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On the Record Range

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Abstract : In this paper we consider some distributional properties of record range of a sequence of i.i.d continuous uniform random variables. More precisely, we calculate the entropy of the record range and evaluate it as function of n. Also, we obtain the joint probability density function of R_n and R_m via a Markov chain and show that the conditional density of R_n given R_{n-1} is independent of n. Finally, some nice equalities related to the moments of the record range are proved.

Keywords : uniform distribution; record range; entropy; Markov chain; joint pdf.
2010 Mathematics Subject Classification : 60E05.

1 Introduction

Let $\{X_i\ ;\ i = 1, 2, 3, ...\}$ be a sequence of iid continuous random variable each distributed according to cumulative distribution function $F(x)$ and probability density function $f(x)$. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. Also, X_j is a lower record if $X_j < X_i$ for every $i < j$. Let $T_{U(1)}, T_{U(2)}, ...$ be the upper records and $T_{L(1)}, T_{L(2)}, ...$ be the lower records of $\{X_i\ ;\ i = 1, 2, 3, ...\}$. Suppose T_{nu} be the largest observation after observing nth record and T_{nl} be the smallest observation after observing the nth record. We say $R_n = T_{nu} - T_{nl} (n \geq 2)$, as nth record range. The joint pdf of $f_{nu,nl}$ of T_{nu} and

T_{nl} is given by [1, 2]

$$f_{nu,nl}(x,y) = \frac{2^{n-1}}{\Gamma(n-1)}[-\ln(\overline{F}(y)+F(x))]^{n-2}f(x)f(y), \tag{1.1}$$

where $-\infty < x < y < +\infty$. Also, the pdf of f_{R_n} of R_n is given by [3, 5]

$$f_{R_n}(r) = \int_{-\infty}^{+\infty} \frac{2^{n-1}}{\Gamma(n-1)}[-\ln(\overline{F}(r+t)+F(t))]^{n-2}f(r+t)f(t)dt, \tag{1.2}$$

where $\overline{F} = 1 - F$.

In this paper we calculate the joint pdf of record ranges and entropy of record range when $\{X_i\ ;\ i = 1,2,3,...\}$ is a sequence of i.i.d continuous uniform variables; i.e., $f(x) = (b-a)^{-1},\ a < x < b$. We also discuss on the beast linear least square predictor $\widehat{R_{n+k}}$ of R_{n+k} based on $R_2, ..., R_n$ and the correlation of R_n and R_m.

The change of variable $y = \ln\Big(1 - \frac{r}{b-a}\Big)$ will be used repeatedly in this paper. Thus for $n \geq 2$,

$$f_{R_n}(r) = \frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)}\Big[-\ln\Big(1-\frac{r}{b-a}\Big)\Big]^{n-2} \tag{1.3}$$

and

$$F_{R_n}(r) = 1 - \Big(1-\frac{r}{b-a}\Big)^2 \sum_{i=0}^{n-2} \frac{\Big(-\ln\Big(1-\frac{r}{b-a}\Big)\Big)^i}{i!}. \tag{1.4}$$

2 Entropy of R_n

Theorem 2.1. *For $n \geq 2$,*

$$Entropy(R_n) = \ln\Big(\frac{b-a}{2}\Gamma(n-1)\Big) + \frac{n-1}{2} - (n-2)\Psi(n-1), \tag{2.1}$$

where $\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}$.

Proof. First,

$$\begin{aligned} -\ln f_{R_n} &= 2\ln(b-a) + \ln\Gamma(n-1) - (n-1)\ln 2 - \ln(b-a-r) \\ &\quad - (n-2)\ln\Big(-\ln\Big(1-\frac{r}{b-a}\Big)\Big). \end{aligned}$$

Thus

$$\begin{aligned} Entropy = {} & 2\ln(b-a) + \ln\Gamma(n-1) - (n-1)\ln 2 \\ & - \int_0^{b-a} \ln(b-a-r)\frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)} \\ & \times [-\ln(1-\frac{r}{b-a})]^{n-2}dr \\ & - \int_0^{b-a} (n-2)\ln\left(-\ln\left(1-\frac{r}{b-a}\right)\right) \\ & \times \frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)}[-\ln(1-\frac{r}{b-a})]^{n-2}dr. \end{aligned}$$

With change of variable y,

$$\int_0^{b-a} \ln(b-a-r)\frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)}[-\ln(1-\frac{r}{b-a})]^{n-2}dr = \ln(b-a) - \frac{n-1}{2}.$$

Also

$$\begin{aligned} (n-2)\int_0^{b-a} & \ln(-\ln(1-\frac{r}{b-a}))\frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)}[-\ln(1-\frac{r}{b-a})]^{n-2}dr \\ & = (n-2)\frac{\Gamma'(n-1)}{\Gamma(n-1)} - \ln 2 = (n-2)\Psi(n-1) - \ln 2 \end{aligned}$$

and proof is completed. □

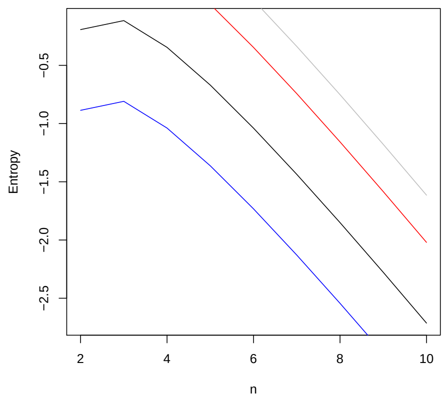

Figure 1: Entropy of R_n.

In Figure 1, we show the behavior of entropy of R_n as function of n for some values a and b $(a < b)$.

3 Joint PDF of R_m and R_n

In the following theorem we obtain the joint pdf of R_m and R_n via a Markov chain.

Theorem 3.1. *For* $2 \leq m < n$ *and* $0 < x < y < b-a$,

$$f_{R_m,R_n}(x,y) = \frac{2^{n-1}}{(b-a)^2\Gamma(m-1)\Gamma(n-m)}\frac{b-a-y}{b-a-x}[-\ln(1-\frac{x}{b-a})]^{m-2} \times [\ln(b-a-x)-\ln(b-a-y)]^{n-m-1}.$$

Proof. Let $S = \Pi_{i=1}^{n} V_i$, where $V_1, ..., V_n$ are i.i.d with $f(v) = 2vI_{(0,1)}(v)$ and V_n is independent of $b-a-R_n$, then $b-a-R_n \stackrel{d}{=} (b-a)S$, where $\stackrel{d}{=}$ means identical in distribution. Because, if $Y := b-a-R_n$, then

$$\begin{aligned} f_Y(y) &= f_{R_n}(b-a-y) \\ &= \frac{2^{n-1}y}{(b-a)^2\Gamma(n-1)}\left(-\ln\left(\frac{y}{b-a}\right)\right)^{n-2}. \end{aligned}$$

Suppose $Z = (b-a-R_n)V_n$ and V_n is independent of $b-a-R_n$. Then

$$\begin{aligned} f_Z(z) &= \int_z^{b-a} \frac{1}{r} f_{b-a-R_n}(r) f_{V_n}\left(\frac{z}{r}\right)dr \\ &= \frac{2^{n-1}z}{(b-a)^2\Gamma(n-1)}\int_z^{b-a}\frac{1}{r}\left(-\ln\left(\frac{r}{b-a}\right)\right)^{n-2}dr. \end{aligned}$$

With change of variable $u = -\ln\left(\frac{r}{b-a}\right)$,

$$\begin{aligned} f_Z(z) &= \frac{2^{n-1}z}{(b-a)^2\Gamma(n-1)}\int_{-\ln(\frac{z}{b-a})}^{0}\frac{u^{n-2}}{be^{-u}}(-be^{-u})du \\ &= \frac{2^n z}{(b-a)^2\Gamma(n)}\left(-\ln\left(\frac{z}{b-a}\right)\right)^{n-1}. \end{aligned}$$

Thus, $b-a-R_{n+1} \stackrel{d}{=} (b-a-R_n)V_n$. Note that the sequence $\{(b-a-R_n)V_n\}_{n=1}^{\infty}$ forms a Markov chain. Since

$$\frac{f_Z(z)}{z} = \frac{2^n}{(b-a)^2\Gamma(n)}\left(-\ln\left(\frac{z}{b-a}\right)\right)^{n-1},$$

we have $b-a-R_n \stackrel{d}{=} (b-a)S$. Now, let

$$Z_1 = \Pi_{j=1}^{m-1} V_j, \quad Z_2 = \Pi_{j=1}^{n-m} V_{(m-1)+j}.$$

Then the joint pdf of $S_1 = (b-a)Z_1$ and $S_2 = (b-a)Z_2$ is given by

$$f_{S_1,S_2}(s_1,s_2) = \frac{2^{n-1}s_1 s_2}{(b-a)^4\Gamma(m-1)\Gamma(n-m)}\Big(-\ln\frac{s_1}{b-a}\Big)^{m-2}\Big(-\ln\frac{s_2}{b-a}\Big)^{n-m-1}.$$

Now, let $T_1 = S_1$ and $T_2 = \frac{1}{b-a}S_1S_2$, then the Jacobian is $\frac{b-a}{t_1}$. Thus

$$f_{T_1,T_2}(t_1,t_2) = \frac{2^{n-1}}{(b-a)^2\Gamma(m-1)\Gamma(n-m)}\frac{t_2}{t_1}\Big(-\ln\frac{t_1}{b-a}\Big)^{m-2}\Big(-\ln\frac{t_2}{t_1}\Big)^{n-m-1}.$$

With substituting $T_1 = b-a-R_m$ and $T_2 = b-a-R_n$ proof is completed. □

Corollary 3.2. *Using $m = n-1$, $a = 0$ and $b = 1$,*

$$f_{R_n|R_{n-1}}(y|x) = \frac{2(1-y)}{(1-x)^2},\ \ 0 < x < y < 1.$$

Lemma 3.3. *Let* $\mu_m^{p,q} := \mathbb{E}(R_m^p R_{m+1}^q)$*, then*

$$(b-a)\mu_m^{p,q} + \frac{2}{q+1}\mu_m^{p+q+1} = \frac{q+3}{q+1}\mu_m^{p,q+1},\quad p,q \geq 0. \tag{3.1}$$

Proof.

$$\begin{aligned}(b-a)\mu_m^{p,q} - \mu_m^{p,q+1} &= \int_0^{b-a}\int_x^{b-a}((b-a)x^py^q - x^py^{q+1})f_{R_m,R_{m+1}}(x,y)dydx\\ &= \int_0^{b-a} x^p\frac{2^m}{(b-a)^2\Gamma(m-1)}\frac{1}{b-a-x}[-\ln(1-\frac{x}{b-a})]^{m-2}dx\\ &\quad\times\int_x^{b-a} y^q(b-a-y)^2dy.\end{aligned}$$

By integration by part with $y^qdy = dv$ and $(b-a-y)^2 = u$, then

$$\int_x^{b-a} y^q(b-a-y)^2dy = -\frac{x^{q+1}}{q+1}(b-a-x)^2 + 2\int_x^{b-a}\frac{y^{q+1}}{q+1}(b-a-y)dy.$$

Thus

$$\begin{aligned}(b-a)\mu_m^{p,q} - \mu_m^{p,q+1} &= \int_0^{b-a} x^p\frac{2^m}{(b-a)^2\Gamma(m-1)}\frac{1}{b-a-x}\\ &\quad\times[-\ln(1-\frac{x}{b-a})]^{m-2}\times -\frac{x^{q+1}}{q+1}(b-a-x)^2\\ &\quad+2\int_x^{b-a}\frac{y^{q+1}}{q+1}(b-a-y)dydx\\ &= -\frac{2}{q+1}\int_0^{b-a}\frac{x^{p+q+1}}{q+1}\frac{2^{m-1}}{(b-a)^2\Gamma(m-1)}(b-a-x)\\ &\quad\times[-\ln(1-\frac{x}{b-a})]^{m-2}dx\\ &\quad+\frac{2}{q+1}\mu_m^{p,q+1} = -\frac{2}{q+1}\mu_m^{p+q+1} + \frac{2}{q+1}\mu_m^{p,q+1}.\end{aligned}$$

□

Lemma 3.4. *Let* $\mu_{m,n}^{p,q} := \mathbb{E}(R_m^p R_n^q)$, *then*

$$\frac{q+3}{q+1}\mu_{m,n}^{p,q+1} = (b-a)\mu_{m,n}^{p,q} + \frac{2}{q+1}\mu_{m,n-1}^{p,q+1}, \quad p,q \geq 0.$$

Proof.

$$\begin{aligned}(b-a)\mu_{m,n}^{p,q} - \mu_{m,n}^{p,q+1} &= \int_0^{b-a}\int_x^{b-a}((b-a)x^p y^q - x^p y^{q+1}) f_{R_m,R_n}(x,y)dydx \\ &= \int_0^{b-a}\int_x^{b-a} x^p y^q (b-a-y)\frac{2^{n-1}}{(b-a)^2\Gamma(m-1)\Gamma(n-m)} \\ &\quad \times \frac{b-a-y}{b-a-x}[-\ln(1-\frac{x}{b-a})]^{m-2} \\ &\quad \times [\ln(b-a-x)-\ln(b-a-y)]^{n-m-1}dydx \\ &= \int_0^{b-a} x^p \frac{2^{n-1}}{(b-a)^2\Gamma(m-1)\Gamma(n-m)}\frac{1}{b-a-x} \\ &\quad \times [-\ln(1-\frac{x}{b-a})]^{m-2}dx \\ &\quad \times \int_x^{b-a}[\ln(b-a-x)-\ln(b-a-y)]^{n-m-1} \\ &\quad \times y^q(b-a-y)^2dy.\end{aligned}$$

Let

$$S(x) = \int_x^{b-a}[\ln(b-a-x)-\ln(b-a-y)]^{n-m-1}y^q(b-a-y)^2dy.$$

If $y^q dy = dv$ and $u = (b-a-y)^2[\ln(b-a-x)-\ln(b-a-y)]^{n-m-1}$, then

$$\begin{aligned}du &= -2(b-a-y)[\ln(b-a-x)-\ln(b-a-y)]^{n-m-1} \\ &\quad + (n-m-1)(b-a-y)[\ln(b-a-x)-\ln(b-a-y)]^{n-m-2}]dy.\end{aligned}$$

Hence

$$\begin{aligned}(b-a)\mu_{m,n}^{p,q} - \mu_{m,n}^{p,q+1} &= \int_0^{b-a} x^p \frac{2^{n-1}}{(b-a)^2\Gamma(m-1)\Gamma(n-m)}\frac{1}{b-a-x} \\ &\quad \times [-\ln(1-\frac{x}{b-a})]^{m-2}\{-\int_x^{b-a}\frac{y^{q+1}}{q+1}[-2(b-a-y) \\ &\quad \times [\ln(b-a-x)-\ln(b-a-y)]^{n-m-1} \\ &\quad + (n-m-1)(b-a-y) \\ &\quad \times [\ln(b-a-x)-\ln(b-a-y)]^{n-m-2}]dy\}dx \\ &= \frac{2}{q+1}\mu_{m,n}^{p,q+1} - \frac{2}{q+1}\mu_{m,n-1}^{p,q+1}\end{aligned}$$

and proof is completed. ∎

4 Some Results

Lemma 4.1. *For $n \geq 2$,*

$$2\Big(F_{R_n}(r) - F_{R_{n+1}}(r)\Big) = (b-a-r)\ f_{R_{n+1}}(r). \tag{4.1}$$

Proof.

$$\begin{aligned} F_{R_n}(r) - F_{R_{n+1}}(r) &= \frac{(b-a-r)^2}{(b-a)^2\Gamma(n)} 2^{n-1}(-\ln(1-\frac{r}{b-a}))^{n-1} \\ &= \frac{f_{R_{n+1}}(r)}{2}(b-a-r). \end{aligned}$$ □

For $j \geq 1$,

$$\begin{aligned} \mu_n^j &= \mathbb{E}(R_n^j) \\ &= \frac{(b-a)^j 2^{n-1} \int_0^\infty (1-e^{-y})^j e^{-2y} y^{n-2} dy}{\Gamma(n-1)}. \end{aligned}$$

Since,

$$(1-e^{-y})^j = \sum_{i=0}^{j} \left(\begin{array}{c} j \\ i \end{array} \right) (-1)^i e^{-iy},$$

by using $j = 1$ and $j = 2$ [4],

$$\mathbb{V}ar(R_n) = (b-a)^2 \left[\Big(\frac{1}{2}\Big)^{n-1} - \Big(\frac{4}{9}\Big)^{n-1} \right].$$

Lemma 4.2. *For $n \geq 2$ and $j = 1, 2, 3, ...$,*

$$(j+2)\mu_n^j - j(b-a)\mu_n^{j-1} = 2\mu_{n-1}^j. \tag{4.2}$$

Proof. By definition,

$$j((b-a)\mu_n^{j-1} - \mu_n^j) = j(b-a)^j 2^{n-1} \sum_{i=0}^{j-1} \left(\begin{array}{c} j-1 \\ i \end{array} \right) \frac{(-1)^i}{(3+i)^{n-1}}.$$

Also

$$\begin{aligned} 2\mu_n^j - 2\mu_{n-1}^j &= 2\int_0^{b-a} r^j \frac{2^{n-1}(b-a-r)}{(b-a)^2\Gamma(n-1)}[-\ln(1-\frac{r}{b-a})]^{n-2} dr \\ &= j(b-a)^j 2^{n-1} \sum_{i=0}^{j-1} \left(\begin{array}{c} j-1 \\ i \end{array} \right) \frac{(-1)^i}{(3+i)^{n-1}}. \end{aligned}$$ □

Theorem 4.3. *The best linear least squares predictor, $\widehat{R_{n+k}}$ of R_{n+k} based on $R_2, ..., R_n$ is*

$$(b-a)\left[1-\left(\frac{2}{3}\right)^k\right]+x\left(\frac{2}{3}\right)^k. \tag{4.3}$$

Proof.

$$\mathbb{E}(b-a-R_n|R_m=x)=\frac{2^{n-m}}{\Gamma(n-m)}\int_x^{b-a}\frac{(b-a-y)^2}{(b-a-x)^2} \times[\ln(b-a-x)-\ln(b-a-y)]^{n-m-1}\,dy,$$

By change of variable $u=\ln(b-a-x)-\ln(b-a-y)$,

$$\mathbb{E}(b-a-R_n|R_m=x)=(b-a-x)(\frac{2}{3})^{n-m}.$$

Thus

$$\begin{aligned}\mathbb{E}(R_n|R_m=x)&=b-a-(b-a-x)\left(\frac{2}{3}\right)^{n-m}\\&=(b-a)\left[1-\left(\frac{2}{3}\right)^{n-m}\right]+x\left(\frac{2}{3}\right)^{n-m}.\end{aligned}$$

By Markov property of $R_2, ..., R_n$,

$$\widehat{R_{n+k}}=\mathbb{E}(R_{n+k}|R_n=x)\quad=(b-a)[1-(\tfrac{2}{3})^k]+x(\tfrac{2}{3})^k.$$ □

Lemma 4.4. *For any fixed m, $\lim_{n\to\infty}\mathbb{C}orr(R_n,R_m)=0$.*

Proof. Since the sequence $\{(b-a-R_n)V_n\}_{n=1}^{\infty}$ forms a Markov chain,

$$\mathbb{E}\Big(b-a-R_n|b-a-R_m=x\Big)=x\left(\frac{2}{3}\right)^{n-m},\quad a<x<b.$$

Thus

$$\begin{aligned}\mathbb{C}ov(R_n,R_m)&=\left(\frac{2}{3}\right)^{n-m}\mathbb{V}ar(R_m)\\&=\left(\frac{2}{3}\right)^{n-m}(b-a)^2\left[\left(\frac{1}{2}\right)^{m-1}-\left(\frac{4}{9}\right)^{m-1}\right].\end{aligned}$$

Thus, by definition for any fixed m,

$$\lim_{n\to\infty}\mathbb{C}orr(R_n,R_m)=0.$$ □

Lemma 4.4 shows that the correlation of between R_n and R_m is independent of $b-a$. Table 1 shows the correlations of between R_n and R_m for $m, n = 2, 3, 4$, $a=0$ and $b=2$.

Table 1: The correlations.

m, n	2	3	4
2	4/18	4/27	8/81
3	4/27	68/324	68/486
4	8/81	68/486	872/5862

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