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On the Diamond Operator Related Nonlinear Beam Equation

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Abstract : In this paper, we study the nonlinear equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(-\diamondsuit)^k u(x,t) = f(x,t,u(x,t)),$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where $u(x,t) \in \mathbb{R}^n \times (0,\infty), \mathbb{R}^n$ is the $n-$ dimensional Euclidean space and $\diamondsuit^k$ is the Diamond operator iterated k times and is defined by (1.1). By ϵ approximation we also obtain the asymptotic solution for such equations. Moreover, if we put $n=1, p=0, q=1$ and $k=1$ we obtain the asymptotic solution of the nonlinear beam equation.

Keywords : diamond operator; beam equation; Furier transform; temperod distribution.

2010 Mathematics Subject Classification : 46F10; 46F12.

1 Introduction

In 1996, A. Kananthai [1] first introduced the operator $\diamondsuit^k$ and is named Diamond operator and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k . \tag{1.1}$$

The operator $\diamondsuit^k$ can be written as the product of the operators in the form

$$\diamondsuit^k = \triangle^k \square^k = \square^k \triangle^k, \tag{1.2}$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k , \tag{1.3}$$

and $\square^k$ is the ultra-hyperbolic operator iterated $k-$ times and is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k . \tag{1.4}$$

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \tag{1.5}$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation where f and g are continuous.

Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) + c^2(-\triangle) u(x,t) = 0, \tag{1.6}$$

with the initial condition $u(x,0) = f(x)$ and $\frac{\partial}{\partial t} u(x,0) = g(x)$ where $\triangle$ is defined by (1.3) with $k = 1$, f and g are given continuous functions. By solving the Cauchy problem for such an equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t) = \widehat{f}(\xi) \cos(2\pi|\xi|)\, t + \widehat{g}(\xi) \frac{\sin(2\pi|\xi|)\, t}{2\pi|\xi|},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ [see [2], p177]. By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{1.7}$$

where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi) = \dfrac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos(2\pi|\xi|)\,t = \dfrac{\partial}{\partial t}\widehat{\Phi}(\xi)$.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\left(-\Diamond\right)^k u(x,t) = f(x,t,u(x,t)), \tag{1.8}$$

with

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where the operator $\Diamond^k$ is defined by (1.1), c is a positive constant, k is a non-negative integer, f and g are continuous functions and absolutely integrable. The equation (1.8) is motivated by replacing the $\triangle$ by $\Diamond$ in (1.6) and extend it to the nonlinear form. We consider (1.8) with the following conditions on u and f as follows:

(1) $u(x,t) \in C^{(4k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(4k)}(\mathbb{R}^n)$ is the space of continuous function with $4k$-derivatives.

(2) f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant with $0 < A < 1$.

(3) $\displaystyle\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\,dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

By $\epsilon-$ approximation and under such conditions of f and u, we obtain asymptotic solution of (1.8) in the convolution form

$$u(x,t) = O(\epsilon^{-\frac{n}{2k}}) * f(x,t,u(x,t)) \tag{1.9}$$

Moreover, if we put $k = 1,\ n = 1,\ p = 0$ and $q = 1$ in (1.8) then (1.8) reduces to the nonlinear beam equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = f(x,t,u(x,t)), \tag{1.10}$$

and also we obtain

$$u(x,t) = O(\epsilon^{-\frac{1}{2}}) * f(x,t,u(x,t)) \tag{1.11}$$

is the asymptotic solution of (1.10). We also study the boundness of $E(x,t)$ where $E(x,t)$ is defined by (2.10) in the Sobelev space. That is in (1.8) by setting the conditions $f(x) \in H_s(\mathbb{R}^n)$ and $g(x) \in H_{s-1}(\mathbb{R}^n)$ then $E(x,t) \in H_s(\mathbb{R}^n \times (0,\infty))$ where $H_s(\mathbb{R}^n)$ is the Sobelev space of order s and is defined by

$$H_k = H_k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n)\}$$

where k is a nonnegative integer and norm

$$\|f\|_k^2 = \int_{\mathbb{R}^n} |f(\xi)|^2(1+|\xi^2|)^k d\xi < \infty$$

$L^2(\mathbb{R}^n)$ is space of the square integrable in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i is a nonnegative integer and

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}} f(x).$$

Before going to that point, the following definitions and some concepts are needed.

2 Preliminaries

We shall need the following definitions

Definition 2.1. Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(x)dx. \tag{2.2}$$

If f is a distribution with compact support by Eq(2.1) can be written as [4]

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i(\xi,x)} \rangle \tag{2.3}$$

Definition 2.2. Let $t > 0$ and p is a real number

$f(t) = O(t^p)$ as $t \to 0 \Leftrightarrow t^{-p}|f(t)|$ is bounded as $t \to 0$

and $f(t) = o(t^p)$ as $t \to 0 \Leftrightarrow t^{-p}|f(t)| \to 0$ as $t \to 0$

Definition 2.3. Let $H_k = H_k(\mathbb{R}^n)$ be the space of the Sobelev space of order k on $\mathbb{R}^n$ and is defined by

$$H_k = H_k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n)\}$$

where k is a nonnegative integer and norm

$$\|f\|_k^2 = \int_{\mathbb{R}^n} |f(\xi)|^2(1+|\xi^2|)^k d\xi < \infty$$

$L^2(\mathbb{R}^n)$ is space of the square integrable in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, α_i is a nonnegative integer and

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} f(x)$$

Lemma 2.4. *Given the function*

$$f(x) = \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2}\,\right]$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad p+q = n, \ \sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2.$

We consider four cases
Case 1 : p odd and q even(n odd).
Case 2: p even and q odd(n odd).
Case 3 : p and q are both (n even), and $n \neq 4k, \quad k = 1, 2, 3, ...$
For case (1)-(3), we obtain

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p \Omega_q}{8} \cdot \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p}{4})\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})}$$

Case 4 : p and q are both even(n even), we obtain

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p \Omega_q}{8} \cdot \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p}{4})\Gamma(\frac{q}{4})}{(-1)^m \Gamma(\frac{q}{4} + m)}$$

where Γ *denotes the Gamma function. That is* $\int_{\mathbb{R}^n} f(x)dx$ *is bounded.*

Proof.

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{-\left(\sum_{i=1}^{p} x_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2}\,\right] dx$$

Let us transform to bipolar coordinates defined by

$$x_1 = r\omega_1, \quad x_2 = r\omega_2, \ldots, \quad x_p = r\omega_p$$

$$dx_1 = rd\omega_1, \quad dx_2 = rd\omega_2, \ldots, \quad dx_p = rd\omega_p$$

and

$$x_{p+1} = s\omega_{p+1}, \quad x_{p+2} = s\omega_{p+2}, \ldots, \quad x_{p+q} = s\omega_{p+q}$$

$$dx_{p+1} = sd\omega_{p+1}, \quad dx_{p+2} = sd\omega_{p+2}, \ldots, \quad dx_{p+q} = sd\omega_{p+q},$$

where $\omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1$.
Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^4 - r^4}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $dx = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively,

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \int_{\mathbb{R}^n} \exp\left[-\sqrt{s^4 - r^4}\right] r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q.$$

By computing directly, we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^4 - r^4}\right] r^{p-1}s^{q-1}drds,$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus

$$|\int_{\mathbb{R}^n} f(x)dx| \leq \Omega_p\Omega_q \int_0^\infty \int_0^s \exp\left[-\sqrt{s^4 - r^4}\right] r^{p-1}s^{q-1}drds.$$

Put $r^2 = s^2\sin\theta$, $2rdr = s^2\cos\theta d\theta$ and $0 \leq \theta \leq \dfrac{\pi}{2}$,

$$\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\leq \frac{\Omega_p\Omega_q}{2} \int_0^\infty \int_0^s e^{-\sqrt{s^4 - s^4\sin^2\theta}} s^{p-2}(\sin\theta)^{\frac{p-2}{2}} s^{q+1}\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{2} \int_0^\infty \int_0^s e^{-s^2\cos\theta} s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta ds. \qquad (2.4)
\end{aligned}$$

Put $y = s^2\cos\theta$, $ds = \dfrac{dy}{2s\cos\theta}$ into (2.4), we obtain

$$\begin{aligned}
|\int_{\mathbb{R}^n} f(x)dx| &\leq \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y}\left(\frac{y}{\cos\theta}\right)^{\frac{n-2}{2}} (\sin\theta)^{\frac{p-2}{2}} \cos\theta d\theta \frac{dy}{\cos\theta} \\
&= \frac{\Omega_p\Omega_q}{4} \int_0^{\pi/2} \int_0^\infty e^{-y} y^{\frac{n-2}{2}} (\cos\theta)^{\frac{2-n}{2}} (\sin\theta)^{\frac{p-2}{2}} dyd\theta \\
&= \frac{\Omega_p\Omega_q}{4}\Gamma\left(\frac{n}{2}\right) \int_0^{\pi/2} (\cos\theta)^{\frac{2-n}{2}} (\sin\theta)^{\frac{p-2}{2}} d\theta \\
&= \frac{\Omega_p\Omega_q}{8}\Gamma\left(\frac{n}{2}\right) \beta\left(\frac{p}{4}, \frac{4-n}{4}\right) \\
|\int_{\mathbb{R}^n} f(x)dx| &\leq \frac{\Omega_p\Omega_q}{8} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p}{4})\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})}. \qquad (2.5)
\end{aligned}$$

We consider the boundness of (2.5) in four cases :
Case 1: p odd and q even(n odd). If $q = 4$ then

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} = \frac{\Gamma(1-\frac{n}{4})}{\Gamma(0)} = 0. \tag{2.6}$$

Thus (2.5) is bounded.
Case 2 : p even and q odd(n odd).
In this case

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} \neq \infty \tag{2.7}$$

Thus (2.5) is bounded.
Case 3 : p and q are both odd(n even and $n \neq 4k$)
For $n \neq 4k, k = 1, 2, 3, \ldots$ Therefore

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} \neq \infty \tag{2.8}$$

Thus (2.5) is bounded.
Case 4 : p and q are both even(n even). In this case using the formula

$$\frac{\Gamma(z)}{\Gamma(z-m)} = \frac{(-1)^m\Gamma(-z+m+1)}{\Gamma(1-z)}, \quad m = 1, 2, 3, \ldots$$

We have

$$\begin{aligned} \frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} &= \frac{\Gamma(\frac{4-q}{4}-\frac{p}{4})}{\Gamma(\frac{4-q}{4})} \\ &= \frac{\Gamma(1-(\frac{4-q}{4}))}{(-1)^m\Gamma(-(\frac{4-q}{4})+m+1)} \\ &= \frac{\Gamma(\frac{q}{4})}{(-1)^m\Gamma(\frac{q}{4}+m)}. \end{aligned} \tag{2.9}$$

Putting (2.9) into (2.4), we obtain

$$\left|\int_{\mathbb{R}^n} f(x)dx\right| \leq \frac{\Omega_p\Omega_q}{8} \cdot \frac{\Gamma(\frac{n}{2})\Gamma(\frac{p}{4})\Gamma(\frac{q}{4})}{(-1)^m\Gamma(\frac{q}{4}+m)},$$

where Γ denotes the Gamma function. By (2.6)-(2.9) we conclude $\int_{\mathbb{R}^n} f(x)dx$ is bounded. □

Lemma 2.5. (The Fourier transform of $\Diamond^k\delta$)

$$\mathcal{F}\Diamond^k\delta = \frac{(-1)^{2k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2 - \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right]^k$$

where $\mathcal{F}$ is the Fourier transform defined by (2.1) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$ then

$$\mathcal{F}\diamondsuit^k\delta \leq \frac{M}{(2\pi)^{n/2}}\|\xi\|^{4k}.$$

Since M is constant thus $\mathcal{F}\diamondsuit^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by Eq.(2.2)

$$\diamondsuit^k\delta = \mathcal{F}^{-1}\frac{(-1)^{2k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^2\right]^k$$

Proof. By Eq. (2.3)

$$\begin{aligned}
\mathcal{F}\diamondsuit^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle \diamondsuit^k\delta, e^{-i\xi.x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \diamondsuit^k e^{-i\xi.x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \Box^k \triangle^k e^{-i\xi.x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, (-1)^k\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)\Box^k e^{-i\xi.x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, (-1)^k\left(\xi_1^2 + \ldots + \xi_n^2\right).(-1)^k \times \left(\xi_1^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2\right)^k e^{-i\xi.x}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}(-1)^{2k}\left(\xi_1^2 + \ldots + \xi_p^2\right)^k \times \left(\xi_1^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \ldots - \xi_{p+q}^2\right)^k \\
&= \frac{1}{(2\pi)^{n/2}}\left(\left(\xi_i^2 + \ldots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 - \ldots - \xi_{p+q}^2\right)^2\right)^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathcal{F}\diamondsuit^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^2\right|^k \\
&\leq \frac{M}{(2\pi)^{n/2}}\left|\xi_1^2 + \ldots + \xi_n^2\right|^k\left|\left(\xi_1^2 + \ldots + \xi_n^2\right)\right|^k \\
&\leq \frac{M}{(2\pi)^{n/2}}\|\xi\|^{4k},
\end{aligned}$$

where M is constant and $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$, $\xi_i(i = 1, 2, \ldots, n) \in \mathbb{R}$. Hence we obtain $\mathcal{F}\diamondsuit\delta$ which is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.2)

$$\diamondsuit^k\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^2\right]^k.$$

That completes the proof. □

Lemma 2.6. *Given the operator*

$$L = \frac{\partial^2}{\partial t^2} + c^2 \left(-\diamondsuit\right)^k, \tag{2.10}$$

where $\diamondsuit^k$ *is the diamond operator and is defined by (1.1). Then we obtain*

$$E(x,t) = O(\epsilon^{-\frac{n}{2k}}) \tag{2.11}$$

an elementary asymptotic solution for the operator defined by (2.10).

Proof. Let

$$LE(x,t) = \delta(x,t),$$

where $E(x,t)$ is the elementary solution of the operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial^2}{\partial t^2} E(x,t) + c^2 \left(-\diamondsuit\right)^k E(x,t) = \delta(x)\delta(t). \tag{2.12}$$

Take applying the Fourier transform defined by (2.1) to both sides of (2.12), we obtain

$$\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t) + c^2 \left(- \left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \widehat{E}(\xi,t) = \frac{1}{(2\pi)^n}\delta(t)$$

The solution of the above equation is

$$\widehat{E}(\xi,t) = H(t)\omega(\xi,t), \tag{2.13}$$

where $H(t)$ is the Heaviside function and $\omega(\xi,t)$ is a solution of homogeneous equation.

Now, we are solving the solution of homogeneous equation. Given the homogeneous equation

$$\frac{\partial^2}{\partial t^2}\widehat{E}(\xi,t) + c^2 \left(- \left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \widehat{E}(\xi,t) = 0 \tag{2.14}$$

Let $\omega(\xi,t)$ be the solution of (2.14), we have

$$\frac{\partial^2}{\partial t^2}\omega(\xi,t) + c^2 \left(- \left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \omega(\xi,t) = 0 \tag{2.15}$$

with the initial condition

$$\omega(x,0) = f(x), \quad \frac{\partial}{\partial t}\omega(x,0) = g(x). \tag{2.16}$$

Now, we put $r^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2$, we obtain

$$\frac{\partial^2}{\partial t^2}\widehat{\omega}(\xi,t) + c^2\left(s^4 - r^4\right)^k \widehat{\omega}(\xi,t) = 0$$

$$\widehat{\omega}(\xi,t) = A(\xi)\cos\left(\sqrt{s^4 - r^4}\right)^k ct + B(\xi)\sin\left(\sqrt{s^4 - r^4}\right)^k ct.$$

By (2.16), $\widehat{\omega}(\xi,0) = A(\xi) = \widehat{f}(\xi)$

$$\frac{\partial\widehat{\omega}(\xi,t)}{\partial t} = -c\left(\sqrt{s^4 - r^4}\right)^k A(\xi)\sin\left(\sqrt{s^4 - r^4}\right)^k ct + c\left(\sqrt{s^4 - r^4}\right)^k B(\xi)\cos\left(\sqrt{s^4 - r^4}\right)^k ct.$$

$$\frac{\partial\widehat{\omega}(\xi,0)}{\partial t} = 0 + c\left(\sqrt{s^4 - r^4}\right)^k B(\xi) = \widehat{g}(\xi)$$

$$B(\xi) = \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^4 - r^4}\right)^k}$$

$$\widehat{\omega}(\xi,t) = \widehat{f}(\xi)\cos\left(\sqrt{s^4 - r^4}\right)^k ct + \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^4 - r^4}\right)^k}\sin\left(\sqrt{s^4 - r^4}\right)^k ct. \tag{2.17}$$

By applying the inverse Fourier transform (2.17), we obtain the solution $\omega(x,t)$ in the convolution form Then (2.15) has a solution in the convolution form

$$\omega(x,t) = f(x) * \psi_t(x) + g(x) * \phi_t(x).$$

Now we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Let us consider the Fourier transform

$$\widehat{\Phi_t}(x) = \frac{\sin\left(\sqrt{s^4 - r^4}\right)^k ct}{c\left(\sqrt{s^4 - r^4}\right)^k} \quad \text{and} \quad \Psi_t(x) = \cos\left(\sqrt{s^4 - r^4}\right)^k ct.$$

They are all tempered distributions but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us define

$$\widehat{\phi_t^\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}\widehat{\phi_t}(\xi) = e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}\frac{\sin\left(\sqrt{s^4 - r^4}\right)^k ct}{c\left(\sqrt{s^4 - r^4}\right)^k} \quad \text{for} \quad \epsilon > 0. \tag{2.18}$$

We see that $\phi_t^\epsilon(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^\epsilon}(x) \to \widehat{\phi_t}(x)$ uniformly as $\epsilon \to 0$. So that

$\phi_t(x)$ will be the limit in the topology of tempered distribution of $\phi_t^\epsilon(x)$. Now

$$\begin{aligned}\Phi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Phi_t^\epsilon}(\xi)d\xi \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}\frac{\sin\left(\sqrt{s^4-r^4}\right)^k ct}{c\left(\sqrt{s^4-r^4}\right)^k}d\xi \\ |\Phi_t^\epsilon(x)| &\leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}}{c\left(\sqrt{s^4-r^4}\right)^k}d\xi\end{aligned}$$

By changing to bipolar coordinates. Now, put

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$$

and $$\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q},\ p+q=n$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$,

$$|\Phi_t^\epsilon(x)| \leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}}{c\left(\sqrt{s^4-r^4}\right)^k}r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, where $\Omega_p = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{(2\pi)^{q/2}}{\Gamma(q/2)}$,

$$|\Phi_t^\epsilon(x)| \leq \frac{\Omega_p\Omega_q}{(2\pi)^{n/2}}\int_0^\infty\int_0^s\frac{e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}}{c\left(\sqrt{s^4-r^4}\right)^k}r^{p-1}s^{q-1}drds,$$

putting $r^2 = s^2\sin\theta$, $2rdr = s^2\cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}|\Phi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{2(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2}\frac{e^{-\epsilon c\left(\sqrt{s^4-s^4\sin^2\theta}\right)^k}}{c\left(\sqrt{s^4-s^4\sin^2\theta}\right)^k}(\sin\theta)^{\frac{p-2}{2}}s^{p+q-1}\cos\theta d\theta ds, \\ &= \frac{\Omega_p\Omega_q}{2c(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2}\frac{e^{-\epsilon c\left(s^2\cos\theta\right)^k}}{\left(s^2\cos\theta\right)^k}s^{p+q-1}(\sin\theta)^{\frac{p-2}{2}}\cos\theta d\theta ds.\end{aligned}$$

Put $y=\epsilon c\left(s^2\cos\theta\right)^k=\epsilon c s^{2k}\cos^k\theta$, $s^{4k}=\dfrac{y}{c\epsilon\cos^k\theta}$, $ds=\frac{sdy}{2ky}$, thus

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{4c(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}s^{n-1}}{y/(\epsilon c)}(\sin\theta)^{\frac{p-2}{2}}\cos\theta\frac{s}{ky}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}\epsilon}{ky^2}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/2k}(\sin\theta)^{\frac{p-2}{2}}\cos\theta dyd\theta \\
&= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}y^{n/2k-2}}{c^{n/2k}k\epsilon^{n/2k-1}}(\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\frac{\Gamma\left(\frac{n}{2k}-1\right)}{k\epsilon^{\frac{n}{2k}-1}c^{n/2k}}\int_0^{\pi/2}(\sin\theta)^{\frac{p-2}{2}}(\cos\theta)^{\frac{2-n}{2}}d\theta \\
&= \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\epsilon^{n/2k-1}}\Gamma\left(\frac{n}{2k}-1\right)\beta\left(\frac{p}{4},\frac{4-n}{4}\right) \\
|\Phi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{8c^{n/2k}(2\pi)^{n/2}k\epsilon^{n/2k-1}}\frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.
\end{aligned}$$

Similarly, we define $\widehat{\Psi_t^\epsilon}(\xi)=e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}\cos\left(\sqrt{s^4-r^4}\right)^k ct$ and

$$\begin{aligned}
\Psi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{\Psi_t^\epsilon}(\xi)d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^4-r^4}\right)^k}\cos\left(\sqrt{s^4-r^4}\right)^k ctd\xi
\end{aligned}$$

and

$$\begin{aligned}
|\Psi_t^\epsilon(x)| &\le \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-\epsilon c\left(\sqrt{s^8-r^8}\right)^k}d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_0^s e^{-\epsilon c\left(\sqrt{s^8-r^8}\right)^k}r^{p-1}s^{q-1}drds,
\end{aligned}$$

put $r^2=s^2\sin\theta$, $2rdr=s^2\cos\theta d\theta$ and $0\le\theta\le\dfrac{\pi}{2}$

$$\begin{aligned}
|\Psi_t^\epsilon(x)| &\le \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2}e^{-\epsilon c\left(s^4\cos\theta\right)^k}(\sin\theta)^{\frac{p-4}{4}}s^{p+q-1}\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2}e^{-\epsilon c\left(s^4\cos\theta\right)^k}s^{p+q-1}(\sin\theta)^{\frac{p-4}{4}}\cos\theta d\theta ds,
\end{aligned}$$

put $y = \epsilon c(s^4 \cos\theta)^k$, $ds = s\dfrac{dy}{4ky}$,

$$\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2}\int_0^\infty \frac{e^{-y}}{y}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/2k} (\sin\theta)^{\frac{p-2}{2}} \cos\theta dy d\theta \\
&= \frac{\Omega_p\Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2}\int_0^\infty \frac{e^{-y}y^{n/2k-1}}{c^{n/2k}\epsilon^{n/2k}} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} dy d\theta \\
&= \frac{\Omega_p\Omega_q}{4(2\pi)^{n/2}kc^{n/2k}\epsilon^{n/2k}} \Gamma\left(\frac{n}{2k}\right) \int_0^{\pi/2} (\sin\theta)^{\frac{p-2}{2}} (\cos\theta)^{\frac{2-n}{2}} d\theta \\
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}\epsilon^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}.
\end{aligned}$$

Set

$$\omega^\epsilon(x,t) = f(x) * \Psi_t^\epsilon(x) + g(x) * \Phi_t^\epsilon(x). \tag{2.19}$$

By $\epsilon-$approximation of $\omega(x,t)$ in (2.15) for $\epsilon \to 0$, $\omega^\epsilon(x,t) \to \omega(x,t)$ uniformly. Now

$$\omega^\epsilon(x,t) = \int_{\mathbb{R}^n} f(r)\Psi_t^\epsilon(x-r)dr + \int_{\mathbb{R}^n} g(r)\Phi_t^\epsilon(x-r)dr$$

Thus

$$\begin{aligned}
|\omega^\epsilon(x,t)| &\leq |\Psi_t^\epsilon(x-r)| \int_{\mathbb{R}^n} |f(r)|dr + |\Phi_t^\epsilon(x-r)| \int_{\mathbb{R}^n} |g(r)|dr \\
&\leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}\epsilon^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M + \\
&\quad \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}\epsilon^{n/2k-1}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N
\end{aligned}$$

$$\begin{aligned}
\epsilon^{n/2k}|\omega^\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M + \\
&\quad \frac{\Omega_p\Omega_q\epsilon}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N, \qquad (2.20)
\end{aligned}$$

where $M = \int_{\mathbb{R}^n} |f(r)|dr$ and $N = \int_{\mathbb{R}^n} |g(r)|dr$, since f and g are absolutely integrable. We consider the boundness of (2.21) in four cases :
Case 1: p odd and q even(n odd). If $q = 8$ then

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} = \frac{\Gamma(1-\frac{n}{4})}{\Gamma(0)} = 0. \tag{2.21}$$

Putting (2.22) into (2.21), we obtain

$$\epsilon^{n/2k}|\omega^\epsilon(x,t)| \leq K(\text{K } is\ constant)$$

. Case 2 : p even and q odd(n odd).

In this case

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} \neq \infty \tag{2.22}$$

Thus

$$\epsilon^{n/2k}|u^{\epsilon}(x,t)| \leq K(\text{K } is\ constant)$$

.

Case 3 : p and q are both odd(n even and $n \neq 4k$)

For $n \neq 8k, k = 1, 2, 3, \ldots$ Therefore

$$\frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} \neq \infty \tag{2.23}$$

Thus

$$\epsilon^{n/2k}|\omega^{\epsilon}(x,t)| \leq K(\text{K } is\ constant).$$

Case 4 : p and q are both even(n even). In this case using the formula

$$\frac{\Gamma(z)}{\Gamma(z-m)} = \frac{(-1)^m\Gamma(-z+m+1)}{\Gamma(1-z)}, \quad m = 1, 2, 3, \ldots$$

We have

$$\begin{aligned} \frac{\Gamma(\frac{4-n}{4})}{\Gamma(\frac{4-q}{4})} &= \frac{\Gamma(\frac{4-q}{4} - \frac{p}{4})}{\Gamma(\frac{4-q}{4})} \\ &= \frac{\Gamma(1-(\frac{4-q}{4}))}{(-1)^m\Gamma(-(\frac{4-q}{4})+m+1)} \\ &= \frac{\Gamma(\frac{q}{4})}{(-1)^m\Gamma(\frac{q}{4}+m)} \end{aligned} \tag{2.24}$$

Putting (2.18) into (2.16), we obtain

$$\begin{aligned} \epsilon^{n/2k}|u^{\epsilon}(x,t)| \leq \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{q}{4}\right)}{(-1)^m\Gamma\left(\frac{q}{4}+m\right)}M + \\ \frac{\Omega_p\Omega_q\epsilon}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{q}{4}\right)}{(-1)^m\Gamma\left(\frac{q}{4}+m\right)}N, \end{aligned} \tag{2.25}$$

By (2.3)-(2.26) we conclude (2.21) is bounded.

By (2.16) we have

$$\begin{aligned} \epsilon^{n/2k}|\omega^{\epsilon}(x,t)| \quad \leq \quad & \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}M + \\ & \frac{\Omega_p\Omega_q\epsilon}{8(2\pi)^{n/2}kc^{n/2k}} \frac{\Gamma\left(\frac{n}{2k}-1\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}N, \end{aligned}$$

and

$$\lim_{\epsilon\to 0}\epsilon^{n/2k}|\omega^{\epsilon}(x,t)| \quad\leq\quad \frac{\Omega_p\Omega_q}{8(2\pi)^{n/2}kc^{n/2k}}\frac{\Gamma\left(\frac{n}{2k}\right)\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}M.$$

By definition (2.2) we obtain the asymptotic solution of (2.15) in the form

$$\omega(x,t) = O\left(\epsilon^{-n/2k}\right) \tag{2.26}$$

for $n \neq k$ as $\epsilon \to 0$.

Thus we obtain an asymptotic elementary solution of the operator by (2.10)

$$\begin{aligned} E(x,t) &= H(t)O(\epsilon^{-\frac{n}{2k}}) \\ &= O(\epsilon^{-\frac{n}{2k}}) \quad , \quad t>0 \end{aligned} \tag{2.27}$$

□

3 Main Results

Theorem 3.1. *Given the equation Given the nonlinear equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(-\diamondsuit)^k u(x,t) = f(x,t,u(x,t)) \tag{3.1}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, k is a positive number and with the following conditions on u and f as follows

(1) $u(x,t)$ is the space of continuous function on $\mathbb{R}^n \times (0,\infty)$.

(2) f satisfies the Lipschitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u-w|$$

where A is constant with $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.2}$$

as a unique solution of (2.21) for $x \in \Omega$ where Ω is a compact subset of $\mathbb{R}^n$ and $0 \leq t \leq T$ with T as constant and $E(x,t)$ as an elementary solution defined by (2.8) and also $u(x,t)$ is bounded for any fixed $t > 0$. In particular, if we put $n = 1, p = 0, q = 1$ and $k = 1$ in (3.1), then (3.1) reduces to the nonlinear beam equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\frac{\partial^4}{\partial x^4}u(x,t) = f(x,t,u(x,t)) \tag{3.3}$$

and we obtain $u(x,t) = \epsilon^{-\frac{1}{2}} * f(x,t,u(x,t))$ as an asymptotic solution of (3.3).

Proof. Convolving both sides of (3.1) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial^2}{\partial t^2}u(x,t) + c^2(-\diamondsuit)^k u(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial^2}{\partial t^2}E(x,t) + c^2(-\diamondsuit)^k E(x,t)\right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$\begin{aligned} u(x,t) &= E(x,t) * f(x,t,u(x,t)) \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s) f(x-r,t-s,u(x-r,t-s))drds, \end{aligned}$$

where $E(r,s)$ is given by definition (**??**). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\le \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds \\ &\le |E(r,s)|N \end{aligned}$$

where $N = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. By condition (3) in Theorem 3.1 and (2.27) we obtain $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$.

To show that $u(x,t)$ is unique. Suppose there is another solution $w(x,t)$ of (3.1). We next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (2.1). Let the operator be

$$L = \frac{\partial^2}{\partial t^2} + c^2(-\diamondsuit)^k$$

then (3.1) can be written in the form

$$Lu(x,t) = f(x,t,u(x,t))$$

Thus

$$Lu(x,t) - Lw(x,t) = f(x,t,u(x,t)) - f(x,t,w(x,t)).$$

By the condition (2) of the theorem 3.1,

$$|Lu(x,t) - Lw(x,t)| \le A|u(x,t) - w(x,t)|. \tag{3.4}$$

Let $\Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $L : C^{(4k)}(\Omega_0) \to C^{(4k)}(\Omega_0)$ for$0 \le t \le T$

Now $\left(C^{(4k)}(\Omega_0), ||..||\right)$ is a Banach space where $u(x,t) \in C^{(4k)}(\Omega_0)$ for $0 \le t \le T$and $||..||$ is given by

$$\|u(x,t)\| = \sup_{\substack{x\in\Omega_0 \\ 0<t\le T}} |u(x,t)|.$$

Then, from (2) with $0 < A < 1$, the operator L is a contraction mapping on $C^{(4k)}(\Omega_0)$. Since $\left(C^{(4k)}(\Omega_0), ||..||\right)$ i a Banach space and $L : C^{(4k)}(\Omega_0) \to C^{(4k)}(\Omega_0)$ is a contraction mapping on $C^{(4k)}(\Omega_0)$, by Contraction Theorem [3], we obtain the operator L which has a fixed point and has uniqueness property. Thus $u(x,t) = w(x,t)$.

In particular, if we put $n = 1, p = 0, q = 1$ and $k = 1$ then (3.1) reduces to the nonlinear beam equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\frac{\partial^2}{\partial x^2})^4 u(x,t) = f(x,t,u(x,t)). \tag{3.5}$$

Thus we obtain $u(x,t) = O(\epsilon^{-1/4}) * f(x,t,u(x,t))$ is an asymptotic solution of (3.5). That complete the proof. □

Theorem 3.2. *A boundness of the elementary solution in Sobelev space.*

Let the condition (1.8) of f and g be $f \in H_s(\mathbb{R}^n)$ and $g \in H_{s-1}(\mathbb{R}^n)$ then $E(x,t) \in H_s(\mathbb{R}^n \times (0,\infty))$ where $H_s(\mathbb{R}^n)$ is a Sobelev space of order s defined by definition 2.3.

Proof. By the Plancherel theorem, $f \in H_s(\mathbb{R}^n)$ if and only if $(1+\sqrt{s^4 - r^4})^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$. Now$\widehat{(\partial_\alpha f)}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$ where

$$\partial^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}$$

for $|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_n \le s$ and s is a nonnegative integer. We have $(i\xi)^\alpha \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$ or equivalent $(1+\sqrt{s^4 - r^4})^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$. We now show that $E(x,t) \in H_s(\mathbb{R}^n \times (0,\infty)))$ with the Sobelev norm

$$\|E(x,t)\|_k^2 = \int_{\mathbb{R}^n} |Ef(\xi,t)|^2(1+\sqrt{s^4 - r^4})^k d\xi < \infty$$

for any given $t \in (0,\infty)$. Now consider $\widehat{(\partial^\alpha\partial_t^j E)}(\xi,t)$ where $\widehat{E}(\xi,t)$ is an elementary solution is given by (2.8),

$$\partial^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}$$

and $\partial_t^j = \frac{\partial^j}{\partial t^j}$, j is a nonnegative integer. We have

$$\begin{aligned}
\widehat{(\partial_x^\alpha\partial_t E)}(\xi,t) &= (i\xi)^\alpha \frac{\partial^j}{\partial t_j}\widehat{E}(\xi,t) \quad \text{for} \quad |\alpha| + j \le s \\
&= (i\xi)^\alpha \frac{\partial^j}{\partial t_j}\left(\widehat{f}(\xi)Cos(c\left(\sqrt{s^4-r^4}\right)^k t) + \widehat{g}(\xi)\frac{Sin(c\left(\sqrt{s^4-r^4}\right)^k t)}{c\sqrt{s^4-r^4}}\right) \\
&= (i\xi)^\alpha \left(c\left(\sqrt{s^4-r^4}\right)^k\right)^j \ trig(c\left(\sqrt{s^4-r^4}\right)^k t) + \\
&\qquad (i\xi)^\alpha \left(c\left(\sqrt{s^4-r^4}\right)^k\right)^{j-1} trig\left(c\left(\sqrt{s^4-r^4}\right)^k t\right)
\end{aligned} \tag{3.6}$$

where trig denotes one of the function $\pm Cos$ or $\pm Sin$. By the Plancherel theorem, if $f \in H_s(\mathbb{R}^n$ and $g \in H_{s-1}(\mathbb{R}^n)$ then on the right hand side of (13) we have $(i\xi)^\alpha (c\sqrt{s^4 - r^4})^j \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$ and $(i\xi)^\alpha \left(c\left(\sqrt{s^4 - r^4}\right)^k\right)^{j-1} \widehat{g}(\xi) \in L^2(\mathbb{R}^n)$. Thus $\widehat{(\partial_x^\alpha \partial_t^j u)}(\xi, t) \in L^2(\mathbb{R}^n \times (0, \infty))$ and it follows that $\partial_x^\alpha \partial_t^j E(x,t) \in L^2(\mathbb{R}^n \times (0, \infty))$ with the Sobelev norm

$$\|E(x,t)\|_s = \left(\int_{\mathbb{R}^n} |E(\xi,t)|^2 (1 + \sqrt{s^4 - r^4})^k d\xi\right)^{1/2}$$

bounded independent of t for $|\alpha| + j \leq s$. It follows that $E(x,t) \in H_s(\mathbb{R}^n \times (0, \infty))$. □

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References

[1] A. Kananthai, On the solution of the n-dimensional diamond operator, Applied Mathematics and Computational (1997) 27-37.

[2] G.B. Folland, Introduction to Partial Differential Equation, Princeton University Press, Princeton, New Jersey, (1995).

[3] E. Kreyzig, Introductory Functional Analysis with Applications, John Willey and Sons Inc., (1978).

[4] A.H. Zemanian, Distribution Theory and Transform Analysis, McGraw-Hill, New York, 1965.

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