Thai Journal of Mathematics
Volume 13 (2015) Number 2 : 399–403

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

A Congruence Relation in Partially Ordered Sets

C. Ganesamoorthy† and SG. Karpagavalli‡,[1]

†Department of Mathematics, Alagappa University,
Karaikudi-630 004, Tamil Nadu, India
e-mail : ganesamoorthyc@gmail.com

‡Department of Mathematics, Dr. Umayal Ramanathan College for Women
Karaikudi-630 004, Tamil Nadu, India
e-mail : sgkarpa@gmail.com

Abstract : A concept of congruence relations in partially ordered sets is introduced in this article. A fundamental theorem for homomorphism is obtained. Further extensions to “doubly directed” sets are obtained.

Keywords : directed set; congruence relation; partition.
2010 Mathematics Subject Classification : 06B10; 18B35.

1 Introduction

Congruence relations are studied in lattices and semilattices(see, for example, [1] and [2]). A definiton for congruence relations in partially ordered sets is proposed in this article. A partial order on a set is an order which is reflexive, anti-symmetric and transitive. A set with a partial order is called a poset(partially ordered set). A poset in which any two elements have a least upper bound and a greatest lower bound is called a lattice. An equivalence relation in a set is a relation which is reflexive, symmetric and transitive. The collection of all equivalence classes on a set corresponding to an equivalence relation form a partition. On the other hand each partition on a set corresponds to an equivalence relation. The books ([3],[4]) are referred to fundamental concepts and results. In a lattice

[1]Corresponding author.

$(L, \leq)$ or $(L, \vee, \wedge)$, an equivalence relation θ, denoted by $x \equiv y$ (mod θ) when x and y are related in L, is called a congruence relation, if it has the substitution properties: $x \wedge y \equiv y \wedge z$ (mod θ) and $x \vee y \equiv y \vee z$ (mod θ) whenever $x \equiv y$ (mod θ) in L and $z \in L$. A definition for congruence relations in posets will be given in the next section. A corresponding fundamental theorem of homomorphism is obtained. The definitions and the theorem are also extended to "doubly directed sets". Let us say that a poset (P, $\leq$) is a doubly directed set, if for given $a, b \in P$, there are c and d in P such that $c \leq a \leq d$ and $c \leq b \leq d$.

2 Some definitions and examples

Definition 2.1. *Let $(P, \leq)$ be a poset. Let θ be an equivalence relation on P. It is called a congruence relation, if the following hold in P.*

(i) If $x_1 \leq x_2$ and $x_1 \equiv y_1$ (mod θ), then there is an element y_2 in P such that $x_2 \equiv y_2$ (mod θ) and $y_1 \leq y_2$.

(ii) If $x_1 \leq x_2$ and $x_2 \equiv y_2$ (mod θ), then there is an element y_1 in P such that $x_1 \equiv y_1$ (mod θ) and $y_1 \leq y_2$.

(iii) If $x \equiv z$ (mod θ) and $x \leq y \leq z$, then $x \equiv y$ (mod θ).

Example 2.1. *Let θ be an usual (lattice) congruence relation on a lattice $(L, \leq)$ or $(L, \vee, \wedge)$ mentioned in the previous section. If $x_1 \leq x_2$ and $x_1 \equiv y_1$ (mod θ) in L , then $x_2 = x_1 \vee x_2 \equiv y_1 \vee x_2$ (mod θ) and $y_1 \leq y_1 \vee x_2$. If $x_1 \leq x_2$ and $x_2 \equiv y_2$ (mod θ) in L , then $x_1 = x_1 \wedge x_2 \equiv x_1 \wedge y_2$ (mod θ) and $x_1 \wedge y_2 \leq y_2$. Thus the conditions (i) and (ii) of the previous definition are satisfied. In view of the lemma in page 22 in [3], the condition (iii) of the previous definition is also satisfied.*

Example 2.2. *Consider the lattice given by the Hasse diagram in figure 1. Consider the partition $\{\{1\}, \{a, b\}, \{c, d\}, \{e, 0\}\}$. This defines a congruence relation θ mentioned in definition 2.1 . However, $e \vee d = b$, $0 \vee d = d$, $e \equiv 0$ (mod θ) and $b \neq d$ (mod θ).*

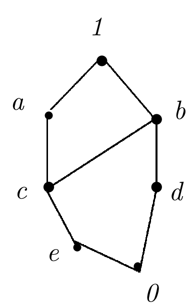

figure 1

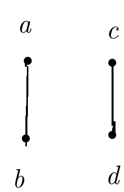

figure 2

Example 2.3. *Consider the poset given by the Hasse diagram in figure 2. Then the partition* $\{\{a,d\},\{b,c\}\}$ *defines an equivalence relation. It is not a congruence relation that satisfies the conditions (i), (ii) and (iii) of definition 2.1.*

Example 2.4. *Let (P,≤) be a poset. Define an equivalence relation* θ *on* P *by* $a \equiv b \pmod{\theta}$ *if and only if* $\{x \in P : x < a\} = \{x \in P : x < b\}$ *and* $\{x \in P : x > a\} = \{x \in P : x > b\}$*, (see definition 4.3 in chapter 2 in [3]) . If* $x_1 < x_2$ *and* $x_1 \equiv y_1 \pmod{\theta}$*, then* $x_2 \in \{x \in P : x > x_1\} = \{x \in P : x > y_1\}$*, so that* $y_1 < x_2$*. If* $x_1 < x_2$ *and* $x_2 \equiv y_2 \pmod{\theta}$*, then* $x_1 \in \{x \in P : x < x_2\}$*=*$\{x \in P : x < y_2\}$ *so that* $x_1 < y_2$*. If* $x_1 \equiv z_1 \pmod{\theta}$ *and* $x_1 \leq y_1 \leq z_1$*, then* $\{x \in P : x < z_1\} = \{x \in P : x < x_1\} = \{x \in P : x < y_1\}$ *and* $\{x \in P : x > x_1\} = \{x \in P : x > z_1\} = \{x \in P : x > y_1\}$ *so that* $x_1 \equiv y_1 \pmod{\theta}$*. These three statements prove that* θ *is a congruence relation of definition 2.1, because the other cases of verification are trivial.*

Let us recall that a mapping $T : P_1 \to P_2$ *from a poset* P_1 *to a poset* P_2 *is said to be order preserving, if* $T(a) \leq T(b)$ *in* P_2*, whenever* $a \leq b$ *in* P_1*.*

Definition 2.2. *A mapping* $T : P_1 \to P_2$ *from a poset* P_1 *to a poset* P_2 *is inversely order preserving if the following are satisfied whenever* $a \leq b$ *for some* $a, b \in T(P_1)$*.*

(I) For given $a_1 \in T^{-1}(a)$*, there is a* $b_1 \in T^{-1}(b)$ *such that* $a_1 \leq b_1$

(II) For given $b_2 \in T^{-1}(b)$*, there is an* $a_2 \in T^{-1}(a)$ *such that* $a_2 \leq b_2$

(III) If $x, z \in T^{-1}(a)$ *and* $x \leq y \leq z$ *in* P_1*, then* $y \in T^{-1}(a)$*.*

3 A fundamental theorem of homomorphism

Theorem 3.1. *Let* θ *be an equivalence relation on a poset (P,≤). If* θ *is a congruence relation mentioned in definition 2.1, then* P/θ *becomes a poset and the natural quotient mapping* $\pi : P \to P/\theta$ *is a surjective, order preserving and inversely order preserving mapping. On the other hand, for a given mapping* $T : P \to P_1$ *from* P *onto a poset* P_1 *which is order preserving and inversely order preserving, the partition* $\{\ T^{-1}(a) : a \in P_1\}$ *leads to a congruence relation of definition 2.1.*

Proof **First part:** When θ is a congruence relation, let $[x]$ denote the equivalence class containing x. Define an order relation $\leq$ on P/θ by the rule: $[x] \leq [y]$ if and only if for given $x_1 \in [x]$ and $y_1 \in [y]$, there are $x_2 \in [x]$ and $y_2 \in [y]$ such that $x_1 \leq y_2$ and $x_2 \leq y_1$. Note that $[x] \leq [x]$, $\forall x \in P$.
To prove anti-symmetricity, suppose $[x] \leq [y]$ and $[y] \leq [x]$ in P/θ. Then there are $y_1 \in [y]$ and $x_1 \in [x]$ such that $x \leq y_1 \leq x_1$. Then, by (iii) of definition 2.1, $y \equiv x \pmod{\theta}$. This proves the anti-symmetricity of the relation in P/θ. To prove transitivity, consider the relations $[x] \leq [y]$ and $[y] \leq [z]$ in P/θ. Then, to given $x_1 \in [x]$, there are $y_1 \in [y]$ and $z_1 \in [z]$ such that $x_1 \leq y_1$ and $y_1 \leq z_1$ so that $x_1 \leq z_1$. Similarly, to given $z_1 \in [z]$, there are $y_1 \in [y]$ and $x_1 \in [x]$ such that $x_1 \leq y_1$ and $y_1 \leq z_1$ so that $x_1 \leq z_1$. This proves transitivity and hence P/θ is also a poset.

Suppose $x \leq y$ in P. Then, by (i) of definition 2.1, for given $x_1 \in [x]$, there is a $y_1 \in [y]$, such that $x_1 \leq y_1$. Similarly, for given $y_2 \in [y]$, there is a $x_2 \in [x]$, such that $x_2 \leq y_2$. So $[x] \leq [y]$. Thus the mapping $\pi : P \to P/\theta$ defined by $\pi(x) = [x]$ is order preserving. The definition 2.1 and the definition 2.6 imply that π is inversely order preserving. **Second part:** Now, let θ denote the equivalence relation defined by $\{T^{-1}(a) : a \in P_1\}$ for the given mapping $T : P \to P_1$. The definition 2.1 and the definition 2.6 imply that θ is a congruence relation. This completes the proof of the theorem.

4 Doubly directed sets

A poset $(P, \leq)$ is a doubly directed set if any two elements in P have an upper bound and a lower bound.

Definition 4.1. *Let $(P, \leq)$ be a doubly directed set. Let θ be an equivalence relation on P. It is called a congruence relation if it satisfies the following:*

(i) If x and y are given, if $z \geq x$, $z \geq y$, and if $x \equiv x_1$ (mod θ), $y \equiv y_1$ (mod θ) in P, then there is a z_1 in P such that $z_1 \geq x_1, z_1 \geq y_1$ and $z \equiv z_1$ (mod θ).

(ii) If x and y are given, if $z \leq x, z \leq y$, and if $x \equiv x_2$(mod θ), $y \equiv y_2$ (mod θ) in P, then there is a z_2 in P such that $z_2 \leq x_2$, $z_2 \leq y_2$, and $z \equiv z_2$ (mod θ).

(iii) If $x \equiv z$ (mod θ) and $x \leq y \leq z$, then $x \equiv y$ (mod θ)

Let us observe that the conditions (i),(ii) and (iii) of definition 4.1 imply the corresponding conditions (i),(ii) and (iii) of definition 2.1, where $z_1 = y_2, z = x_2$ and $x = y = x_1$ (for(i)). Let us now rephrase the definition 2.6.

Definition 4.2. *A mapping $T : P_1 \to P_2$ from a doubly directed set P_1 to a doubly directed set P_2 is inversely direction preserving, if the following are satisfied:*

(I) If a, b, c are in T(P_1) satisfying $a \leq c$ and $b \leq c$, then for given $a_1 \in T^{-1}(a), b_1 \in T^{-1}(b)$, there is a $c_1 \in T^{-1}(c)$ such that $a_1 \leq c_1$ and $b_1 \leq c_1$.

(II) If a, b, c are in T(P_1) satisfying $a \geq c$ and $b \geq c$, then for given $a_2 \in T^{-1}(a), b_2 \in T^{-1}(b)$, there is a $c_2 \in T^{-1}(c)$ such that $a_2 \geq c_2$ and $b_2 \geq c_2$.

(III) If $a \in T(P_1)$ and $x, z \in T^{-1}(a)$, and $x \leq y \leq z$ in P_1, then $y \in T^{-1}(a)$.

Let us again observe that the conditions (I),(II) and (III) of definition 4.2 imply that corresponding conditions (I),(II) and (III) of definition 2.6, where $c_1 = b, c = b, b_1 = a_1$ and $b = a$ (for(I)). Let us now rephrase the theorem 3.1.

Theorem 4.1. *Let θ be an equivalence relation on a doubly directed set $(P,\leq)$. If θ is a congruence relation mentioned in definition 4.1, then P/θ becomes a doubly directed set and the natural quotient mapping $\pi : P \to P/\theta$ is a surjective, order preserving and inversely direction preserving mapping. On the other hand, for a given mapping T from P onto a doubly directed set P_1 which is order preserving and inversely direction preserving, the partition $\{T^{-1}(a) : a \in P_1\}$ leads to a congruence relation of definition 4.1.*

Proof **First part:** Let us follow the notations and definitions used in the proof for the first part of the theorem 3.1. Then P/θ is a poset. Let us fix x and y in P and hence $[x]$ and $[y]$ in P/θ to verify that P/θ is a doubly directed set. Since P is a doubly directed set, there are a and b in P such that $a \leq x \leq b$ and $a \leq y \leq b$. Then it follows from the definition 4.1 that $[a] \leq [x] \leq [b]$ and $[a] \leq [y] \leq [b]$ in P/θ. Thus P/θ is a doubly directed set; and π is inversely direction preserving. **Second part:** The definitions 4.1 and 4.2 imply the second part to complete the proof of the theorem.

References

[1] R. Giacobazzi, F. Ranzato, Some properties of complete congruence lattices, Algebra Univers. 40 (1998) 189-200.

[2] D. Papert, Congruence relations in semilattices, J. London Math.Soc. 39 (1964) 723-729.

[3] G. Birkoff, Lattice Theory, Second edition, Amer.Math.Soc., New York, 1948.

[4] E.Harzheim, Ordered sets, Springer, New York, 2005.

(Received 5 December 2013)
(Accepted 2 January 2015)