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A Generalization of Ćirić Quasi-Contractions for Maps on S-Metric Spaces[1]

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Abstract : In this paper, we prove a fixed point theorem for a class of maps depending on another map on S-metric spaces. As applications, we get the fixed point theorems in [1] and [2]. Also, examples are given to analyze the results.

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1 Introduction and Preliminaries

In [2], Sedghi et al. have introduced a new structure of generalized metric spaces as follows.

Definition 1.1 ([2, Definition 2.1])**.** Let X be a nonempty set. An S-*metric* on X is a function $S : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S(x, y, z) = 0$ if and only if $x = y = z$.
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-*metric space*.

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The following is the intuitive geometric example for S-metric spaces.

Example 1.2 ([2, Example 2.4]). Let $X = \mathbb{R}^2$ and d be the ordinary metric on X. Put

$$S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in X$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

An interesting work relating to this notion is to state fixed point theorems for maps on S-metric spaces. In this line, some results have been proved in [2–4]. Recently, Karapinar et al. have proved a fixed point theorem for a class of maps on metric spaces that satisfy the Ćirić's quasi-contraction depending on another map in [1]. This result gives rise to stating an analogue for maps on S-metric spaces.

In this paper, we prove a fixed point theorem for a class of maps depending on another map on S-metric spaces. As applications, we get the fixed point theorems in [1, 2]. Also, examples are given to analyze the results.

We recall some notions, lemmas and examples which will be useful later.

Lemma 1.3 ([2, Lemma 2.5]). *Let (X, S) be an S-metric space. Then*

$$S(x, x, y) = S(y, y, x)$$

for all $x, y \in X$.

Lemma 1.4. *Let (X, S) be an S-metric space. Then*

$$S(x, x, z) \le 2S(x, x, y) + S(y, y, z)$$

and

$$S(x, x, z) \le 2S(x, x, y) + S(z, z, y)$$

for all $x, y, z \in X$.

Proof. It is a direct consequence of Definition 1.1 and Lemma 1.3. □

Definition 1.5 ([2]). Let (X, S) be an S-metric space.

1. A sequence $\{x_n\} \subset X$ is said to *converge* to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.
2. A sequence $\{x_n\} \subset X$ is said to be *Cauchy* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.
3. The S-metric space (X, S) is said to be *complete* if every Cauchy sequence is a convergent sequence.

From [2, Examples in page 260] we have the following example.

Example 1.6.

1. Let $\mathbb{R}$ be the real line. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the *usual S-metric* on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.

2. Let Y be a nonempty subset of $\mathbb{R}$. Then

$$S(x, y, z) = |x - z| + |y - z|$$

for all $x, y, z \in Y$ is an S-metric on Y. If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Lemma 1.7 ([2, Lemma 2.10])**.** *Let (X, S) be an S-metric space. If $x_n \to x$ in X then the limit point x is unique.*

Lemma 1.8 ([2, Lemma 2.12])**.** *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then $S(x_n, x_n, y_n) \to S(x, x, y)$.*

Definition 1.9 ([2])**.** Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$, we define the *open ball* $B_S(x, r)$ with center x and radius r as follows.

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}.$$

The *topology induced by the S-metric* or the *S-metric topology* is the topology generated by the base of all open balls in X.

Lemma 1.10. *Let $\{x_n\}$ be a sequence in X. Then $x_n \to x$ in the S-metric space (X, S) if and only if $x_n \to x$ in the S-metric topological space X.*

Proof. It is a direct consequence of Definition 1.5(1) and Definition 1.9. □

Lemma 1.11 ([4, Corollary 2.4])**.** *Let $f : X \longrightarrow Y$ be a map from an S-metric space X to an S-metric space Y. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

The following lemma states the relation between a metric and an S-metric.

Lemma 1.12. *Let (X, d) be a metric space. Then we have*

1. *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).*
4. *(X, d) is complete if and only if (X, S_d) is complete.*

Proof. (1) See [2, Example (3), page 260].

(2) $x_n \to x$ in (X, d) if and only if $d(x_n, x) \to 0$, if and only if

$$S_d(x_n, x_n, x) = 2d(x_n, x) \to 0$$

that is, $x_n \to x$ in (X, S_d).

(3) $\{x_n\}$ is Cauchy in (X, d) if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, if and only if

$$S_d(x_n, x_n, x_m) = 2d(x_n, x_m) \to 0$$

as in $n, m \to \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d).

(4) It is a direct consequence of (2) and (3). □

The following example shows that there exists an S-metric S satisfying $S \neq S_d$ for all metrics d.

Example 1.13. Let $X = \mathbb{R}$ and $S(x, y, z) = |y+z-2x|+|y-z|$ for all $x, y, z \in X$. By [2, Example (1), page 260], (X, S) is an S-metric space. We shall prove that there does not exist any metric d such that $S = S_d$. Indeed, suppose to the contrary that there exists a metric d with $S(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then $d(x, z) = \frac{1}{2}S(x, x, z) = |x-z|$ and $d(x, y) = S(x, y, y) = 2|x-y|$ for all $x, y, z \in X$. It is a contradiction.

2 Main Results

In the line of notations and definitions in [5, 6] we have the following definitions.

Definition 2.1. Let (X, S) be an S-metric space, $T, F : X \longrightarrow X$ be two maps and $Y \subset X$, $x \in X$. Then we denote

1. $\delta(Y) = \sup\{S(x, x, y) : x, y \in Y\}$;
2. $O_{T,F}(x, n) = \{Tx, TFx, TF^2x, \ldots, TF^nx\}$;
3. $O_{T,F}(x, \infty) = \{Tx, TFx, TF^2x, \ldots\}$;
4. $O_F(x, n) = O_{T,F}(x, n)$ and $O_F(x, \infty) = O_{T,F}(x, \infty)$ if T is the identify.

Definition 2.2. Let (X, S) be an S-metric space and $T : X \longrightarrow X$ be a map. T is said to be *sequentially convergent* if every sequence $\{y_n\}$ is convergent provided that each sequence $\{Ty_n\}$ is convergent.

The main result of the paper is as follows.

Theorem 2.3. *Let (X, S) be an S-metric space and $T, F : X \longrightarrow X$ be two maps such that*

1. *T is one-to-one, continuous and sequentially convergent;*

2. *Every Cauchy sequence of the form $\{TF^n x\}$ is convergent in X for all $x \in X$;*
3. *There exists $q \in [0,1)$ satisfying*

$$S(TFx,TFx,TFy) \leq q \max\Big\{S(Tx,Tx,Ty), S(Tx,Tx,TFx), S(Ty,Ty,TFy), S(Tx,Tx,TFy), S(Ty,Ty,TFx)\Big\} \tag{2.1}$$

for all $x, y \in X$.

Then we have

1. *$S(TF^i x, TF^i x, TF^j x) \leq q\delta[O_{T,F}(x,n)]$ for all $i,j \leq n$, $n \in \mathbb{N}$ and $x \in X$;*
2. *$\delta[O_{T,F}(x,\infty)] \leq \dfrac{2}{1-q} S(Tx,Tx,TFx)$ for all $x \in X$;*
3. *F has a unique fixed point b;*
4. *$\lim_{n\to\infty} TF^n x = Tb$.*

Proof. (1) For each $x \in X$ and all $1 \leq i,j \leq n$, $n \in \mathbb{N}$, we have

$$TF^{i-1}x, TF^i x, TF^{j-1}x, TF^j x \in O_{T,F}(x,n)$$

where $F^0 x = x$. It follows from (2.1) that

$$\begin{aligned} S(TF^i x, TF^i x, TF^j x) &= S\big(TF(F^{i-1}x), TF(F^{i-1}x), TF(F^{j-1}x)\big) \\ &\leq q \max\Big\{S(TF^{i-1}x, TF^{i-1}x, TF^{j-1}x), \\ &\qquad S(TF^{i-1}x, TF^{i-1}x, TF^i x), S(TF^{j-1}x, TF^{j-1}x, TF^j x), \\ &\qquad S(TF^{i-1}x, TF^{i-1}x, TF^j x), S(TF^{j-1}x, TF^{j-1}x, TF^i x)\Big\} \\ &\leq q\delta[O_{T,F}(x,n)]. \end{aligned}$$

That is $S(TF^i x, TF^i x, TF^j x) \leq q.\delta[O_{T,F}(x,n)]$.

(2) We have

$$\delta[O_{T,F}(x,n)] \leq \delta[O_{T,F}(x,n+1)]$$

for all $n \in \mathbb{N}$. Then

$$\delta[O_{T,F}(x,\infty)] = \sup\{\delta[O_{T,F}(x,n)] : n \in \mathbb{N}\}.$$

So we only need to prove that $\delta[O_{T,F}(x,n)] \leq \frac{1}{1-q} S(Tx,Tx,TFx)$ for all $n \in \mathbb{N}$. For all $1 \leq i,j \leq n$, it follows from the conclusion (1) that

$$S(TF^i x, TF^i x, TF^j x) \leq q\delta[O_{T,F}(x,n)]$$

and the fact $O_{T,F}(x,n)$ is finite, there exists $k \leq n$ such that

$$S(Tx, Tx, TF^k x) = \delta[O_{T,F}(x,n)].$$

Applying Lemma 1.4 and the conclusion (1) we get

$$\begin{aligned} S(Tx, Tx, TF^k x) &\leq 2S(Tx, Tx, TFx) + S(TFx, TFx, TF^k x) \\ &\leq 2S(Tx, Tx, TFx) + q.\delta[O_{T,F}(x,n)] \\ &= 2S(Tx, Tx, TFx) + q.S(Tx, Tx, TF^k x). \end{aligned}$$

Then $\delta[O_{T,F}(x,n)] = S(Tx, Tx, TF^k x) \leq \frac{2}{1-q} S(Tx, Tx, TFx)$.

(3) For each $x_0 \in X$, we define two iterative sequences $\{x_n\}$ and $\{y_n\}$ as follows

$$x_{n+1} = Fx_n = F^{n+1}x_0, \quad y_n = Tx_n = TF^n x_0$$

for all $n \in \mathbb{N}$. We will prove that $\{y_n\}$ is a Cauchy sequence. For all $n < m$, by using the conclusion (1) we have

$$\begin{aligned} S(y_n, y_n, y_m) &= S(TF^n x_0, TF^n x_0, TF^m x_0) \\ &= S(TFF^{n-1}x_0, TFF^{n-1}x_0, TF^{m-n+1}F^{n-1}x_0) \\ &\leq q\delta[O_{T,F}(F^{n-1}x_0, m-n+1)]. \end{aligned}$$

Note that there exists $1 \leq l \leq m-n+1$ satisfying

$$\delta[O_{T,F}(F^{n-1}x_0, m-n+1)] = S(TF^{n-1}x_0, TF^{n-1}x_0, TF^l F^{n-1}x_0).$$

On the other hand we have

$$\begin{aligned} S(TF^{n-1}x_0, TF^{n-1}x_0, TF^l F^{n-1}x_0) &= S(TFF^{n-2}x_0, TFF^{n-2}x_0, TF^{l+1}F^{n-2}x_0) \\ &\leq q\delta[O_{T,F}(F^{n-2}x_0, l+1)]. \end{aligned}$$

Therefore,

$$\begin{aligned} S(y_n, y_n, y_m) &= S(TF^n x_0, TF^n x_0, TF^m x_0) \\ &\leq q\delta[O_{T,F}(F^{n-1}x_0, m-n+1)] \\ &\leq q^2\delta[O_{T,F}(F^{n-2}x_0, m-n+2)] \\ &\vdots \\ &\leq q^{n-1}\delta[O_{T,F}(Fx_0, m)] \\ &\leq q^n\delta[O_{T,F}(x_0, m)]. \end{aligned} \tag{2.2}$$

Using the conclusion (2) we have

$$\delta[O_{T,F}(x_0, m)] \leq \delta[O_{T,F}(x_0, \infty)] \leq \frac{2}{1-q} S(Tx_0, Tx_0, TFx_0). \tag{2.3}$$

By combining (2.2) and (2.3) we get

$$S(y_n, y_n, y_m) \leq \frac{2q^n}{1-q} S(Tx_0, Tx_0, TFx_0).$$

Taking the limit as $n, m \to \infty$ we obtain $\lim_{n,m\to\infty} S(y_n, y_n, y_m) = 0$, that is, $\{y_n\}$ is a Cauchy sequence. By the assumption (2), there exists $a \in X$ such that $\lim_{n\to\infty} TF^n x_0 = a$. By the assumption (1), there exists $b \in X$ such that $\lim_{n\to\infty} F^n x_0 = b$. Since T is continuous, by Lemma 1.11 we have

$$\lim_{n\to\infty} TF^n x_0 = Tb. \tag{2.4}$$

It follows from Lemma 1.7 that $Tb = a$.

Next we shall prove that $Fb = b$. By using Lemma 1.3, Lemma 1.4 and (2.1) we have

$$\begin{aligned}
S(Tb, Tb, TFb) &\leq 2S(Tb, Tb, TF^{n+1}x_0) + S(TF^{n+1}x_0, TF^{n+1}x_0, TFb) \\
&= 2S(Tb, Tb, TF^{n+1}x_0) + S(TFF^n x_0, TFF^n x_0, TFb) \\
&\leq 2S(Tb, Tb, TF^{n+1}x_0) \\
&\quad + q \max\Big\{S(TF^n x_0, TF^n x_0, Tb), S(TF^n x_0, TF^n x_0, TFF^n x_0), \\
&\quad S(Tb, Tb, TFb), S(TF^n x_0, TF^n x_0, TFb), S(Tb, Tb, TFF^n x_0)\Big\} \\
&\leq 2S(Tb, Tb, TF^{n+1}x_0) + q \max\Big\{S(TF^n x_0, TF^n x_0, Tb), \\
&\quad S(TF^n x_0, TF^n x_0, TFF^n x_0), S(Tb, Tb, TFb), \\
&\quad 2S(TF^n x_0, TF^n x_0, Tb) + S(Tb, Tb, TFb), S(Tb, Tb, TFF^n x_0)\Big\} \\
&\leq 2S(Tb, Tb, TF^{n+1}x_0) + q\Big(S(TF^n x_0, TF^n x_0, TF^{n+1}x_0) \\
&\quad + S(Tb, Tb, TF^{n+1}x_0) + 2S(TF^n x_0, TF^n x_0, Tb) \\
&\quad + S(Tb, Tb, TFb)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S(Tb, Tb, TFb) \leq \frac{1}{1-q}\Big[&(2+q)S(Tb, Tb, TF^{n+1}x_0) \\
&+ qS(TF^n x_0, TF^n x_0, TF^{n+1}x_0) + 2qS(TF^n x_0, TF^n x_0, Tb)\Big].
\end{aligned} \tag{2.5}$$

By using Lemma 1.8, (2.4) and taking the limit as $n \to \infty$ in (2.5) we get

$$S(Tb, Tb, TFb) = 0.$$

That is, $TFb = Tb$. Since T is one-to-one, we get $Fb = b$.

Now we prove that b is the unique fixed point of F. Let b and b' be two fixed points of F. Then $Fb = b$ and $Fb' = b'$. By using (2.1) and Lemma 1.3 we have

$$\begin{aligned} S(Tb, Tb, Tb') &= S(TFb, TFb, TFb') \\ &\leq q \max\Big\{S(Tb, Tb, Tb'), S(Tb, Tb, TFb), \\ &\qquad S(Tb', Tb', TFb'), S(Tb, Tb, TFb'), S(Tb', Tb', TFb)\Big\} \\ &= qS(Tb, Tb, Tb'). \end{aligned}$$

Since $0 \leq q < 1$, we get $S(Tb, Tb, Tb') = 0$, that is, $Tb = Tb'$. Note that T is one-to-one, then $b = b'$.

(4) It is straightforward from (2.4). □

We get the following corollary which is similar to [1, Theorem 2.1] except for the conclusion (2).

Corollary 2.4. *Let (X, d) be a metric space and $T, F : X \longrightarrow X$ be two maps such that*

1. *T is one-to-one, continuous and sequentially convergent;*
2. *Every Cauchy sequence of the form $\{TF^n x\}$ is convergent in X for all $x \in X$;*
3. *There exists $q \in [0, 1)$ satisfying*

$$\begin{aligned} d(TFx, TFy) \leq q \max\Big\{&d(Tx, Ty), d(Tx, TFx), \\ &d(Ty, TFy), d(Tx, TFy), d(Ty, TFx)\Big\} \end{aligned} \tag{2.6}$$

for all $x, y \in X$.

Then we have

1. *$d(TF^i x, TF^j x) \leq q.\delta[O_{T,F}(x, n)]$ for all $i, j \leq n$, $n \in \mathbb{N}$ and $x \in X$;*
2. *$\delta[O_{T,F}(x, \infty)] \leq \frac{2}{1-q} d(Tx, TFx)$ for all $x \in X$;*
3. *F has a unique fixed point b;*
4. *$\lim_{n\to\infty} TF^n x = Tb$.*

Proof. By using Lemma 1.12 and Theorem 2.3 where S_d plays the role of S we get the conclusion. □

By using Theorem 2.3 where T is the identity we get the following corollary which is a generalization of [6, Theorem 1] into the structure of S-metric.

Corollary 2.5. *Let (X, S) be an S-metric space and $F : X \longrightarrow X$ be a map such that*

1. *Every Cauchy sequence of the form* $\{F^n x\}$ *is convergent in* X *for all* $x \in X$*;*
2. *There exists* $q \in [0, 1)$ *satisfying*

$$S(Fx, Fx, Fy) \leq q \max\Big\{S(x, x, y), S(x, x, Fx), S(y, y, Fy), S(x, x, Fy), S(y, y, Fx)\Big\} \tag{2.7}$$

for all $x, y \in X$.

Then we have

1. $S(F^i x, F^i x, F^j x) \leq q.\delta[O_F(x, n)]$ *for all* $i, j \leq n$, $n \in \mathbb{N}$ *and* $x \in X$*;*
2. $\delta[O_F(x, \infty)] \leq \frac{2}{1-q} S(x, x, Fx)$ *for all* $x \in X$*;*
3. F *has a unique fixed point* b*;*
4. $\lim_{n\to\infty} F^n x = b$.

The following corollary is a generalization of the main result of [7] into the structure of S-metric.

Corollary 2.6. *Let* (X, S) *be an* S*-metric space and* $T, F : X \longrightarrow X$ *be two maps such that*

1. T *is one-to-one, continuous and sequentially convergent;*
2. *Every Cauchy sequence of the form* $\{TF^n x\}$ *is convergent in* X *for all* $x \in X$*;*
3. *There exist* $a_i \geq 0$, $i = 1, \ldots, 5$, *satisfying* $\sum_{i=1}^{n} a_i < 1$ *and*

$$\begin{aligned} S(TFx, TFx, TFy) \leq\ & a_1 S(Tx, Tx, Ty) + a_2 S(Tx, Tx, TFx) \\ & + a_3 S(Ty, Ty, TFy) + a_4 S(Tx, Tx, TFy) \\ & + a_5 S(Ty, Ty, TFx) \end{aligned} \tag{2.8}$$

for all $x, y \in X$.

Then we have

1. F *has a unique fixed point* b*;*
2. $\lim_{n\to\infty} TF^n x = Tb$.

Proof. Since (2.1) is a consequence of (2.8), we get the corollary. □

Remark 2.7. *By choosing* T *is the identity and* $a_2 = a_3 = a_4 = a_5 = 0$ *in Corollary 2.6 we get [2, Theorem 3.1].*

The following example shows that Corollary 2.6 is a proper generalization of [2, Theorem 3.1].

Example 2.8. Let $X = [0,1]$ and $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in X$. Then (X,S) is a complete S-metric space by Example 1.6. Put

$$Fx = \begin{cases} \dfrac{1}{2} & \text{if } x \in [0,1) \\ \dfrac{1}{4} & \text{if } x = 1. \end{cases}$$

We have $S(F\frac{3}{4}, F\frac{3}{4}, F1) = \frac{1}{2}$ and $S(\frac{3}{4}, \frac{3}{4}, 1) = \frac{1}{2}$. This proves that [2, Theorem 3.1] is not applicable to F. By choosing $Tx = x$ for all $x \in X$ we have

$$S(TFx, TFx, TF1) = S(Fx, Fx, F1) = \frac{1}{2}$$

$$S(T1, T1, TF1) = S(1, 1, F1) = \frac{3}{2}.$$

Then for $a_1 = a_2 = a_4 = a_5 = 0$ and $a_3 = \frac{1}{2}$ we see that the condition (2.8) in Corollary 2.6 is satisfied. Also, the other conditions in Corollary 2.6 are. Then Corollary 2.6 is applicable to F and T, and $x = \frac{1}{2}$ is the unique fixed point of F.

By adapting [1, Example 2.3] we have the following example that proves Theorem 2.3 is a proper generalization of Corollary 2.5.

Example 2.9. Let $X = [0,\infty)$ and $S(x,y,z) = |x-z| + |y-z|$. Then (X,S) is a complete S-metric space by Example 1.6. Put $Fx = \frac{x^2}{x+1}$ for all $x \in X$. Then we have

$$\begin{aligned} S\big(Fx, Fx, F(2x)\big) &= \frac{2x^2(2x+3)}{(2x+1)(x+1)} \\ S(x, x, Fx) &= \frac{2x}{x+1} \\ S\big(2x, 2x, F(2x)\big) &= \frac{4x}{2x+1} \\ S\big(x, x, F(2x)\big) &= 2\left|\frac{2x^2 - x}{2x+1}\right| \\ S(2x, 2x, Fx) &= 2\frac{x^2+2x}{x+1} \\ S(x, x, 2x) &= 2x. \end{aligned}$$

Therefore, if x is large enough we have

$$\max\big\{S(x,x,2x), S(x,x,Fx), S(2x,2x,F(2x)), S(x,x,F(2x)), S(2x,2x,Fx)\big\} = 2\frac{x^2+2x}{x+1}.$$

This implies that the condition (2.7) is equivalent to

$$\frac{x(2x+3)}{(2x+1)(x+2)} \leq q. \tag{2.9}$$

Taking the limit as $x \to \infty$ in (2.9) we get $q \geq 1$. It is a contradiction. Then Corollary 2.5 is not applicable to F.

By choosing $Tx = \mathrm{e}^x - 1$ for all $x \in X$ and $q = \frac{1}{2}$. Then we have T is one-to-one, continuous and sequentially convergent on X and

$$S(TFx, TFx, TFy) = 2\left|\mathrm{e}^{\frac{x^2}{x+1}} - \mathrm{e}^{\frac{y^2}{y+1}}\right|$$
$$S(Tx, Tx, Ty) = 2|\mathrm{e}^x - \mathrm{e}^y|.$$

Now we will show that

$$S(TFx, TFx, TFy) \leq \frac{1}{2}S(Tx, Tx, Ty) \tag{2.10}$$

for all $x, y \in X$. The case of $x = y$ is trivial. We may assume that $x > y$. Then (2.10) is equivalent to

$$\left|\mathrm{e}^{\frac{x^2}{x+1}} - \mathrm{e}^{\frac{y^2}{y+1}}\right| \leq \frac{1}{2}|\mathrm{e}^x - \mathrm{e}^y|$$

that is

$$\mathrm{e}^{\frac{x^2}{x+1}} - \frac{\mathrm{e}^x}{2} \leq \mathrm{e}^{\frac{y^2}{y+1}} - \frac{\mathrm{e}^y}{2}.$$

This is true because the function $\varphi(t) = \mathrm{e}^{\frac{t^2}{t+1}} - \frac{\mathrm{e}^t}{2}$ is decreasing on X. Therefore, (2.10) and then (2.6) holds.

By the above, Theorem 2.3 is applicable to F and T, and $x = 0$ is the unique fixed point of F.

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