



Approximation of Fixed Points of Nonexpansive Mappings by Modified Krasnoselski-Mann Iterative Algorithm in Banach Space

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Abstract : Let E be a real uniformly convex Banach space which is also uniformly smooth. For each $n = 1, 2, \dots$, let $T_n : E \rightarrow E$ be a nonexpansive mapping such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. A strong convergence theorem is proved for approximation of common fixed points of $\{T_n\}$ using a modified Krasnoselski-Mann iterative algorithm introduced by Yao et al. [Y. Yao, H. Zhou, Y.C. Liou, Strong convergence of a modified Krasnoselski-Mann iterative algorithm for nonexpansive mappings, *J. Appl. Math. Comput.* 29 (2009) 383–389]. As applications, we prove strong convergence theorems for approximation of common zeroes of a finite family of continuous accretive mappings of E into E and approximation of common fixed point (assuming existence) of a finite family of continuous pseudocontractive mappings in a real uniformly convex and uniformly smooth Banach space. Our result extends many important recent results in the literature.

Keywords : nonexpansive mapping; fixed point; Krasnoselski-Mann iterations; strong convergence.

2010 Mathematics Subject Classification : 47H09; 47H10.

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1 Introduction

Let E be a real Banach space and K a nonempty, closed and convex subset of E . We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E) defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

A mapping $T : K \rightarrow K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. We denote by $F(T) = \{x \in K : Tx = x\}$ the set of fixed points of T . It is assumed throughout that $F(T) \neq \emptyset$.

One of the most important fixed point theorems and application is the classical contraction mapping principle, or, in other words, the Banach-Cacciopoli fixed point theorem which is the following

Theorem 1.1 (Banach [1], Cacciopoli [2]). *Let (X, ρ) be a complete metric space and let $T : (X, \rho) \rightarrow (X, \rho)$ satisfy*

$$\rho(T(x), T(y)) \leq \gamma \rho(x, y). \quad (1.1)$$

for some nonnegative constant $\gamma < 1$ and for each $x, y \in X$. Then T has a unique fixed point in X . Moreover, starting with arbitrary $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0, \quad (1.2)$$

converges strongly to the unique fixed point.

The iterative technique (1.2) is due to Picard [3]. A mapping T satisfying (1.1) is called a *strict contraction*. If $\gamma = 1$ in the relation (1.1), then T is called nonexpansive. For the iterative formula, it was observed that if the condition $\gamma < 1$ on the operator T is weakened to $\gamma = 1$, the sequence $\{x_n\}$ defined by (1.2) may fail to converge to a fixed point of T . This can be seen by considering an anticlockwise rotation of the unite disc of \mathbb{R}^2 about the origin through an angle of say, $\frac{\pi}{4}$. This map is nonexpansive but the Picard sequence fails to converge. Krasnosel'ski [4], however, showed that in this example, if the Picard iteration formula is replaced by the following formula

$$x_0 \in X, \quad x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad n \geq 0,$$

then the iterative sequence converges to the fixed point. However, in 1953, the most general iterative formula for approximation of fixed points of nonexpansive mapping which is called Krasnoselski-Mann iterative algorithm where introduced (in the light of [5]) as follows:

$$x_0 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \quad (1.3)$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$. Though simple in form, the Krasnoselski-Mann iteration is remarkably useful for finding fixed points of a nonexpansive mapping and provides a unified framework for many algorithms from various different fields. In this respect, the following result is basic and important.

Theorem 1.2 (Genel and Lindenstrass [6]). *Let T be a nonexpansive mapping on a real Hilbert space H . Then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T , provided $\alpha_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$, whenever each fixed points exist.*

However, we note that all results in the literature on the Krasnoselki-Mann iterative algorithm for nonexpansive mappings have only weak convergence even in a real Hilbert space. For more details, please see [6]. In [7, 8], Yang and Zhao further generalized the Krasnoselki-Mann iteration and proposed the generalized KM theorems. For the details, please see [7, 8].

Construction of fixed points of nonexpansive mappings is an important subject in nonlinear mapping theory and its applications; in particular, in image recovery and signal processing (see, for example, [9–11]). Many authors have worked extensively on the approximation of fixed points of nonexpansive mappings. For example, the reader can consult the recent monographs of Berinde [12] and Chidume [13].

Recently, Yao et al. [14] introduced a modified Krasnoselki-Mann iterative algorithm for nonexpansive mapping in the framework of a real Hilbert space and proved the following theorem.

Theorem 1.3 (Yao et al. [14]). *Let H be a real Hilbert space. Let $T : H \rightarrow H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x_0 \in H$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} y_n = (1 - \alpha_n)x_n; \\ x_{n+1} = (1 - \beta_n)y_n + \beta_nTy_n, \quad n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ satisfied the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a point in $F(T)$.

A mapping $A : D(A) \subseteq E \rightarrow E$ is called *accretive* if, for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \tag{1.5}$$

By the results of Kato [15], equation (1.5) is equivalent to

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|, \quad \forall s > 0. \tag{1.6}$$

If E is a Hilbert space, accretive mappings are also called *monotone*. A mapping A is called *m-accretive* if it is accretive and $R(I + rA)$, range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $cl(D(A)) \subseteq R(I + rA), \forall r > 0$,

where $cl(D(A))$ denotes the closure of the domain of A . A is said to be *maximal accretive* if it is accretive and the inclusion $G(A) \subset G(B)$, where $G(A)$ is a graph of A , with B accretive, implies $G(A) = G(B)$. It is known (see e.g., [16]) that every maximal accretive mapping is m -accretive and the converse holds if E is a Hilbert space. Interest in accretive mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., [17]) that many physically significant problems can be modelled by initial-value problems of the form

$$u'(t) + Au(t) = 0, \quad u(0) = u_0, \quad (1.7)$$

where A is an accretive mapping in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. One of the fundamental results in the theory of accretive mappings, due to Browder [18], states that if A is locally Lipschitzian and accretive, then A is m -accretive. This result was subsequently generalized by Martin [19] to the continuous accretive mappings. If in (1.7), $u(t)$ is independent of t , then (1.7) reduces to

$$Au = 0, \quad (1.8)$$

whose solutions correspond to the equilibrium points of the system (1.7). Consequently, considerable research efforts have been devoted, especially within the past 40 years or so, to iterative methods for approximating these equilibrium points.

Closely related to the class of accretive mappings is the class of pseudocontractive mappings. An mapping T with domain $D(T)$ in E and range $R(T)$ in E is called *pseudocontractive* if $A := I - T$ is accretive. It is then clear that any zero of A is a fixed point of T . Consequently, the study of approximating fixed points of pseudocontractive mappings, which correspond to equilibrium points of the system (1.7), became a flourishing area of research for numerous mathematicians (see, e.g., [20–22] and the references therein).

It is not difficult to deduce from (1.6) that the mapping A is accretive if and only if $(I + rA)^{-1}$, $\forall r > 0$, is nonexpansive on the range of $(I + rA)$. Thus, in particular, $J_A := (I + A)^{-1}$ is nonexpansive and single valued on the range of $(I + A)$. Furthermore, $F(J_A) := N(A) := \{x \in D(A) : Ax = 0\}$. It is well known that every nonexpansive mapping is pseudocontractive and the converse does not, however, hold.

It is our purpose in this paper to strong convergence theorem for approximation of common fixed points of an infinite family $\{T_n\}$ of nonexpansive mappings using a modified Krasnoselki-Mann iterative algorithm introduced by Yao et al. [14] in a real uniformly convex and uniformly smooth Banach space. As applications, we prove strong convergence theorem for approximation of common zeroes of a finite family of continuous accretive mappings of E to E and approximation of common fixed point (assuming existence) of a finite family of continuous pseudocontractive mappings. Our theorem, is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$. Our result extends the results of Yao et al. [14] from Hilbert space to uniformly convex and uniformly smooth Banach space.

2 Preliminaries

Let K be a nonempty, closed, convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [23], see also Lim [24] and the references contained therein) is defined by $N(E) := \inf\{\frac{d(K)}{r(K)} : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structure*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., [13, 25]).

In the sequel, we shall also make use of the following lemmas.

Lemma 2.1. *Let E be a real normed linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall x, y \in E, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2 (Xu [26], Zalinescu [27, 28]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0, 1]$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \rightarrow \mathbb{R}, \quad g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|).$$

Lemma 2.3 (Browder [29], Goebel and Kirk [30]). *Let E be a uniformly convex Banach space, K a closed convex subset of E , and $T : K \rightarrow K$ a nonexpansive mapping with a fixed point. Assume that a sequence $\{x_n\}$ in K is such that $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow y$. Then $x - Tx = y$.*

Lemma 2.4 (Lim and Xu [25]). *Suppose E is a Banach space with uniform normal structure, K a nonempty bounded subset of E and $T : K \rightarrow K$ is a uniformly L -Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset C of K with the following property (P):*

$$x \in C \text{ implies } \omega_w(x) \in C,$$

(where $\omega_w(x)$ is the ω -limit set of T at x , that is, the set $\{y \in E : y = \text{weak } \omega - \lim T^{n_j}x \text{ for some } n_j \rightarrow \infty\}$). Then T has a fixed point in C .

Lemma 2.5 (Shioji and Takahashi [31]). *Let $(x_0, x_1, x_2, \dots) \in l_\infty$ be such that $\mu_n x_n \leq 0$ for all Banach limits μ . If $\limsup_{n \rightarrow \infty} (x_{n+1} - x_n) \leq 0$, then $\limsup_{n \rightarrow \infty} x_n \leq 0$.*

Lemma 2.6 (Cioranescu [16]). *Let A be a continuous accretive mapping defined on a real Banach space E with $D(A) = E$. Then A is m -accretive.*

Lemma 2.7 (Zegeye and Shahzad [32]). *Let K be a nonempty, closed and convex subset of a real strictly convex Banach space E . For each $r = 1, 2, \dots, N$ let $A_r : K \rightarrow E$ be an m -accretive mapping such that $\bigcap_{r=1}^N N(A_i) \neq \emptyset$. Let $a_0, a_1, a_2, \dots, a_N$ be real numbers in $(0, 1)$ such that $\sum_{i=0}^N a_i = 1$ and let $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \dots + a_N J_{A_N}$, with $J_{A_r} := (I + A_r)^{-1}$. Then S_N is nonexpansive and $F(S_N) = \bigcap_{r=1}^N N(A_r)$.*

Lemma 2.8 (Xu [33]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; ($n \geq 0$), $\sum \gamma_n < \infty$. Then, $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2.9 (Nilsrakoo and Saejung [34], Aoyama et al. [35]). *Let K be a nonempty closed subset of a Banach space and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of K into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_nx\| : x \in K\} < \infty$. Then, for each $x \in K$, $\{T_nx\}_{n=1}^{\infty}$ converges strongly to some point of K . Moreover, let T be a mapping of K into itself defined by $Tx := \lim_{n \rightarrow \infty} T_nx$, for all $x \in K$. Then, $\lim_{n \rightarrow \infty} \sup\{\|T_nx - Tx\| : x \in K\} = 0$.*

3 Main Results

Theorem 3.1. *Let E be a real uniformly convex Banach space which is also uniformly smooth. For each $n = 1, 2, \dots$, let $T_n : E \rightarrow E$ be a nonexpansive mapping such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in E$,*

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, \quad n \geq 1, \end{cases} \quad (3.1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ satisfy the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_nx\| : x \in B\} < \infty$ for any bounded subset B of E . Let T be a mapping of K into itself defined by $Ty = \lim_{n \rightarrow \infty} T_ny$ for all $y \in E$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ converges strongly to a point in $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. First we show that $\{x_n\}$ is bounded. For any $p \in \bigcap_{n=1}^{\infty} F(T_n)$, from (3.1) we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \beta_n)\|y_n - p\| + \beta_n\|T_n y_n - p\| \\ &\leq \|y_n - p\| \\ &= \|(1 - \alpha_n)x_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|, \|p\|\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \|p\|\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded and also are $\{y_n\}$ and $\{T_n y_n\}$.

Using Lemma 2.2 and (3.1), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \beta_n)(y_n - p) + \beta_n(T_n y_n - p)\|^2 \\ &\leq (1 - \beta_n)\|y_n - p\|^2 + \beta_n\|T_n y_n - p\|^2 - \beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|) \\ &\leq (1 - \beta_n)\|y_n - p\|^2 + \beta_n\|y_n - p\|^2 - \beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|) \\ &= \|y_n - p\|^2 - \beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|). \end{aligned}$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned} \beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|) &\leq \|y_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n \langle x_n, j(y_n - p) \rangle. \end{aligned} \tag{3.2}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists a constant $M \geq 0$ such that

$$\langle x_n, j(y_n - p) \rangle \leq M \quad \text{for all } n \geq 1.$$

So, from (3.2) we have

$$\beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M. \tag{3.3}$$

To prove that $\{x_n\}$ converges to p , we have two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \rightarrow 0.$$

It then implies from (3.3) that

$$\lim_{n \rightarrow \infty} \beta_n(1 - \beta_n)g(\|T_n y_n - y_n\|) = 0.$$

Using the condition $\beta_n \in [a, b] \subset (0, 1)$ and the property of g , we have

$$\lim_{n \rightarrow \infty} \|T_n y_n - y_n\| = 0. \quad (3.4)$$

Now from (3.4), we obtain

$$\|y_n - x_{n+1}\| = \beta_n \|T_n y_n - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

From (3.1), we know that

$$\|y_n - x_n\| = \alpha_n \|x_n\| \rightarrow 0. \quad (3.6)$$

Therefore, from (3.6) and (3.5), we have

$$\|x_{n+1} - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Also, from (3.4) and (3.6), we have

$$\begin{aligned} \|T_n x_n - x_n\| &\leq \|T_n x_n - T_n y_n\| + \|T_n y_n - y_n\| + \|y_n - x_n\| \\ &\leq 2\|x_n - y_n\| + \|T_n y_n - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.7)$$

At the same time, we observe that

$$\|x_n - T x_n\| \leq \|T x_n - T_n x_n\| + \|x_n - T_n x_n\|. \quad (3.8)$$

By (3.8) and Lemma 2.9, we have that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0.$$

Furthermore,

$$\begin{aligned} \|y_n - T y_n\| &\leq \|y_n - x_n\| + \|T y_n - T x_n\| + \|x_n - T x_n\| \\ &\leq 2\|y_n - x_n\| + \|x_n - T x_n\| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $p \in E$. Furthermore, since $I - T$ is demiclosed, then $p \in F(T)$.

We now prove that

$$\limsup_{n \rightarrow \infty} \langle -p, j(y_n - p) \rangle \leq 0.$$

Define a map $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(x) := \mu_n \|y_n - x\|^2, \quad \forall x \in E.$$

Then, $\phi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $y^* \in E$ such that $\phi(y^*) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \left\{ x \in E : \phi(x) = \min_{u \in E} \phi(u) \right\} \neq \emptyset.$$

We shall make use of Lemma 2.4. If $x \in K^*$ and $y := \omega - \lim_{j \rightarrow \infty} T^{m_j}x$, then from weak lower semi-continuity of ϕ and $\lim_{n \rightarrow \infty} \|y_n - Ty_n\| = 0$, we have (since $\lim_{n \rightarrow \infty} \|y_n - Ty_n\| = 0$ implies $\lim_{n \rightarrow \infty} \|y_n - T^m y_n\| = 0$, $m \geq 1$, this is easily proved by induction),

$$\begin{aligned} \phi(y) &\leq \liminf_{j \rightarrow \infty} \phi(T^{m_j}x) \leq \limsup_{m \rightarrow \infty} \phi(T^m x) \\ &= \limsup_{m \rightarrow \infty} (\mu_n \|y_n - T^m x\|^2) \\ &= \limsup_{m \rightarrow \infty} (\mu_n \|y_n - T^m y_n + T^m y_n - T^m x\|^2) \\ &\leq \limsup_{m \rightarrow \infty} (\mu_n \|T^m y_n - T^m x\|^2) \\ &\leq \limsup_{m \rightarrow \infty} (\mu_n \|y_n - x\|^2) \\ &= \phi(x) = \inf_{u \in E} \phi(u). \end{aligned}$$

So, $y^* \in K^*$. By Lemma 2.4, T has a fixed point in K^* and so $K^* \cap F(T) \neq \emptyset$. Without loss of generality, assume that $y^* = p \in K^* \cap F(T)$. Let $t \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - tp)$ and using Lemma 2.1, we obtain that

$$\|y_n - p + tp\|^2 \leq \|y_n - p\|^2 + 2t \langle p, j(y_n - p + tp) \rangle$$

which implies that

$$\mu_n \langle -p, j(y_n - p + tp) \rangle \leq 0.$$

Moreover,

$$\begin{aligned} \mu_n \langle -p, j(y_n - p) \rangle &= \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp) \rangle \\ &\quad + \mu_n \langle -p, j(y_n - p + tp) \rangle \\ &\leq \mu_n \langle -p, j(y_n - p) - j(y_n - p + tp) \rangle. \end{aligned}$$

Since j is norm-to-norm continuous on bounded subsets of E , we have that

$$\mu_n \langle -p, j(y_n - p) \rangle \leq 0.$$

Observe that

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

Then by (3.6) and (3.5), we have

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0.$$

This implies that

$$\limsup_{n \rightarrow \infty} \left[\langle -p, j(y_n - p) \rangle - \langle -p, j(y_{n+1} - p) \rangle \right] \leq 0$$

and so we obtain by Lemma 2.5 that

$$\limsup_{n \rightarrow \infty} \langle -p, j(y_n - p) \rangle \leq 0.$$

Finally, from the recursion formula (3.1) and Lemma 2.1, we have the following;

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \beta_n)(y_n - p) + \beta_n(T_n y_n - p)\|^2 \\ &\leq (1 - \beta_n)\|y_n - p\|^2 + \beta_n\|T_n y_n - p\|^2 \\ &\leq \|y_n - p\|^2 \\ &= \|(1 - \alpha_n)x_n - p\|^2 \\ &= \|(1 - \alpha_n)(x_n - p) - \alpha_n p\|^2 \\ &\leq (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle -p, j(y_n - p) \rangle. \end{aligned}$$

By using Lemma 2.8, we have that $\{x_n\}$ converges strongly to $p \in \cap_{n=1}^\infty F(T_n)$.

Case 2. Assume that $\{\|x_n - p\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and let $\tau : N \rightarrow N$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.3), we see that

$$\beta_{\tau(n)}(1 - \beta_{\tau(n)})g(\|T_{\tau(n)}y_{\tau(n)} - y_{\tau(n)}\|) \leq 2\alpha_{\tau(n)}M \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore, we have

$$\|T_{\tau(n)}y_{\tau(n)} - y_{\tau(n)}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the same argument as in Case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ converge weakly to p as $\tau(n) \rightarrow \infty$ and $\limsup_{\tau(n) \rightarrow \infty} \langle -p, j(y_{\tau(n)} - p) \rangle \leq 0$. We know that for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \leq 2\alpha_n[\langle -p, j(y_{\tau(n)} - p) \rangle - \|x_{\tau(n)} - p\|^2],$$

which implies that

$$\|x_{\tau(n)} - p\|^2 \leq \langle -p, j(y_{\tau(n)} - p) \rangle.$$

Then we conclude that

$$\lim_{n \rightarrow \infty} \|x_{\tau(n)} - p\| = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} \Gamma_{\tau(n)} = \lim_{n \rightarrow \infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n \rightarrow \infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to p . Consequently, it is easy to prove that $\{y_n\}$ converges strongly to p . This complete the proof. \square

Remark 3.1. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive mappings of E into E and let $\{\xi_n\}_{n=1}^\infty$ be a sequence of real numbers such that $0 \leq \xi_n \leq 1$. For each $n \geq 1$, define a mapping W_n of E into E as follows:

$$\left\{ \begin{array}{l} U_{n,n+1} = I, \\ U_{n,n} = \xi_n T_n U_{n,n+1} + (1 - \xi_n)I, \\ U_{n,n-1} = \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\ \dots \\ U_{n,k} = \xi_k T_k U_{n,k+1} + (1 - \xi_k)I, \\ U_{n,k-1} = \xi_{k-1} T_{k-1} U_{n,k} + (1 - \xi_{k-1})I, \\ \dots \\ U_{n,2} = \xi_2 T_2 U_{n,3} + (1 - \xi_2)I, \\ W_n = U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1)I. \end{array} \right.$$

Such mapping is called W_n is called W -mapping generated by T_n, T_{n-1}, \dots, T_1 and $\xi_n, \xi_{n-1}, \dots, \xi_1$, see, for example, [36]. Clearly, W_n is nonexpansive and from [36], we know that $F(W_n) = \cap_{n=1}^\infty F(T_n)$. Furthermore, from [37], we have the sequence $\{W_n\}$ satisfies the condition

$$\sum_{n=1}^\infty \sup\{\|W_{n+1}x - W_nx\| : x \in D\} < \infty \text{ for any bounded subset } D \text{ of } E$$

imposed in Theorem 3.1.

By the above remark, we obtain the following theorem using Theorem 3.1.

Theorem 3.2. Let E be a real uniformly convex Banach space which is also uniformly smooth. For each $n = 1, 2, \dots$, let $T_n : E \rightarrow E$ be a nonexpansive mapping such that $\cap_{n=1}^\infty F(T_n) \neq \emptyset$. Let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in E$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n W_n y_n, \end{cases} \quad n \geq 1, \tag{3.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ satisfy the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^\infty \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a point in $\cap_{n=1}^\infty F(T_n)$.

We next prove a strong convergence theorem for approximation common zeroes of a finite family of continuous accretive mappings of E into E . Let $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \dots + a_N J_{A_N}$, with $J_{A_r} := (I + A_r)^{-1}$, $r = 1, 2, \dots, N$ for $0 < a_i < 1$, $i = 0, 1, 2, \dots, N$, $\sum_{i=0}^N a_i = 1$.

Theorem 3.3. Let E be a real uniformly convex Banach space which is also uniformly smooth. For each $r = 1, 2, \dots, N$, let $A_r : E \rightarrow E$ be a continuous accretive mapping such that $\cap_{r=1}^N N(A_r) \neq \emptyset$ and $J_{A_r} := (I + A_r)^{-1}$ for each $r = 1, 2, \dots, N$. For given $x_1 \in E$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S_N y_n, \quad n \geq 1, \end{cases} \quad (3.10)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ satisfy the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a point in $\cap_{r=1}^N N(A_r)$.

Proof. By Lemma 2.6, we have that A_r is m -accretive for each $r = 1, 2, \dots, N$. By Lemma 2.7, S_N is nonexpansive and $F(S_N) = \cap_{r=1}^N N(A_r)$. By Theorem 3.1, we obtain the desired result. \square

The following theorems give strong convergence to a common fixed point of a finite family of pseudocontractive mappings.

Theorem 3.4. Let E be a real uniformly convex Banach space which is also uniformly smooth. For each $r = 1, 2, \dots, N$, let $T_r : E \rightarrow E$ be a continuous pseudocontractive mapping on E such that $\cap_{r=1}^N F(T_r) \neq \emptyset$ and $J_{T_r} := (2I - T_r)^{-1}$. For given $x_1 \in E$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n S_N y_n, \quad n \geq 1, \end{cases} \quad (3.11)$$

where $S_N := a_0 I + a_1 J_{T_1} + a_2 J_{T_2} + \dots + a_N J_{T_N}$ for $0 < a_i < 1$, $i = 0, 1, 2, \dots, N$, $\sum_{i=0}^N a_i = 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$ satisfy the following conditions:

(C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a point in $\cap_{r=1}^N F(T_r)$.

Proof. Let $A_r := (I - T_r)$ for each $r = 1, 2, \dots, N$. Then, clearly, $F(T_r) = N(A_r)$ and hence $\cap_{r=1}^N N(A_r) = \cap_{r=1}^N F(T_r) \neq \emptyset$. Furthermore, each A_r for $r = 1, 2, \dots, N$ is m -accretive. The result follows from Theorem 3.3. \square

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(Received 24 March 2012)

(Accepted 26 March 2013)