2f(x)=7f(-x)+7f(x)+7f(x).$$

Using the previously derived equation, $f(3x) = 9f(x)$ for all $x \in X$, the above equation simplifies to $f(-x) = f(x)$ for all $x \in X$. Thus, f is an even function. Setting $(x, y, z) = (x, 0, 0)$ in (2.1), we get

$$f(2x)+f(0)+f(-x)+2f(x)=7f(x)+7f(0)+7f(0),$$

which yields $f(2x) = 4f(x)$ for all $x \in X$. Letting $z = 0$ in (2.1), we obtain

$$f(2x-y)+f(2y)+f(-x)+2f(x+y)=7f(x)+7f(y)+7f(0),$$

or simply

$$f(2x-y)+2f(x+y)=6f(x)+3f(y). \quad (2.3)$$

Letting $z = -y$ in (2.1), we have

$$f(2x-y)+f(3y)+f(-2y-x)+2f(x)=7f(x)+7f(y)+7f(-y),$$

which simplifies to

$$f(2x-y)+f(x+2y)=5f(x)+5f(y). \tag{2.4}$$

Eliminating $f(2x-y)$ from (2.3) and (2.4), we are left with

$$f(x+2y)+f(x)=2f(x+y)+2f(y).$$

The functional equation (2.2) follows from putting $(x,y)=(x-y,y)$ in the above equation.

Suppose that a function $f : X \to Y$ satisfies (2.2). Setting $(x,y)=(x+y,z)$ as well as the other two cyclic permutations of the variables x, y, and z in (2.2), we obtain a set of equations :

$$\begin{aligned} f(x+y+z)+f(x+y-z) &= 2f(x+y)+2f(z), \\ f(x+y+z)+f(x-y+z) &= 2f(x+z)+2f(y), \\ f(x+y+z)+f(-x+y+z) &= 2f(y+z)+2f(x). \end{aligned} \tag{2.5}$$

Setting $(x,y)=(x,y-z)$ and all cyclic permutations of the variables in (2.2), we have another set of equations :

$$\begin{aligned} f(x+y+z)+f(x-y+z) &= 2f(x)+2f(y-z), \\ f(-x+y+z)+f(x+y-z) &= 2f(y)+2f(z-x), \\ f(x-y+z)+f(-x+y+z) &= 2f(z)+2f(x-y). \end{aligned} \tag{2.6}$$

Subtracting half the sum of all equations in (2.6) from the sum of all equations in (2.5), we are left with

$$\begin{aligned} 3f(x+y+z) &= 2\left(f(x+y)+f(y+z)+f(z+x)\right) \\ &\quad -\left(f(x-y)+f(y-z)+f(z-x)\right)+\left(f(x)+f(y)+f(z)\right). \end{aligned} \tag{2.7}$$

If we rewrite (2.2) as $f(x+y)=2f(x)+2f(y)-f(x-y)$ and perform cyclic permutation of all variables, then (2.7) simplifies to

$$3f(x+y+z)=9\left(f(x)+f(y)+f(z)\right)-3\left(f(x-y)+f(y-z)+f(z-x)\right). \tag{2.8}$$

Setting $(x,y)=(x,x-y)$ and all cyclic permutations of variables in (2.2), we have

$$\begin{aligned} f(2x-y)+f(y) &= 2f(x)+2f(x-y), \\ f(2y-z)+f(z) &= 2f(y)+2f(y-z), \\ f(2z-x)+f(x) &= 2f(z)+2f(z-x). \end{aligned} \tag{2.9}$$

If we use (2.9) to eliminate $f(x-y), f(y-z)$, and $f(z-x)$ in (2.8), then (2.1) follows, and the proof is complete. □

3 The Generalized Hyers-Ulam-Rassias Stability

The following theorem gives a general condition for which a *true* quadratic function exists near an *approximately quadratic* function.

Theorem 3.1 *Let X be a real vector space and Y be a Banach space. Let $\phi : X^3 \to [0,\infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} \text{ converges and } \lim_{n\to\infty} \frac{\phi(3^n x, 3^n y, 3^n z)}{9^n} = 0 \tag{3.1}$$

for all $x, y, z \in X$. If a function $f : X \to Y$ satisfies

$$\|f(2x-y) + f(2y-z) + f(2z-x) + 2f(x+y+z) - 7f(x) - 7f(y) - 7f(z)\| \le \phi(x,y,z) \tag{3.2}$$

for all $x, y, z \in X$, then there exists a unique function $T : X \to Y$ which satisfies (2.1) and

$$\|f(x) - T(x)\| \le \frac{1}{18} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} \quad \forall x \in X. \tag{3.3}$$

The function T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{9^n} \quad \forall x \in X. \tag{3.4}$$

Proof. Setting $x = y = z$ in (3.2), we have

$$\|3f(x) + 2f(3x) - 21f(x)\| \le \phi(x,x,x) \quad \forall x \in X.$$

Dividing the above inequality by 18, we get

$$\left\| \frac{f(3x)}{9} - f(x) \right\| \le \frac{1}{18}\phi(x,x,x) \tag{3.5}$$

Suppose that

$$\left\| \frac{f(3^n x)}{9^n} - f(x) \right\| \le \frac{1}{18} \sum_{i=0}^{n-1} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i}$$

for a positive integer n and for all $x \in X$. Then

$$\begin{aligned}
\left\| \frac{f(3^{n+1} x)}{9^{n+1}} - f(x) \right\| &\le \left\| \frac{f(3^{n+1} x)}{9^{n+1}} - \frac{f(3^n x)}{9^n} \right\| + \left\| \frac{f(3^n x)}{9^n} - f(x) \right\| \\
&= \frac{1}{9^n} \left\| \frac{f(3 \cdot 3^n x)}{9} - f(3^n x) \right\| + \left\| \frac{f(3^n x)}{9^n} - f(x) \right\| \\
&\le \frac{1}{9^n} \cdot \frac{1}{18}\phi(3^n x, 3^n x, 3^n x) + \frac{1}{18} \sum_{i=0}^{n-1} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} \\
&= \frac{1}{18} \sum_{i=0}^{n} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i}
\end{aligned}$$

By mathematical induction, we conclude that

$$\left\| \frac{f(3^n x)}{9^n} - f(x) \right\| \leq \frac{1}{18} \sum_{i=0}^{n-1} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i}$$

for every positive integer n and for all $x \in X$.

We have to show that the sequence $\left\{ \frac{f(3^n x)}{9^n} \right\}$ converges for all $x \in X$. For every positive integer m, consider

$$\begin{aligned} \left\| \frac{f(3^{n+m} x)}{9^{n+m}} - \frac{f(3^n x)}{9^n} \right\| &= \frac{1}{9^n} \left\| \frac{f(3^m \cdot 3^n x)}{9^m} - f(3^n x) \right\| \\ &\leq \frac{1}{18 \cdot 9^n} \sum_{i=0}^{m-1} \frac{\phi(3^i \cdot 3^n x, 3^i \cdot 3^n x, 3^i \cdot 3^n x)}{9^i} \\ &\leq \frac{1}{18} \sum_{i=0}^{\infty} \frac{\phi(3^{i+n} x, 3^{i+n} x, 3^{i+n} x)}{9^{i+n}}. \end{aligned}$$

By condition (3.1), the right-hand side approaches 0 when $n \to \infty$. Thus, the sequence is a Cauchy sequence. Due to the completeness of the Banach space, Y,

$$T(x) = \lim_{n \to \infty} \frac{f(3^n x)}{9^n} \quad \forall x \in X$$

is well-defined. We can see that (3.3) holds.

To show that T indeed satisfies (2.1), we set $(x, y, z) = (3^n x, 3^n y, 3^n z)$ in (3.2) and divide the resulting equation by 9^n. The result is

$$\begin{aligned} &\frac{1}{9^n} \| f(3^n(2x - y)) + f(3^n(2y - z)) + f(3^n(2z - x)) \\ &\quad + 2f(e^n(x + y + z)) - 7f(3^n x) - 7f(3^n y) - 7f(3^n z) \| \leq \frac{\phi(3^n x, 3^n y, 3^n z)}{9^n}. \end{aligned}$$

Taking the limit as $n \to \infty$ and noting the definition of T, the above equation becomes

$$\|T(2x - y) + T(2y - z) + T(2z - x) + 2T(x + y + z) - 7T(x) - 7T(y) - 7T(z)\| \leq 0$$

for all $x, y, z \in X$. Therefore, T satisfies (2.1).

To prove the uniqueness of T, suppose that there exists another quadratic function $S : X \to Y$ such that S satisfies (2.1) and

$$\|f(x) - S(x)\| \leq \frac{1}{18} \sum_{i=0}^{\infty} \frac{\phi(3^i x, 3^i x, 3^i x)}{9^i} \quad \forall x \in X.$$

By Theorem 2.1, every solution of (2.1) is also a solution of the classical quadratic functional equation and, thus, exhibits the quadratic property; i.e., $T(nx) =$

$n^2T(x)$ for every positive integer n and for all $x \in X$. Therefore,

$$\begin{aligned}\|S(x)-T(x)\| &= \frac{1}{9^n}\|S(3^nx)-T(3^nx)\| \\ &\le \frac{1}{9^n}\|S(3^nx)-f(3^nx)\| + \frac{1}{9^n}\|T(3^nx)-f(3^nx)\| \\ &\le 2\cdot\frac{1}{9^n}\cdot\frac{1}{18}\sum_{i=0}^{\infty}\frac{\phi(3^i\cdot 3^nx, 3^i\cdot 3^nx, 3^i\cdot 3^nx)}{9^i} \\ &= \frac{1}{9}\sum_{i=0}^{\infty}\frac{\phi(3^{i+n}x, 3^{i+n}x, 3^{i+n}x)}{9^{i+n}} \quad \forall x \in X.\end{aligned}$$

By condition (3.1), the right-hand side goes to 0 as $n \to \infty$, and it follows that $S(x) = T(x)$ for all $x \in X$. Hence, T is unique. □

Theorem 3.2 *Let X be a real vector space and Y be a Banach space. Let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{i=1}^{\infty} 9^i\phi(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}) \text{ converges and } \lim_{n\to\infty} 9^n\phi(\frac{x}{3^n}, \frac{x}{3^n}, \frac{x}{3^n}) = 0 \tag{3.6}$$

for all $x, y, z \in X$. If a function $f : X \to Y$ satisfies

$$\begin{aligned}\|f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z) \\ -7f(x)-7f(y)-7f(z)\| \le \phi(x,y,z)\end{aligned} \tag{3.7}$$

for all $x, y, z \in X$, then there exists a unique function $T : X \to Y$ which satisfies (2.1) and

$$\|f(x)-T(x)\| \le \frac{1}{18}\sum_{i=1}^{\infty} 9^i\phi(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}) \quad \forall x \in X. \tag{3.8}$$

The function T is given by

$$T(x) = \lim_{n\to\infty} 9^n f(\frac{x}{3^n}) \quad \forall x \in X. \tag{3.9}$$

Proof. The proof begins in the same manner as that of Theorem 3.1. The only difference starts with the replacement of inequality (3.5) by

$$\left\| f(x) - 9f(\frac{x}{3})\right\| \le \frac{1}{2}\phi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}),$$

which can be extended by mathematical induction to

$$\left\| f(x) - 9^n f(\frac{x}{3^n})\right\| \le \frac{1}{18}\sum_{i=1}^{n} 9^i\phi(\frac{x}{3^i}, \frac{x}{3^i}, \frac{x}{3^i}).$$

The rest of the proof follows in the same fashion and will be omitted here. □

It should be noted that the Hyers-Ulam stability can be obtained from Theorem 3.1 with an appropriate choice of the function ϕ in the following corollary.

Corollary 3.3 *If a function $f : X \to Y$ satisfies the functional equation*

$$\|f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z)-7f(x)-7f(y)-7f(z)\| \le \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y, z \in X$, then there exists a unique quadratic function $T : X \to Y$ such that

$$\|f(x)-T(x)\| \le \frac{\varepsilon}{16} \quad \forall x \in X.$$

Proof. We choose $\phi(x, y, z) = \varepsilon$ for all $x, y, z \in X$. Being in accordance with condition (3.1) in Theorem 3.1, it follows that

$$\|f(x)-T(x)\| \le \frac{1}{18}\sum_{i=0}^{\infty}\frac{\varepsilon}{9^i} = \frac{\varepsilon}{16}$$

for all $x \in X$ as desired. □

For the Hyers-Ulam-Rassias stability, we again choose an appropriate ϕ function, then apply Theorems 3.1 and 3.2.

Corollary 3.4 *If a function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned}\|f(2x-y)+f(2y-z)+f(2z-x)+2f(x+y+z)\\ -7f(x)-7f(y)-7f(z)\| \le \varepsilon\left(\|x\|^p+\|y\|^p+\|z\|^p\right)\end{aligned}$$

with $0 < p < 2$ or $p > 2$ for some $\varepsilon > 0$ and for all $x, y, z \in X$, then there exists a unique quadratic function $T : X \to Y$ such that

$$\|f(x)-T(x)\| \le \frac{3\varepsilon}{2|9-3^p|}\|x\|^p \quad \forall x \in X.$$

Proof. We choose $\phi(x, y, z) = \varepsilon\left(\|x\|^p+\|y\|^p+\|z\|^p\right)$ for all $x, y, z \in X$. If $0 < p < 2$, then condition (3.1) in Theorem 3.1 is fulfilled, and, consequently,

$$\|f(x)-T(x)\| \le \frac{1}{18}\sum_{i=0}^{\infty}\frac{3\varepsilon\cdot\|3^i x\|^p}{9^i} = \frac{3\varepsilon}{2(9-3^p)}\|x\|^p.$$

If $p > 2$, then condition (3.6) in Theorem 3.2 is fulfilled, and, consequently,

$$\|f(x)-T(x)\| \le \frac{1}{18}\sum_{i=1}^{\infty}\left(9^i\cdot 3\varepsilon\|3^{-i}x\|^p\right) = \frac{3\varepsilon}{2(3^p-9)}\|x\|^p.$$

This completes the proof. □

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(Received 23 January 2006)

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