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Analytical Method for the Solution of Inverse Parabolic Problem

Abdolsadeh Neisy

Department of Mathematics, Computer and Statistics
Faculty of Economics, Allameh Tabataba'i University, Iran
e-mail : a_neisy@atu.ac.ir

Abstract : In this paper, an inverse problem for a parabolic partial differential equation will be discussed. Using an overspecified condition, it is shown that the solution to this problem exists and is unique.

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1 Introduction

Inverse problems are encountered in various branches of science and engineering. Mechanical, aerospace and chemical engineers; mathematicians, astrophysicists, geophysicists, statisticians and specialists of many other disciplines are all interested in inverse problems, each with different applications in mind. In the field of heat transfer, using inverse analysis for the estimation of surface conditions such as temperature and heat flux, or the determination of thermal properties such as thermal conductivity and heat capacity of solids by utilizing the transient temperature measurements taken within the medium has numerous practical applications. By principle of conservation, we have [1, 2]

$$a_0 u_t + \nabla J = 0, \tag{1.1}$$

where a_0 is a heat conduction coefficient.

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In *théorie Analytique de la Chaleur*, Fourier stated his famous law

$$J = -D_0 \nabla u,$$

where J is the *rate of flow of heat energy* per unit time through a unit area and D_0 is the conductivity. By consideration the rotational curent we have [3–5]

$$J = -D_0 \nabla u + \mu_0 u. \tag{1.2}$$

Substitution (1.2) into (1.1) yields

$$u_t - \nabla.(a(t)\nabla u) + b(t)\nabla u = 0,$$

where $a(t) = \frac{D_0(t)}{a_0(t)}$ is the diffusivity and $b(t) = \frac{\mu_0(t)}{a_0(t)}$ is a transport coefficient.

Now we consider the following one-dimensional parabolic partial differential equation in a slab

$$u_t - a(t)u_{xx} + b(t)u_x = 0, \tag{1.3}$$

in the domain D=$\{(x,t) \mid 0 < x < 1,\, 0 < t < T\}$ for some $T > 0$. We require that the solution of (1.3) satisfies the primary initial and boundary conditions

$$u(x,0) = f_0(x), \quad 0 \le x \le 1, \tag{1.4}$$
$$u_x(0,t) = G_0(u(0,t)), \quad 0 \le t \le T, \tag{1.5}$$
$$u_x(1,t) = H_0(u(1,t)), \quad 0 \le t \le T, \tag{1.6}$$

where a, b, G_0, H_0 and f_0 are all considered known and sufficiently smooth on their domains. Then the problem is concerned with the determination of temperature distribution $u(x,t)$ in the interior region of the solid as a function of time and position. We shall refer to such traditional problems as the *direct parabolic problems.*

We now consider a problem similar to that given by (1.3)-(1.6), but the boundary coundition functions H_0, G_0 at boundary surfaces are unknown. To compensate for the lack of information on the boundary conditions, measured functions $u(0,t) = g_0(t)$ and $u(1,t) = h_0(t)$ are given at the boundary points $x = 0$, $x = 1$ at any time t, $0 \le t \le T$, where T is the final time. This is an *inverse parabolic problem* because it is concerned with the estimation of the unknown H_0, G_0 [6]. The mathematical formulation of this problem is to find (u, H_0, G_0) so that

$$u_t - a(t)u_{xx} + b(t)u_x = 0, \quad \text{in } D, \tag{1.7}$$
$$u(x,0) = f_0(x), \quad 0 \le x \le 1, \tag{1.8}$$
$$u_x(0,t) = G_0(u(0,t)), \quad 0 \le t \le T, \tag{1.9}$$
$$u_x(1,t) = H_0(u(1,t)), \quad 0 \le t \le T, \tag{1.10}$$

and the overspecified data

$$u(0,t) = g_0(t), \quad 0 \le t \le T, \tag{1.11}$$
$$u(1,t) = h_0(t), \quad 0 \le t \le T. \tag{1.12}$$

This is an inverse parabolic problem with *unknown boundary conditions.*

2 The Inverse Problem (1.7)-(1.12)

For equation of the form

$$u_t - a(t)u_{xx} + b(t)u_x = 0,$$

set

$$x = \xi + \int_0^{\phi(\tau)} b(\eta)d\eta,$$

and

$$t = \phi(\tau),$$

where ϕ is the inverse of the mapping

$$\tau = \int_0^{\phi(\tau)} a(\eta)d\eta.$$

Clearly,

$$\phi'(\tau) = [a(\phi(\tau))]^{-1} \equiv [a(t)]^{-1}.$$

Follows from $t = \phi(\tau)$. Using $U(\xi, \tau) = u(\xi + \int_0^{\phi(\tau)} b(\eta)d\eta, \phi(\tau))$, The problem (1.7)-(1.10) becomes

$$U_\tau = U_{\xi\xi}, \quad s_1(\tau) < \xi < s_2(\tau), \ 0 < \tau < T_1, \tag{2.1}$$
$$U(\xi, 0) = f(\xi), \quad 0 \le \xi \le 1, \tag{2.2}$$
$$U_\xi(s_1(\tau), \tau) = G(U(s_1(\tau), \tau)), \quad 0 \le \tau \le T_1, \tag{2.3}$$
$$U_\xi(s_2(\tau), \tau) = H(U(s_2(\tau), \tau)), \quad 0 \le \tau \le T_1, \tag{2.4}$$

and, similarly (1.11) and (1.12) yield

$$U(s_1(\tau), \tau)) = g(\tau), \quad 0 \le \tau \le T_1, \tag{2.5}$$
$$U(s_2(\tau), \tau)) = h(\tau), \quad 0 \le \tau \le T_1, \tag{2.6}$$

where

$$s_1(\tau) = -\int_0^{\phi(\tau)} b(\eta)d\eta,$$

$$s_2(\tau) = 1 - \int_0^{\phi(\tau)} b(\eta)d\eta,$$

and

$$T_1 = \int_0^T a(\eta)d\eta.$$

In the next section it will be shown that the above inverse problem has a unique solution.

3 Existence and Uniqueness

We consider now the problem of determining the (U, G, H) that satisfies in the (2.1)-(2.4), where the data f, G, and H sufficiently smooth, $s_i \in C^{\gamma}([0, T_1])$, $\gamma > \frac{1}{2}$, $i = 1, 2$. Since,

$$\delta = \inf_{0 \le \tau \le T_1} \mid s_1(\tau) - s_2(\tau) \mid = 1 > 0,$$

and

$$\Delta = \sup_{0 \le \tau \le T_1} \mid s_1(\tau) - s_2(\tau) \mid = 1 < \infty.$$

Then the problem (2.1)-(2.4) has a unique solution of the form [7].

$$\begin{aligned} U(\xi, \tau) = v(\xi, \tau) + \int_0^{\tau} K(\xi - s_1(t), \tau - t) G(U(s_1(t), t)) dt \\ + \int_0^{\tau} K(\xi - s_2(t), \tau - t) H(U(s_2(t), t)) dt, \end{aligned} \tag{3.1}$$

where

$$v(\xi, \tau) = \int_{-\infty}^{\infty} K(\xi - x, \tau) f(x) dx,$$

and f denotes a continuous extension of f with compact support,

$$K(\xi, \tau) = \frac{1}{\sqrt{4\pi\tau}} \exp\left(-\frac{\xi^2}{4\tau} \right). \tag{3.2}$$

Differentiating equation (3.1) with respect to ξ and using (2.5) and (2.6) we obtain respectively,

$$\begin{aligned} G(g(\tau)) = v_{\xi}(s_1(\tau), \tau) - 2^{-1} G(g(\tau)) + \int_0^{\tau} \frac{\partial K}{\partial \xi}(s_1(\tau) - s_1(t), \tau - t) G(g(t)) dt \\ + \int_0^{\tau} \frac{\partial K}{\partial \xi}(s_1(\tau) - s_2(t), \tau - t) H(h(t)) dt, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} H(h(\tau)) = v_{\xi}(s_2(\tau), \tau) + 2^{-1} H(h(\tau)) + \int_0^{\tau} \frac{\partial K}{\partial \xi}(s_2(\tau) - s_1(t), \tau - t) G(g(t)) dt \\ + \int_0^{\tau} \frac{\partial K}{\partial \xi}(s_2(\tau) - s_2(t), \tau - t) H(h(t)) dt. \end{aligned} \tag{3.4}$$

Lemma 3.1. *For $s_i \in C^{\gamma}([0, T_1])$, $\gamma > \frac{1}{2}$, $i = 1, 2$, there exists a positive constant $c_1 = c_1(T_1, \mid s_2 \mid_{\gamma}, \gamma)$ such that*

$$\left| \frac{\partial K}{\partial \xi}(s_2(\tau) - s_2(t), \tau - t) \right| \le c_1 (\tau - t)^{-(1/2)}.$$

Proof. From inequality,

$$\exp\{-\xi\} \leq 1/\xi,$$

we have

$$\begin{aligned}\left|\frac{\partial K}{\partial \xi}(s_2(\tau)-s_2(t),\tau-t)\right| &= \left|\frac{-2(s_2(\tau)-s_2(t))}{4(\tau-t)\sqrt{4\pi(\tau-t)}}\right| \exp\left\{\frac{-(s_2(\tau)-s_2(t))^2}{4(\tau-t)}\right\} \\ &\leq \frac{1}{\sqrt{\pi}} \mid (s_2(\tau)-s_2(t))^{-1}(\tau-t)^{-(1/2)} \mid,\end{aligned}$$

and $s_2 \in C^\gamma([0,T_1])$, thus, there exists a positive constant $c_1 = c_1(T_1, \mid s_2 \mid_\gamma, \gamma)$ such that

$$\left|\frac{\partial K}{\partial \xi}(s_2(\tau)-s_2(t),\tau-t)\right| \leq c_1(\tau-t)^{-(1/2)}. \qquad \square$$

Lemma 3.2. *For* $s_i \in C^\gamma([0,T_1]), \gamma > \frac{1}{2}$*,* $i = 1,2$*, there exists a positive constant* $c_2 = c_2(T_1, \mid s_2 \mid_\gamma, \gamma)$ *such that*

$$\left|\frac{\partial K}{\partial \xi}(s_1(\tau)-s_2(t),\tau-t)\right| \leq c_2(\tau-t)^{-(1/2)}.$$

Proof. Consider

$$\begin{aligned}(s_1(\tau)-s_2(t))^2 &= ([s_1(\tau)-s_2(\tau)]+[s_2(\tau)-s_2(t)])^2 \\ &= (s_1(\tau)-s_2(\tau)^2 + 2(s_1(\tau)-s_2(\tau))(s_2(\tau)-s_2(t)) + (s_2(\tau)-s_2(t))^2 \\ &\geq \delta^2 - \epsilon\Delta^2 + (1-\epsilon^{-1})(s_2(\tau)-s_2(t))^2.\end{aligned}$$

Selecting $\epsilon = \frac{\delta^2}{2\Delta^2} = \frac{1}{2}$, we get

$$(s_1(\tau)-s_2(t))^2 \geq 2^{-1} - (s_2(\tau)-s_2(t))^2 \geq 2^{-1} - \mid s_2 \mid_\gamma^2 \mid \tau - t \mid^{2\gamma}.$$

Consequently,

$$\exp\left\{-\frac{(s_1(\tau)-s_2(t))^2}{4(\tau-t)}\right\} \leq \exp\{4^{-1} \mid s_2 \mid_\gamma^2 T_1^{2\gamma-1}\} \exp\left\{-\frac{1}{8(\tau-t)}\right\}.$$

Since $\delta = 1 > 0$, it follows, from $\exp\{-\xi\} \leq 1/\xi$ that there exists a constant $c_2 = c_2(T_1, \mid s_2 \mid_\gamma, \gamma)$ such that

$$\left|\frac{\partial K}{\partial \xi}(s_1(\tau)-s_2(t),\tau-t)\right| \leq c_2(\tau-t)^{-(1/2)},$$

and in a similar manner there exists a constant $c_3 = c_3(T_1, \mid s_1 \mid_\gamma, \gamma)$ such that

$$\left|\frac{\partial K}{\partial \xi}(s_2(\tau)-s_1(t),\tau-t)\right| \leq c_3(\tau-t)^{-(1/2)}. \qquad \square$$

Theorem 3.3 (Main Theorem). *For $f(x) \in C([0,1])$, $g(\tau) \in C^0_{(\nu)}((0,T_1])$, and $h(\tau) \in C^0_{(\nu)}((0,T_1])$, there exists unique functions $G(g(\tau)) \in C^0_{(\nu)}((0,T_1])$, and $H(h(\tau)) \in C^0_{(\nu)}((0,T_1])$, which satisfy (3.3)-(3.4) for some $0 < \tau \le T_1$. Moreover, the solution to the problem (2.1)-(2.6) exists and is unique.*

Proof. Set

$$H_1 = \frac{\partial K}{\partial \xi}(s_1(\tau) - s_1(t), \tau - t)G(g(t)) + \frac{\partial K}{\partial \xi}(s_1(\tau) - s_2(t), \tau - t)H(h(t)),$$

and

$$H_2 = \frac{\partial K}{\partial \xi}(s_2(\tau) - s_1(t), \tau - t)G(g(t)) + \frac{\partial K}{\partial \xi}(s_2(\tau) - s_2(t), \tau - t)H(h(t)).$$

We have

$$\begin{aligned} \mid H_1^1 - H_1^2 \mid = \Big| &\frac{\partial K}{\partial \xi}(s_1(\tau) - s_1(t), \tau - t)(G^1(g(t)) - G^2(g(t))) \\ &+ \frac{\partial K}{\partial \xi}(s_1(\tau) - s_2(t), \tau - t)(H^1(h(t)) - H^2(h(t)))\Big|, \\ \mid H_2^1 - H_2^2 \mid = \Big| &\frac{\partial K}{\partial \xi}(s_2(\tau) - s_1(t), \tau - t)(G^1(g(t)) - G^2(g(t))) \\ &+ \frac{\partial K}{\partial \xi}(s_2(\tau) - s_2(t), \tau - t)(H^1(h(t)) - H^2(h(t)))\Big|. \end{aligned}$$

Now, from lemmas (3.1) and (3.2) we see that

$$\mid H_1^1 - H_1^2 \mid \le L(\tau, t)\{\mid G^1(g(t)) - G^2(g(t)) \mid + \mid H^1(h(t)) - H^2(h(t)) \mid\},$$

$$\mid H_2^1 - H_2^2 \mid \le L(\tau, t)\{\mid G^1(g(t)) - G^2(g(t)) \mid + \mid H^1(h(t)) - H^2(h(t)) \mid\}.$$

Where

$$L(\tau, t) = c_4(T_1, \mid s_1 \mid_\gamma, \mid s_2 \mid_\gamma, \gamma)(\tau - t)^{-(1/2)},$$

and

$$\int_{\tau_1}^{\tau_2} L(\tau_2, t)dt = 2c_4(T_1, \mid s_2 \mid_\gamma, \gamma)(\tau_2 - \tau_1)^{(1/2)} = \alpha(\tau_2 - \tau_1).$$

For some monotone-increasing function α with

$$\lim_{\eta \to 0^+} \alpha(\eta) = 0.$$

Then the system of linear Volterra integral equations of the first kind (3.3)-(3.4) has a unique solution (G, H). Therefore, the solution to the problem (2.1)-(2.6) exists and is unique ([7–10]). □

In the next section, the stability of solution will be discussed.

4 Stability Solutions $(U,\, G,\, H)$

By demonstrating the following theorem we prove the stability of solution $(U,\, G,\, H)$.

Theorem 4.1. *Let $(U_1,\, G_1,\, H_1)$ and $(U_2,\, G_2,\, H_2)$ be two solutions of problem (2.1)-(2.6) corresponding to two given data $(f_1,\, g_1,\, h_1)$ and $(f_2,\, g_2,\, h_2)$. Then these solutions are stable.*

Proof. Using (3.2) we obtain

$$\begin{aligned}\left|\int_0^\tau K(\xi - s_2(t), \tau - t))dt\right| &= \left|\int_0^\tau \frac{1}{\sqrt{4\pi(\tau - t)}} \exp\{\frac{-(\xi - s_2(t))^2}{4(\tau - t)}\}dt\right| \\ &\le \left|\int_0^\tau \frac{1}{\sqrt{4\pi(\tau - t)}} \frac{4(\tau - t)}{(\xi - s_2(t))^2} dt\right| \\ &\le M T_1^{3/2},\end{aligned}$$

where $M = M(\mid s_1 \mid_\gamma, \mid s_2 \mid_\gamma, \gamma)$. Now by Lemma 3.2 we have

$$\left|\int_0^\tau \left\{\frac{\partial K}{\partial \xi}(s_1(\tau) - s_2(t), \tau - t)\right\} dt\right| \le \left|\int_0^\tau c_2(\tau - t)^{-(1/2)} dt\right| \le 2c_1 T_1^{1/2}.$$

Putting

$$M_1 = M_1(T_1,\ \mid s_1 \mid_\gamma,\ \mid s_2 \mid_\gamma,\ \|G\|_{T_1},\ \gamma),$$

and

$$M_2 = M_2(T_1,\ \mid s_1 \mid_\gamma,\ \mid s_2 \mid_\gamma,\ \|H\|_{T_1},\ \gamma),$$

we obtain

$$\mid U_1(\xi, \tau) - U_2(\xi, \tau) \mid \le \mid f_1 - f_2 \mid + M_1\|g_1 - g_2\|_{T_1} + M_2\|h_1 - h_2\|_{T_1},$$

therefore, the solution for U is stable. Similar results may be obtained for the solutions G and H. This completes the proof of stability solutions. □

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