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Common Coupled Coincidence and Coupled Fixed Point of C-Contractive Mappings in Generalized Metric Spaces

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Abstract : In this paper, study of necessary conditions for existence of common coupled coincidence and coupled fixed point results for C-contractive type mappings in the context of generalized metric space equipped with a partial order is initiated. These results generalize comparable results from the current literature. We also provide illustrative example in support of our new results.

Keywords : generalized metric spaces; coupled fixed point; ordered metric spaces.
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1 Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity [1–3]. Mustafa and Sims [4] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [5–7] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [8] initiated

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the study of common fixed point theory in generalized metric spaces (see also [9, 10]). While Gajić and Crvenković [11, 12] initiated the study of fixed point results for mappings with contractive iterate at a point in G-metric spaces. Recently, many mathematicians have considered fixed point and common fixed point problem in generalized metric spaces (see, e.g., [13–17]). The existence of fixed points in partially ordered metric spaces has been investigated in 2004 by Ran and Reurings [18], and then further results in this direction were proved (see [1, 19, 20]). Results on weak contractive mappings on such spaces, together with applications to differential equations, were obtained by Harjani and Sadarangani in [21].

Bhashkar and Lakshmikantham in [22] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed applications of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Afterwards, Lakshmikantham and Ćirić [2] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered complete metric spaces. Then, later [23] and [24] obtained interesting results in this direction. Abbas et al. [25] have proved coupled coincidence and coupled common fixed point results in cone metric spaces for $w-$ compatible mappings.

Very recently, Cho et al [26] obtained some coupled fixed point results in generalized metric spaces (see also, [17, 27–32] and references therein). Recently, Harjani et al. [33] obtained some fixed point theorems for weakly $C-$contractive mappings in ordered metric spaces.

The aim of this paper is to prove some common coupled coincidence and coupled fixed points results for $C-$contractive mappings defined on a partial ordered set equipped with a generalized metric. Our results extend and unify various comparable results.

Consistent with Mustafa and Sims [4], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;

(b) $0 < G(x, y, z)$ for all $x, y \in X$, with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables); and

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a $G-$*metric* on X and (X, G) is called a $G-$ *metric space*.

Definition 1.2. A sequence $\{x_n\}$ in a $G-$metric space X is:

(i) a $G-$*Cauchy sequence* if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,

(ii) a $G-$*convergent sequence* if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A $G-$metric space on X is said to be $G-$complete if every $G-$Cauchy sequence in X is $G-$convergent in X. It is known that $\{x_n\}$ $G-$converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$ [4].

Proposition 1.3 ([4]). *Let X be a $G-$metric space. Then the following are equivalent:*

1. $\{x_n\}$ *is $G-$convergent to x.*
2. $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$.
3. $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
4. $G(x_n, x_m, x) \to 0$ *as* $n,\ m \to \infty$.

Proposition 1.4. *A $G-$metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Proposition 1.5. *Every $G-$metric on X will define a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x),\ \forall\ x, y \in X. \tag{1.1}$$

For a symmetric $G-$metric

$$d_G(x, y) = 2G(x, y, y),\ \forall\ x, y \in X. \tag{1.2}$$

However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y),\ \forall\ x, y \in X. \tag{1.3}$$

Recall that if $(X, \leq)$ is a partially ordered set and $f : X \to X$ is such that for $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$, then a mapping f is said to be nondecreasing. Similarly, a nonincreasing mapping is defined.

Definition 1.6 ([22]). An element $(x, y) \in X \times X$ is called a *coupled fixed point* of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.7 ([13]). An element $(x, y) \in X \times X$ is called:

(c_1) a *coupled coincidence point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called *coupled point of coincidence.*

(c_2) a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.8 ([2]). Let $(X, \leq)$ be a partially ordered set. A map $F : X\times X \to X$ is said to has a *g-mixed monotone property* where $g : X \to X$ if for $x_1, x_2, y_1, y_2 \in X$

$$gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y) \text{ for all } y \in X$$

and

$$gy_1 \leq gy_2 \text{ implies } F(x, y_2) \leq F(x, y_1) \text{ for all } x \in X.$$

If we take $g = I_X$ (an identity mapping on X), then F is said to has the *mixed monotone property* ([22]).

2 Main Results

We obtain common coupled coincidence and coupled fixed points results for C-contractive mappings defined on a partial ordered set equipped with generalized metric space. We also extend some recent results of Choudhury and Maity [34] for two maps in generalized metric space.

We start with following result.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G- metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be continuous mappings such that F has the mixed g-monotone property, g commutes with F and $F(X \times X) \subseteq g(X)$. Suppose that there exist a continuous function $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $\phi(t, s) = 0$ if and only if $t = s = 0$ such that*

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(gx, gu, gw), G(gy, gv, gz)\} \\ - \phi(G(gx, gu, gw), G(gy, gv, gz)) \end{aligned} \tag{2.1}$$

for all $x, y, u, v, w, z \in X$ with $gw \leq gu \leq gx$ and $gy \leq gv \leq gz$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then F and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Similarly we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \leq gx_1 \leq gx_2$ and $gy_2 \leq gy_1 \leq gy_0$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_n = F(x_{n-1}, y_{n-1}) \leq gx_{n+1} = F(x_n, y_n)$$

and

$$gy_{n+1} = F(y_n, x_n) \leq gy_n = F(y_{n-1}, x_{n-1}).$$

If for some integer k, we have $(gx_{k+1}, gy_{k+1}) = (gx_k, gy_k)$, then $F(x_k, y_k) = gx_k$ and $F(y_k, x_k) = gy_k$, therefore (x_k, y_k) is a coincidence point of F and g. So, we assume that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \in \mathbb{N}$, that is, either $gx_{n+1} \neq gx_n$ or $gy_{n+1} \neq gy_n$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} G(gx_{n+1}, gx_{n+1}, gx_n) &= G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\} \\ &\qquad - \phi(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})) \\ &\leq \max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\}. \end{aligned} \tag{2.2}$$

On other hand,

$$\begin{aligned} G(gy_n, gy_{n+1}, gy_{n+1}) &= G(F(y_{n-1}, x_{n-1}), F(y_n.x_n), F(y_n, x_n)) \\ &\leq \max\{G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n)\} \\ &\qquad - \phi(G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n)) \\ &\leq \max\{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n)\}. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} \max\{&G(gx_{n+1},gx_{n+1}, gx_n), G(gy_n, gy_{n+1}, gy_{n+1})\} \\ &\leq \max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\} \\ &\qquad - \min\{\phi(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}), \\ &\qquad \phi(G(gy_{n-1}, gy_n, gy_n), G(gx_{n-1}, gx_n, gx_n))\} \\ &\leq \max\{G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})\}. \end{aligned} \tag{2.4}$$

Thus $\{\max\{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n)\}\}$ is a nonnegative decreasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \max\{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n)\} = r.$$

On taking limit as $n \to \infty$ in (2.4), we get

$$\begin{aligned} r &\leq r - \min\{\lim_{n\to\infty} \phi(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})), \\ &\qquad \lim_{n\to\infty} \phi(G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n))\} \\ &\leq r. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \phi(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1})) = 0.$$

By using the properties of ϕ, we have

$$\lim_{n\to\infty} G(gx_n, gx_n, gx_{n-1}) = 0$$

and

$$\lim_{n\to\infty} G(gy_{n-1}, gy_n, gy_n) = 0.$$

Therefore, $r = 0$ and hence

$$\lim_{n\to\infty} \max\{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n)\} = 0. \tag{2.5}$$

Now we shill show that $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy sequences.

Assume on Contrary that $\{gx_n\}$ or $\{gy_n\}$ is not a G-Cauchy sequence, that is

$$\lim_{n,m\to\infty} G(gx_m, gx_n, gx_n) \neq 0$$

or

$$\lim_{n,m\to\infty} G(gy_m, gy_n, gy_n) \neq 0.$$

This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers m_k and n_k with $n_k > m_k > k$ such that

$$\max\{G(gx_{n_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\} \geq \varepsilon. \tag{2.6}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ which satisfy (2.6). Then

$$\max\{G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{n_k}, gy_{n_k-1}, gy_{n_k-1})\} < \varepsilon. \tag{2.7}$$

By using the proper (e) of generalized metric and (2.7), we have

$$\begin{aligned}
&G(gx_{n_k}, gx_{n_k}, gx_{n_k})\\
&\quad \leq G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k})\\
&\quad \leq G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})\\
&\qquad + G(gx_{m_k-1}, gx_{n_k}, gx_{n_k})\\
&\quad \leq 2G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})\\
&\qquad + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k})\\
&\quad < 2G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + \varepsilon + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}),
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&G(gy_{m_k}, gy_{n_k}, g_{n_k})\\
&\quad \leq G(gy_{m_k}, gy_{n_k-1}, gy_{n_k-1}) + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\\
&\quad \leq G(gy_{m_k}, gy_{m_k-1}, gy_{m_k-1}) + G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})\\
&\qquad + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\\
&\quad \leq 2G(gy_{m_k}, gy_{m_k}, gy_{m_k-1}) + G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})\\
&\qquad + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\\
&\quad < 2G(gy_{m_k}, gy_{m_k}, gy_{m_k-1}) + \varepsilon + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k}).
\end{aligned} \tag{2.9}$$

By (2.6)-(2.9), we have

$$\begin{aligned}\varepsilon &\le \max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\}\\ &\le 2\max\{G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}), G(gy_{m_k}, gy_{m_k}, gy_{m_k-1})\}\\ &\quad + \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})\}\\ &\quad + \max\{G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}), G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\}\\ &\le 2\max\{G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}), G(gy_{m_k}, gy_{m_k}, gy_{m_k-1})\} + \varepsilon\\ &\quad + \max\{G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}), G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\}.\end{aligned}$$

Letting $k \to \infty$ in above inequalities and using (2.5), we obtain

$$\begin{aligned}&\lim_{k\to\infty} \max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\}\\ &\quad = \lim_{k\to\infty} \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})\}\\ &\quad = \varepsilon. \qquad (2.10)\end{aligned}$$

Since $gx_{n_k-1} \ge gx_{n_k-1} \ge gx_{m_k-1}$ and $gy_{n_k-1} \le gy_{n_k-1} \le gy_{m_k-1}$, by (2.1) we have

$$\begin{aligned}&G(gx_{n_k}, gx_{n_k}, gx_{m_k})\\ &\quad = G(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1}), F(x_{m_k-1}, y_{m_k-1}))\\ &\quad \le \max\{G(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{n_k-1}, gy_{n_k-1}, gy_{m_k-1})\}\\ &\qquad - \phi(G(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{n_k-1}, gy_{n_k-1}, gy_{m_k-1})) \qquad (2.11)\end{aligned}$$

and

$$\begin{aligned}&G(gy_{m_k}, gy_{n_k}, gy_{n_k})\\ &\quad = G(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k-1}, x_{n_k-1}))\\ &\quad \le \max\{G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})\}\\ &\qquad - \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})). \qquad (2.12)\end{aligned}$$

By (2.11) and (2.12), we get

$$\begin{aligned}&\max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\}\\ &\quad \le \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})\}\\ &\qquad - \min\{\phi(G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})),\\ &\qquad \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}))\}\\ &\quad \le \max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\}.\end{aligned}$$

On taking limit as $k \to \infty$ in the above inequalities and using (2.10), we have

$$\begin{aligned}\varepsilon &\le \varepsilon - \min\{\lim_{k\to\infty} \phi(G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1})),\\ &\qquad \lim_{k\to\infty} \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}))\}\\ &\le \varepsilon.\end{aligned}$$

Hence

$$\lim_{k\to\infty}\phi(G(gx_{m_k-1},gx_{n_k-1},gx_{n_k-1}),G(gy_{m_k-1},gy_{n_k-1},gy_{n_k-1}))=0$$

or

$$\lim_{k\to\infty}\phi(G(gy_{m_k-1},gy_{n_k-1},gy_{n_k-1}),G(gx_{m_k-1},gx_{n_k-1},gx_{n_k-1}))=0.$$

It now follows that

$$\lim_{k\to\infty}G(gx_{m_k-1},gx_{n_k-1},gx_{n_k-1})=0$$

By (2.10), we obtain that $\varepsilon=0$, a contradiction. Therefore $\{gx_n\}$ and $\{gy_n\}$ are both G-Cauchy sequences in X. Since (X,G) is G-complete, there are $x,y\in X$ such that $\{gx_n\}$ and $\{gy_n\}$ are G-convergent to x and y respectively, that is,

$$\lim_{n\to\infty}G(gx_n,gx_n,x)=\lim_{n\to\infty}G(gx_n,x,x)=0 \tag{2.13}$$

and

$$\lim_{n\to\infty}G(gy_n,gy_n,y)=\lim_{n\to\infty}G(gy_n,y,y)=0. \tag{2.14}$$

Using (2.13), (2.14) and the continuity of g, we have

$$\lim_{n\to\infty}G(g(gx_n),g(gx_n),gx)=\lim_{n\to\infty}G(g(gx_n),gx,gx)=0 \tag{2.15}$$

and

$$\lim_{n\to\infty}G(g(gy_n),g(gy_n),gy)=\lim_{n\to\infty}G(g(gy_n),gy,gy)=0. \tag{2.16}$$

Therefore $\{g(gx_n)\}$ is G-convergent to gx and $\{g(gy_n)\}$ is G-convergent to gy. Since F and g commute, we get

$$g(gx_{n+1})=g(F(x_n,y_n))=F(gx_n,gy_n) \tag{2.17}$$

and

$$g(gy_{n+1})=g(F(y_n,x_n))=F(gy_n,gx_n). \tag{2.18}$$

As F is continuous, so taking limit as $n\to\infty$ in (2.17) and (2.18) implies that $gx=F(x,y)$ and $gy=F(y,x)$. That is, (gx,gy) is a coupled coincidence point of F and g. □

If we take $u=w$ and $v=z$ in Theorem 2.1, then we obtain the following corollary.

Corollary 2.2. *Let $(X,\leq)$ be a partially ordered set such that there exists a complete G-metric space on X. Let $F:X\times X\to X$ and $g:X\to X$ be continuous mappings such that F has the mixed g-monotone property, g commutes with F and $F(X\times X)\subseteq g(X)$.. Suppose that there exist a continuous function $\phi:[0,\infty)\times[0,\infty)\to[0,\infty)$ with $\phi(t,s)=0$ if and only if $t=s=0$ such that*

$$\begin{aligned}G(F(x,y),F(u,v),F(u,v))\leq{}&\max\{G(gx,gu,gu),G(gy,gv,gv)\}\\&-\phi(G(gx,gu,gu),G(gy,gv,gv))\end{aligned} \tag{2.19}$$

for all $x, y, u, v \in X$ *with* $gw \leq gu$ *and* $gy \leq gv$*. If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq gy_0$*, then* F *and* g *have a coupled coincidence point.*

If we take $g = I_X$ (the identity mapping) in Theorem 2.1, we obtain the following coupled fixed point result.

Corollary 2.3. *Let* $(X, \leq)$ *be a partially ordered set such that there exists a complete* G*-metric space on* X*. Let* $F : X \times X \to X$ *be a continuous mapping satisfying the mixed monotone property. Suppose that there exist a continuous function* $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ *with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$ *such that*

$$G\left(F(x,y), F(u,v), F(w,z)\right) \leq \max\{G(x,u,w), G(y,v,z)\} - \phi\left(G(x,u,w), G(y,v,z)\right) \tag{2.20}$$

for all $x, y, u, v, w, z \in X$ *with* $w \leq u \leq x$ *and* $y \leq v \leq z$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq y_0$*, then* F *has a coupled fixed point.*

Corollary 2.4. *Let* $(X, \leq)$ *be a partially ordered set such that there exists a complete* G*-metric space on* X*. Let* $F : X \times X \to X$ *be a continuous mapping satisfying the mixed monotone property. Suppose that there exist a continuous function* $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ *with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$ *such that*

$$G\left(F(x,y), F(u,v), F(w,z)\right) \leq \frac{1}{2}(G(x,u,w) + G(y,v,z)) - \phi\left(G(x,u,w), G(y,v,z)\right) \tag{2.21}$$

for all $x, y, u, v, w, z \in X$ *with* $w \leq u \leq x$ *and* $y \leq v \leq z$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq y_0$*, then* F *has a coupled fixed point.*

Proof. Follows from Corollary 2.3 by noting that

$$\frac{1}{2}(G(x,u,w) + G(y,v,z)) \leq \max\{G(x,u,w), G(y,v,z)\}. \tag{2.22}$$

□

In our next result, we drop the continuity of F.

Theorem 2.5. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *such that there exists a complete* G*-metric space on* X*. Suppose that there exist a continuous function* $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ *with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$ *such that*

$$G(F(x,y), F(u,v), F(w,z)) \leq \max\{G(gx,gu,gw), G(gy,gv,gz)\} - \phi(G(gx,gu,gw), G(gy,gv,gz)) \tag{2.23}$$

for all $x, y, u, v, w, z \in X$ *with* $gw \leq gu \leq gx$ *and* $gy \leq gv \leq gz$*. Assume that* X *satisfies:*

1. *if a non-decreasing sequence* $\{x_n\}$ *is such that* $x_n \to x$, *then* $x_n \leq x$ *for all* n,
2. *if a non-increasing sequence* $\{y_n\}$ *is such that* $y_n \to y$, *then* $y \leq y_n$ *for all* n.

Suppose also that $(g(X), G)$ *is* G*-complete,* F *has the mixed* g*-monotone property and* $F(X \times X) \subseteq g(X)$. *If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq gy_0$, *then* F *and* g *have a coupled coincidence point.*

Proof. Following the proof of Theorem 2.1, we construct two G-Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in $g(X)$ with

$$gx_n \leq gx_{n+1} \text{ and } gy_n \geq gy_{n+1}$$

for all $n \in \mathbb{N}$. Since $(g(X), G)$ is G-complete, then there are $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$ as $n \to \infty$. By the properties of X, we have $gx_n \leq gx$ and $gy \leq gy_n$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} G(F(x,y), gx_{n+1}, gx_{n+1}) &= G(F(x,y), F(x_n, y_n), F(gx_n, gy_n)) \\ &\leq \max\{G(gx, gx_n, gx_n), G(gy, gy_n, gy_n)\} \\ &\quad - \phi(G(gx, gx_n, gx_n), G(gy, gy_{n+1}, gy_n)). \end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality and using the continuity of ϕ, we obtain $G(F(x,y), gx, gx) = 0$, which implies that $F(x,y) = gx$. Similarly, one can show that $F(y,x) = gy$. Thus (x,y) is a coupled coincidence point of F and g. □

If we take $u = w$ and $v = z$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let* $(X, \leq)$ *be a partially ordered set such that there exists a complete* G*-metric space on* X. *Suppose that there exist a continuous function* $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ *with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$ *such that*

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) &\leq \max\{G(gx, gu, gw), G(gy, gv, gz)\} \\ &\quad - \phi(G(gx, gu, gw), G(gy, gv, gz)) \end{aligned} \tag{2.24}$$

for all $x, y, u, v, w, z \in X$ *with* $gw \leq gu \leq gx$ *and* $gy \leq gv \leq gz$. *Assume that* X *satisfies:*

1. *if a non-decreasing sequence* $\{x_n\}$ *is such that* $x_n \to x$, *then* $x_n \leq x$ *for all* n,
2. *if a non-increasing sequence* $\{y_n\}$ *is such that* $y_n \to y$, *then* $y \leq y_n$ *for all* n.

Suppose also that $(g(X), G)$ *is* G*-complete,* F *has the mixed* g*-monotone property and* $F(X \times X) \subseteq g(X)$. *If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0)$ *and* $F(y_0, x_0) \leq gy_0$, *then* F *and* g *have a coupled coincidence point.*

If we take $g = I_X$ (identity map) in Theorem 2.5, we obtain the following result.

Corollary 2.7. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G-metric space on X. Suppose that there exist a continuous function $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $\phi(t, s) = 0$ if and only if $t = s = 0$ such that*

$$G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(x, u, w), G(y, v, z)\} - \phi(G(x, u, w), G(y, v, z)) \quad (2.25)$$

for all $x, y, u, v, w, z \in X$ with $w \leq u \leq x$ and $y \leq v \leq z$. Assume that X satisfies:

1. *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all n.*

Suppose F has the mixed monotone property. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point.

Corollary 2.8. *Let $(X, \leq)$ be a partially ordered set such that there exists a complete G-metric space on X. Suppose that there exist a continuous function $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ with $\phi(t, s) = 0$ if and only if $t = s = 0$ such that*

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{1}{2}(G(x, u, w) + G(y, v, z)) - \phi(G(x, u, w), G(y, v, z)) \quad (2.26)$$

for all $x, y, u, v, w, z \in X$ with $w \leq u \leq x$ and $y \leq v \leq z$. Assume that X satisfies:

1. *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all n.*

Suppose F has the mixed monotone property. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point.

Proof. Since

$$\frac{1}{2}(G(x, u, w) + G(y, v, z)) \leq \max\{G(x, u, w), G(y, v, z)\}. \quad (2.27)$$

So that the result follows from Corollary 2.7. □

Remark 2.9.

1) [34, Theorem 3.1] *is a special case of Corollary 2.4 (by taking $\phi(t, s) = (\frac{1}{2} - \frac{1}{k})(s + t)$.*

2) [34, Theorem 3.2] *is a special case of Corollary 2.8 (by taking* $\phi(t,s) = (\frac{1}{2} - \frac{1}{k})(s+t)$.

Example 2.10. Let $X = [0,1]$ be partially ordered set with the natural ordering of real numbers and

$$G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$$

be a complete $G-$metric on X. Let $F : X \times X \to X$ and $g : X \to X$ be defined by

$$F(x,y) = \begin{cases} \dfrac{x^2-y^2}{4}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases} \qquad g(x) = \frac{3}{4}x^2$$

and $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ be given by

$$\phi(s,t) = \frac{1}{10}(s+t), \text{ for } s,t \in [0,\infty).$$

Notice that $F(X \times X)$ is contained in the set $g(X)$.
Now for $g(x) \leq g(u)$ and $g(y) \geq g(v)$,

$$\begin{aligned}
&G(F(x,y), F(u,v), F(u,v)) \\
&\qquad = \frac{1}{4}\left|(x^2-y^2) - (u^2-v^2)\right| \\
&\qquad = \frac{1}{4}\left|(x^2-u^2) - (y^2-v^2)\right| \\
&\qquad \leq \frac{3}{10}\left[\left|x^2-u^2\right| + \left|y^2-v^2\right|\right] \\
&\qquad = \frac{3}{4}\left[\frac{\left|x^2-u^2\right| + \left|y^2-v^2\right|}{2}\right] - \frac{1}{10}\left(\frac{3}{4}\left|x^2-u^2\right| + \frac{3}{4}\left|y^2-v^2\right|\right) \\
&\qquad \leq \max\left\{\frac{3}{4}\left|x^2-u^2\right|, \frac{3}{4}\left|y^2-v^2\right|\right\} - \phi\left(\frac{3}{4}\left|x^2-u^2\right|, \frac{3}{4}\left|y^2-v^2\right|\right) \\
&\qquad = \max\left\{G\left(\frac{3}{4}x^2, \frac{3}{4}u^2, \frac{3}{4}u^2\right), G\left(\frac{3}{4}y^2, \frac{3}{4}v^2, \frac{3}{4}v^2\right)\right\} \\
&\qquad\quad - \phi\left(G\left(\frac{3}{4}x^2, \frac{3}{4}u^2, \frac{3}{4}u^2\right), G\left(\frac{3}{4}y^2, \frac{3}{4}v^2, \frac{3}{4}v^2\right)\right) \\
&\qquad = \max\{G(gx,gu,gu), G(gy,gv,gv)\} - \phi(G(gx,gu,gu), G(gy,gv,gv)).
\end{aligned}$$

Thus mappings F, g and ϕ satisfy all the conditions of Corollary 2.2. Moreover $(0,0)$ is a coupled coincidence point of F and g.

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