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On Near Armendariz Ideals

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Abstract : We say a left ideal I is a near Armendariz whenever polynomials $(x)f = f_0 + f_1x + \ldots + f_mx^m$, $(x)g = g_0 + g_1x + \ldots + g_nx^n \in R[x]$ satisfy $(x)fo(x)g \in r_{R[x]}(I[x])$ then $g_jf_i \in r_R(I)$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(f_0)g \in r_R(I)$. The behavior of the left near Armendariz ideal condition is investigated with respect to various constructions. It is shown that every left ideal of a near Armendariz ring is a near Armendariz left ideal. Examples are included to illustrate and delimit the theory.

Keywords : near Armendariz ideal; abelian ideal; IFP ideal.
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1 Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring. The polynomial ring with an indeterminate x over R and the n-by-n upper triangular matrix ring over R are denoted by $R[x]$ and $U_n(R)$, respectively. In [1], M. B. Rege and S. Chhawchharia introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i and j. (The converse is always true). The term of an Armendariz ring was chosen because E. Armendariz [2, lemma 1] had noted that a reduced ring satisfies this condition. In [3], Sh. Ghalandarzadeh et al. introduced the notion of an Armendariz ideal and abelian ideal. A left ideal I of R is called Armendariz

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if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in r_{R[x]}(I[x])$ we have $a_i b_j \in r_R(I)$ for all i and j. Also a left ideal I of R is called abelian if for each idempotent element $e \in R, er - re \in r_R(I)$ for any $r \in R$. Over a reduced ring R, G. F. Birkenmeier, [4, lemma 3.4], proved that $g_j f_i = 0$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(fo)g = 0$ whenever $(x)fo(x)g = 0$ where $(x)f = \sum_{i=0}^{m} f_i x^i$, $(x)g = \sum_{j=0}^{n} g_j x^j \in R[x]$. Due to Ghalandarzadeh et al. [5], such rings (possibly not reduced) that satisfy Birkenmeier's result, are called near Armendariz. The binary operation of substitution, denoted by o, of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for $(x)f$, $(x)g \in R[x]$, with $(x)g = \sum_{j=0}^{n} g_j x^j$, $(x)fo(x)g = \sum_{j=0}^{n} g_j((x)f)^j$; However, the operation "o" left distributes but does not right distribute over addition. Thus $(\mathrm{R[x]}, +, o)$ forms a left nearring but not a ring. Henceforth, unless indicated otherwise, $R[x]$ denotes the nearing of polynomials $(\mathrm{R[x]}, +, o)$ and $R_0[x]$ the subnearring of polynomials with zero constant term. In this paper we study near Armendariz ideals; this concept is related to that of near Armendariz rings.

2 On Near Armendariz Ideals

In this section we define and study near Armendariz (one-sided) ideals. All our left-sided concepts and results have right-sided counterparts. The right annihilator of a subset A of a ring R is denoted by $r_R(A)$ or $r(A)$ (when R is clear from the contex). We begin with the following definition.

Definition 2.1. Let R be a ring.
A left ideal I of R is called *near Armendariz* if whenever polynomials $(x)f = \sum_{i=0}^{m} f_i x^i$, $(x)g = \sum_{j=0}^{n} g_j x^j \in R[x]$ satisfy $(x)fo(x)g \in r_{R[x]}(I[x])$ then $g_j f_i \in r_R(I)$ for each $1 \leq i \leq m$, $1 \leq j \leq n$ and $(f_0)g \in r_R(I)$.

For any $(x)f \in R[x]$, we denote by C_f the set of all coefficients of $(x)f$.

Proposition 2.2. *Let R be a ring and I be a near Armendariz left ideal of R. If $(x)f_1, (x)f_2, \ldots, (x)f_n \in R[x]$ are such that $(x)f_1 o \ldots o(x)f_n \in r_{R[x]}(I[x])$, then if $a_k \in C_{f_k}$ for $k \in \{1, \ldots, n\}$, we have $a_n a_{n-1} \ldots a_1 \in r_R(I)$*

Proof. We use induction on n. The case $n = 2$ is clear by definition of near Armendariz ideal. Suppose $n > 2$. Consider $(x)h = (x)f_1 o \ldots o(x)f_{n-1}$. Then $(x)ho(x)f_n \in r_{R[x]}(I[x])$ and hence, since I is a near Armendariz left ideal of R, $a_n a_h \in r_R(I)$ where $a_h \in C_{f_1 o \ldots o f_{n-1}}$ and $a_n \in C_{f_n}$. It follows $a_n((x)f_1 o \ldots o(x)f_{n-1}) = a_n(x)h \in r_{R[x]}(I[x])$. So $a_n(((x)f_1 o \ldots o(x)f_{n-2})o(x)f_{n-1}) \in r_{R[x]}(I[x])$. Then $a_n a_{n-1} C_{f_1 o \ldots o f_{n-2}} \in r_R(I)$ where $a_{n-1} \in C_{f_{n-1}}$. Therefore similarly by induction, we obtain $a_n a_{n-1} \ldots a_1 \in r_R(I)$ for $a_k \in C_{f_k}, k \in \{1, \ldots, n\}$. ☐

Next, we show that every near Armendariz left ideal is an abelian left ideal.

Proposition 2.3. *If I is a near Armendariz left ideal of R, then I is an abelian left ideal.*

Proof. Let I is a near Armendariz left ideal of R and $e^2 = e \in R$. Consider $(x)f = er(1-e)x, (x)g = er(1-e)x + ex^2 \in R[x]$ for any $r \in R$. Then $(x)fo(x)g \in r_{R[x]}(I[x])$. By hypothesis $er(1-e) \in r_R(I)$ for any $r \in R$. Similarly, Consider $(x)t = (1-e)rex, (x)h = (1-e)rex + (1-e)x^2 \in R[x]$ for any $r \in R$. Then $(x)to(x)h \in r_{R[x]}(I[x])$. As before $(1-e)re \in r_R(I)$ for any $r \in R$. Since $r_R(I)$ is an ideal, $er - re \in r_R(I)$. Then I is an abelian left ideal. □

The following example shows that the converse of proposition 2.3, does not hold.

Example 2.4. Let S an abelian ring and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \Big| a, b, c, d \in S \right\}.$$

Notice that R has trivial idempotent.Thus R is abelian. Next let

$$I = \begin{pmatrix} 0 & S & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then I is an abelian left ideal, because $er - re \in r_R(I)$. Consider

$$(x)f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x, \quad (x)g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x^2$$

in $R[x]$. Then $(x)fo(x)g \in r_{R[x]}(I[x])$, but

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin r_R(I).$$

So I is not near Armendariz left ideal.

We know that any subring of a near Armendariz ring is also near Armendariz. In the following theorem we prove that any left ideal of R is a near Armendariz left ideal provided that R is near Armendariz.

Proposition 2.5. *Let R be a near Armendariz ring and I be an ideal of R. Then $R/r_R(I)$ is a near Armendariz ring.*

Proof. We denote $\bar{R} = R/r_R(I)$ and let $\bar{f} = f + r_R(I)$ for $f \in R$. Suppose that $(x)\bar{f} = \bar{f}_0 + \bar{f}_1 x + \ldots + \bar{f}_n x^n$, $(x)\bar{g} = \bar{g}_0 + \bar{g}_1 x + \ldots + \bar{g}_n x^n \in \bar{R}[x]$. Then $(x)\bar{f}o(x)\bar{g} = 0$ Thus, $0 = (x)\bar{f}o(x)\bar{g} = (\bar{g}_0 + \bar{g}_1\bar{f}_0 + \ldots + \bar{g}_n\bar{f}_0^n) + \sum_{p=1}^{n^2} \left(\sum_{j=[\frac{p}{n}]}^{n} \bar{g}_j \bar{c}_p^{(j)} \right) x^p$ where $\bar{c}_p^{(j)} = \sum_{u_1+\ldots+u_j=p} \bar{f}_{u_1}\bar{f}_{u_2} \ldots \bar{f}_{u_j}$ for $p \in \{1, 2, \ldots, n^2\}$. Hence $(\bar{f}_0)\bar{g} = \bar{g}_n\bar{f}_0^n + \ldots +$

$\bar{g}_1\bar{f}_0+\bar{g}_0=0$. It follows $\left(\sum_{j=[\frac{p}{n}]}^{n} g_j c_p^{(j)}\right) \in r_R(I)$ and so $\left(\sum_{j=[\frac{p}{n}]}^{n} v g_j c_p^{(j)}\right)=0$ for all $v\in I$. This means that $(x)fov(x)g=0$ in $R[x]$. Since R is a near Armendariz ring, for $1\le i,j\le n$, $vg_jf_i=0$ for all $v\in I$. So $\bar{g}_j\bar{f}_i=0$ for $1\le i,j\le n$. This proves the proposition. □

Theorem 2.6. *If R is a near Armendariz ring, then each left ideal of R is a near Armendariz left ideal.*

Proof. Let R be a near Armendariz ring and I be a left ideal of R. Suppose $(x)f=\sum_{i=0}^{m} f_i x^i$, $(x)g=\sum_{j=0}^{n} g_j x^j$ are elements of $R[x]$ such that $(x)fo(x)g\in r_{R[x]}(I[x])$. Next consider $\bar{R}=R/r_R(I)$. Suppose that $(x)\bar{f}=\bar{f}_0+\bar{f}_1x+\ldots+\bar{f}_nx^n$, $(x)\bar{g}=\bar{g}_0+\bar{g}_1x+\ldots+\bar{g}_nx^n\in\bar{R}[x]$ such that $(x)\bar{f}o(x)\bar{g}=0$. Since $R/r_R(I)$ is near Armendariz by proposition 2.5, we obtain $\left(\sum_{j=[\frac{p}{n}]}^{n} g_j c_p^{(j)}\right)\in r_R(I)$ and hence $\left(\sum_{j=[\frac{p}{n}]}^{n} v g_j c_p^{(j)}\right)=0$ for all $v\in I$. This means that $(x)fov(x)g=0$ in $R[x]$. Since R is a near Armendariz ring, for $1\le i,j\le n$, $vg_jf_i=0$ for all $v\in I$. Then $g_jf_i\in r_R(I)$. Also from hypothesis $(x)\bar{f}o(x)\bar{g}=0$, it follows $(\bar{f}_0)\bar{g}=0$ so $(f_0)g\in r_R(I)$. □

It is obvious that the converse of Theorem 2.6, is true, because R is an ideal of R and $r_R(R)=0$.

Let R be a ring. The trivial extension of R is defined to be the ring $T(R,R)=R\oplus R$ with the usual addition and the multiplication $(r_1,r_2)(r_1',r_2')=(r_1r_1',r_1r_2'+r_2r_1')$. This is isomorphic to the ring of all matrices $\binom{r\ r'}{0\ r}$, where $r,r'\in R$. Next we give an example of a nonzero near Armendariz left ideal of a non-near Armendariz ring.

Example 2.7. Let $S=\mathbb{Z}_8$ and $R=T(S,S)$. Then R is not near Armendariz because $(x)f=(\overline{4},\overline{2})x$, $(x)g=(\overline{2},0)x^2\in R[x]$. Then $(x)fo(x)g=0$, but $(\overline{2},0)(\overline{4},\overline{2})\neq 0$. Write $a=\begin{pmatrix}0 & \overline{2}\\ 0 & 0\end{pmatrix}$, $I=Ra$ and $r(I)=r_R(I)$, then

$$r(I)=\left\{\begin{pmatrix}r & b\\ 0 & r\end{pmatrix} \mid r\in\{0,\overline{2}\}, b\in\mathbb{Z}_8\right\}.$$

Since $r(I)$ is an ideal of R and $R/r(I)$ is a reduced ring, I is a near Armendariz left ideal of R.

Recall that a one-sided ideal I of a ring R has the insertion of factors property (or simply, IFP) if $ab\in I$ implies $aRb\subseteq I$ for $a,b\in R$.

By the following example, we show that I has the IFP, but I is not near Armendariz ideal.

Example 2.8. Let D be a domain, $R=U_2(D)$ and $I=\begin{pmatrix}D & D\\ 0 & 0\end{pmatrix}$. Let

$$0 \neq \alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, 0 \neq \beta = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R.$$

It is easily shown that $\alpha\beta \in I$ if and only if $c = 0$. Thus if $c = 0$ then $\alpha R\beta \subseteq I$. This implies that I has the IFP. Next consider

$$(x)f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x, \quad (x)g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x^2 \in R[x]$$

such that $(x)fo(x)g \in r_{R[x]}(I[x])$, but

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin r_R(I).$$

Thus I is not a near Armendariz left ideal.

Proposition 2.9. *Let R be a ring and let I be a near Armendariz left ideal of R, then $r_R(I)$ has the IFP.*

Proof. let $ab \in r_R(I)$ with $a, b, r \in R$. Let $(x)f = bx + r$, $(x)g = ax^2 - arx$ be polynomials $\in R[x]$. Then $(x)fo(x)g \in r_{R[x]}(I[x])$. Since the left ideal I of R is near Armendariz, $arb \in r_R(I)$ for all $r \in R$. □

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