



Linear Maps Preserving the Generalized Projection

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Abstract : Let \mathcal{H} be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . We give the concrete forms of surjective linear maps $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which preserve the generalized projections, operator pairs whose products are nonzero generalized projections or operator pairs whose triple Jordan products are nonzero generalized projections in both directions.

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1 Introduction

Linear preserver problems is an active research area in matrix, operator theory and Banach algebras. It has attracted the attention of many mathematicians in the last few decades ([1]-[11]). By a linear preserver we mean a linear map of an algebra \mathcal{A} into itself which, roughly speaking, preserves certain properties of some elements in \mathcal{A} . Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. In this paper, we shall concentrate on the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators

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on a complex Hilbert space \mathcal{H} . We should point out that a great deal of work has been devoted to the case when \mathcal{H} is finite dimensional, that is, the case when \mathcal{A} is a matrix algebra (see survey articles [6], [12], [13]), and that the first papers concerning this case date back to the previous century [5].

The aim of this paper is to characterize a continuous linear maps

$$\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

which preserve the generalized projections, operator pairs whose products are nonzero generalized projections or operator pairs whose triple Jordan products are nonzero generalized projections in both directions.

2 Preliminaries

First we introduce some notation and terminology, for more we can refer to [14].

Definition 2.1. Let \mathcal{H} be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be a algebra of all bounded linear operators on \mathcal{H} . Operator $T \in \mathcal{B}(\mathcal{H})$ is a *generalized projection* if $T^2 = T^*$. The set of all generalized projection on \mathcal{H} is denoted by $\mathcal{GP}(\mathcal{H})$.

Proposition 2.2. *If T is a generalized projection in $\mathcal{B}(\mathcal{H})$ then T^3 is a projection. If T is a projection in $\mathcal{B}(\mathcal{H})$ then T is a generalized projection.*

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

1. T is a generalized projection.
2. T is a normal operator and $T^4 = T$.
3. T is a partial-isometry and $T^4 = T$.

Theorem 2.4. *Let $S, T \in \mathcal{GP}(\mathcal{H})$. Then $S + T \in \mathcal{GP}(\mathcal{H})$ if and only if $ST = TS = 0$.*

A linear map ϕ from algebra \mathcal{A} into an algebra \mathcal{B} is called a n -Jordan homomorphism if $\phi(x^n) = \phi(x)^n$ for every $x \in \mathcal{A}$ where n is a natural fixed numbers. A well known result of Herstein [15] shows that a n -Jordan homomorphism on prime algebra \mathcal{A} is either an homomorphism or an anti-homomorphism, multiplied by $a \in \mathbb{C}$ where $a^{n-1} = 1$.

Throughout \mathcal{H} denotes a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all linear bounded operators on \mathcal{H} . Also, for $T \in \mathcal{B}(\mathcal{H})$, $R(T)$ denotes the range of T and $N(T)$ the kernel of T .

3 Main Results

Let ϕ be a linear map on $\mathcal{B}(\mathcal{H})$. If for any $A \in \mathcal{B}(\mathcal{H})$, $\phi(A)$ is a generalized projection if and only if A is, then we say that ϕ preserves generalized projections in both directions. If for any $A, B \in \mathcal{B}(\mathcal{H})$, $\phi(A)\phi(B)$ is a nonzero projection if and only if AB is, then we say that ϕ preserves operator pairs whose products are nonzero projections in both directions. If for any $A, B \in \mathcal{B}(\mathcal{H})$, $\phi(A)\phi(B)\phi(A)$ is a nonzero projection if and only if ABA is, then we say that ϕ preserves operator pairs whose triple Jordan products are nonzero projections in both directions.

Theorem 3.1. *Let \mathcal{H} be a complex Hilbert space and let $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a linear continuous surjective map. If ϕ preserves generalized projections in both directions. Then, there exists a unitary operator U and a constant a with $a^3 = 1$ such that ϕ takes one of the following forms.*

$$\phi(A) = aUAU^*$$

or

$$\phi(A) = aUA^tU^*$$

for all $A \in \mathcal{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of \mathcal{H} .

We first prove this lemma.

Lemma 3.2. *ϕ is injective.*

Proof. Let A in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = 0$. Then A is a generalized projection, so

$$A^4 = A, \tag{3.1}$$

since $\phi(2A) = 0$ and ϕ preserves generalized projection in both directions, then $2A$ is a generalized projection, hence

$$16A^4 = 2A. \tag{3.2}$$

By 3.1 and 3.2 $8A = A$, so $A = 0$, consequently ϕ is injective. □

Now we will prove The Theorem 3.1.

Proof. We consider two mutually orthogonal projections p and q , so $p + q$ is a projection. It follows that $p + q$, p and q are generalized projections. Since ϕ preserves generalized projections then $\phi(p + q)$, $\phi(p)$ and $\phi(q)$ are generalized projections. We have

$$\phi(p + q) = \phi(p) + \phi(q),$$

from Theorem 2.4

$$\phi(p)\phi(q) = \phi(q)\phi(p) = 0.$$

We know that p and q are generalized projection, then $\phi(p)$ and $\phi(q)$ are generalized projections, this implies that

$$\phi(p) = \phi(p)^{*2}$$

and

$$\phi(q) = \phi(q)^{*2},$$

consequently

$$\phi((ap + bq)^2) = (a\phi(p) + b\phi(q))^{*2} \text{ for all } a, b \in \mathbb{R}. \quad (3.3)$$

Now by spectral theorem that the real linear combination of mutually orthogonal projections is dense in the set of all self-adjoint operators $\mathcal{B}(\mathcal{H})$. Then for every self-adjoint A , there is a sequence of operators P_n , such that $A = \lim P_n$ and

$$P_n = \sum_{i=1}^n \alpha_i E_i$$

where $\alpha_i \in \mathbb{R}$ and E_i are mutually orthogonal projections. Using 3.3, we get

$$\phi((P_n)^2) = (\phi(P_n))^{*2}.$$

Since ϕ is continuous, so by passing to limit

$$\phi(A)^{*2} = \phi(A^2)$$

for all self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$. Indeed for S, T are self-adjoint operator, then $S + T$ is self-adjoint. Thus

$$\phi((S + T)^2) = (\phi(S) + \phi(T))^{*2},$$

so

$$\phi(ST + TS) = \phi(S)^* \phi(T)^* + \phi(T)^* \phi(S)^*.$$

Now, we can write any operator $A \in \mathcal{B}(\mathcal{H})$ in the form $A = S + iT$ where S, T are self-adjoint operator. then

$$\begin{aligned} \phi(A^2) &= \phi(S^2 - T^2 + i(ST + TS)), \\ &= \phi(S)^2 - \phi(T)^2 + i\phi(ST + TS) \\ &= \phi(S)^{*2} - \phi(T)^{*2} + i(\phi(S)^* \phi(T)^* + \phi(T)^* \phi(S)^*) \\ &= \phi(A^*)^{*2} \end{aligned}$$

for all A in $\mathcal{B}(\mathcal{H})$. It follows that $\phi(A^4) = \phi(A^{*2})^{*2} = \phi(A)^4$ for all $A \in \mathcal{B}(\mathcal{H})$, (i.e., ϕ is a 4-Jordan-homomorphism). It is well-known that every 4-Jordan-homomorphism of a prime algebra is a homomorphism or an anti-homomorphism multiplied by a such that $a^3 = 1$. Since $\mathcal{B}(\mathcal{H})$ is a prime algebra. Thus ϕ is a homomorphism or an anti-homomorphism multiplied by a such that $a^3 = 1$,

by Lemma 3.2 ϕ is injective so ϕ is an automorphism or an anti-automorphism multiplied by a . Then there is an invertible operator U such that

$$\phi(A) = aUBU^{-1}$$

for all A in $\mathcal{B}(\mathcal{H})$ or

$$\phi(A) = aUA^tU^{-1}$$

for all A in $\mathcal{B}(\mathcal{H})$. We only consider the first form of ϕ , the proof of the second form is similar to the first. Moreover, we have $\phi(p)^2 = \phi(p)^*$, for all orthogonal projection $p \in \mathcal{B}(\mathcal{H})$, so

$$a^2UpU^{-1} = \bar{a}U^{*-1}pU^*,$$

this implies that

$$U(cp + dq)U^{-1} = U^{*-1}(cp + dq)U^*,$$

for all $c, d \in \mathbb{R}$.

Now by using the spectral theorem that the real linear combination of mutually orthogonal projections is dense in the set of all self-adjoint operators $\mathcal{B}(\mathcal{H})$, we obtain

$$USU^{-1} = U^{*-1}SU^*,$$

for all S self-adjoint operator, or any operator $A \in \mathcal{B}(\mathcal{H})$ is written in the form $A = S + iT$ where S, T are self-adjoint operator, then we get

$$\begin{aligned} UAU^{-1} &= U(S + iT)U^{-1} \\ &= USU^{-1} + iUTU^{-1} \\ &= U^{*-1}SU^* + iU^{*-1}TU^* \\ &= U^{*-1}(S + iT)U^* \\ &= U^{*-1}AU^*, \end{aligned}$$

for all A in $\mathcal{B}(\mathcal{H})$. Consequently,

$$U^*UA = AU^*U \text{ for all } A \in \mathcal{B}(\mathcal{H}),$$

since the center of $\mathcal{B}(\mathcal{H})$ are scalar operators, then $U^*U = \lambda I$. Since U^*U is a positive operator, then λ is a positive real number, hence you can reduce a $\lambda = 1$, but U is an invertible operator, so

$$U^*U = UU^* = I.$$

We get the existence of a unitary operator U such as the form of ϕ in the desired theorem. □

Theorem 3.3. *Let \mathcal{H} be a complex Hilbert space with $\dim \mathcal{H} \geq 2$ and let $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a linear continue and surjective map. If ϕ preserves operator pairs whose products are nonzero generalized projections in both directions, then*

there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a constant a with $a^6 = 1$ such that one of the following holds.

$$\phi(A) = aUAU^*$$

or

$$\phi(A) = aUA^tU^*$$

for all $A \in \mathcal{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of \mathcal{H} .

For proof of Theorem we will need to prove this lemma.

Lemma 3.4. ϕ is injective and $\phi(I) = aI$, for some a such that $a^6 = 1$.

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = 0$. if $A \neq 0$, then there exist a vector $x \in \mathcal{B}(\mathcal{H})$ such that $Ax \neq 0$. let $B = \frac{x \otimes Ax}{\|Ax\|^2}$. AB is an non zero projection generalized projection, which implies that $\phi(A)\phi(B)$ is an non zero generalized projection. But $\phi(A)\phi(B) = 0$ a contradiction, thus ϕ is injective.

Suppose that $\phi(A) = I$. If $A \notin \mathbb{C}I$, then there exist an non zero vectors $x \in \mathcal{H}$ such that x and Ax are linearly independent. Put $B = \frac{x \otimes Ax}{\|Ax\|^2}$, then AB is a non zero generalized projection. So is $\phi^{-1}(A)\phi^{-1}(B) = \phi^{-1}(B)$ is a non-zero generalized projection, so $\phi^{-1}(B)^2 = \phi^{-1}(B)^*$. This implies that $\phi^{-1}(B)^*$ is a generalized projection, so $\phi^{-1}(B)^*$ is a generalized projection and so B^2 is an non zero generalized projection. It follows that B^6 is a non zero projection. Now $B^6 = \langle x, Ax \rangle^5 \frac{x \otimes Ax}{\|Ax\|^2}$. Then x and Ax are linearly dependent. This contradiction shows that $A = aI$ for some constant a . Since $A^6 = a^6I$ must be a non-zero projection. Then $a^6 = 1$ which completes the proof. \square

Now we prove the Theorem 3.3.

Proof. From Lemma 3.4, there are two cases to consider.

Case1, $\phi(I) = I$. So ϕ preserves generalized projections, by Theorem 3.1 we will get the result.

Case2, $\phi(I) = aI$. We can instead work with the linear map ψ defined by $\psi(A) = \bar{a}\phi(A)$, for all $A \in \mathcal{B}(\mathcal{H})$. This map clearly is unital linear continuous and bijective map that preserves generalized projections in both directions. \square

Theorem 3.5. Let \mathcal{H} be a complex Hilbert space with $\dim \mathcal{H} \geq 2$ and let $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a linear continuous surjective map. If ϕ preserves operator pairs whose triple Jordan products are nonzero generalized projection in both directions, then there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a constant a with $a^9 = 1$ such that one of the following holds.

$$\phi(A) = aUAU^*$$

or

$$\phi(A) = aUA^tU^*$$

for all $A \in \mathcal{B}(\mathcal{H})$, A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of \mathcal{H} .

For proof of Theorem we will need to proof this lemma.

Lemma 3.6. ϕ is injective and $\phi(I) = aI$, for some a such that $a^9 = 1$.

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = 0$. If $A \neq 0$, then there exists a unit vector $x \in \mathcal{H}$ such that $\langle Ax, x \rangle \neq 0$. Let $B = \frac{x \otimes x}{\sqrt{\langle Ax, x \rangle}}$. It is clear that $ABA = x \otimes x$ is a non zero-projection so a non-zero generalized projection. Then $\phi(A)\phi(B)\phi(A)$ is a non-zero generalized projection. However, $\phi(A)\phi(B)\phi(A) = 0$, contradiction. Thus, ϕ is injective.

On the other hand, suppose that $\phi(I) = A$. We claim that $A \in \mathbb{C}I$. In fact we have $A^3 = E$ is a non-zero generalized projection. If E is not unitary operator. By [14], $R(E)$ is closed, then $\mathcal{H} = R(E) \oplus N(E)$. Since $AE = EA$, then

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

with A_{11}^3 is a unitary operator in $R(E)$ and $A_{22}^3 = 0$. Let p be the projection from $N(E)$ onto the kernel of A_{22} . Then $p \neq 0$ and $A_{22}pA_{22} = 0$. Let $B_z = A_{11} \oplus zp$ for all $z \in \mathbb{C}$. Note that $AB_zA = E \neq 0$. Then $\phi^{-1}(A)\phi^{-1}(B_z)\phi^{-1}(A) = \phi^{-1}(B_z)$ is a non-zero generalized projection so $\phi^{-1}(B_z)^3 = \phi^{-1}(B_z)^{*3}$ is a non-zero generalized projection. Thus $B_z^3 = A_{11}^3 \oplus z^3p$ is a non-zero generalized projection. This Contradiction. It follows that E is a unitary operator and A is invertible, since E is a non-zero generalized projection, then $A^9 = I$.

Now we prove that $A = aI$ for some constant $a \in \mathbb{C}$, $a^9 = 1$. For any unit vector $x \in \mathcal{H}$, there is a non zero vector $y \in \mathcal{H}$ such that $Ax = A^*y$ since A is invertible. Put $B = \frac{x \otimes y}{\|Ax\|^2}$. Then $ABA = \frac{Ax \otimes Ax}{\|Ax\|^2}$ is a non zero generalized projection, which implies that $\phi^{-1}(A)\phi^{-1}(B)\phi^{-1}(A) = \phi^{-1}(B)$ is a non-zero generalized projection, so $\phi^{-1}(B)^3 = \phi^{-1}(B)^{*3}$ is a non-zero generalized projection. Therefore B^3 is a non-zero generalized projection and so $B^9 = \frac{\langle x, y \rangle^8}{\|Ax\|^{18}} x \otimes y$ is a non-zero projection. It follows that $y = a_x x$ for some non-zero constant $a_x \in \mathbb{C}$. Hence, $A^*y = a_x A^*x = Ax$. Note that A is invertible. It follows that $A^* = aA$ for some constant $a \in \mathbb{C}$ from Theorem 2.3 in [16]. Thus, $A^{*9} = a^9 A^9 = I$ so $a^9 = 1$. The proof is complete. \square

Now we prove the Theorem 3.5.

Proof. From Lemma 3.6, we have two cases.

In the first case $\phi(I) = I$, then ϕ preserves generalized projections, from Theorem 3.1 the desired result follows.

In the second case $\phi(I) = aI$ where $a \neq 1$. Repeating the same with $\psi = \bar{a}\phi$, completes the proof. \square

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