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Common Fixed Point Theorems for the Finite Family of Uniformly Quasi–sup(f_n) Lipschitzian Mappings and g_n–Expansive Mappings in Convex Metric Spaces

S. Kamkong and O. Tripak[1]

Department of Mathematics and Statistics, Faculty of Science,
Prince of Songkla University, Songkhla 90110, Thailand
Centre of Excellence in Mathematics,
CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand
e-mail : satornn@hotmail.com (S. Kamkong),
orawan.t@psu.ac.th (O. Tripak)

Abstract : In this paper we introduce a new $n-$steps iterative process, which converges to a common fixed point of finite families of uniformly quasi−sup (f_n) Lipschitzian mappings and g_n-expansive mappings in convex metric spaces.

Keywords : convex metric space; common fixed point theorems; uniformly quasi−sup (f_n) Lipschitzian mapping; g_n-expansive mapping.
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1 Introduction

In 2011, Byung−Soo Lee [1] defined new family of mappings: infinite family $\{T_i\}_{i=1}^{\infty}$ of uniformly quasi−sup (f_n) Lipschitzian mappings and infinite family $\{S_i\}_{i=1}^{\infty}$ of g_n-expansive mappings for approximating a common fixed point in convex metric spaces using a Noor−type iterative. The iterative is defined as follows: let C be a nonempty convex subset of (X, d, W), $\{T_i\}_{i=1}^{\infty}$ an infinite family of uniformly quasi−sup (f_n) Lipschitzian mappings and $\{S_i\}_{i=1}^{\infty}$ a g_n-expansive

[1]Corresponding author.

mappings of C. Suppose that $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{a_n\},\{b_n\},\{c_n\},\{d_n\},\{e_n\},\{l_n\}$ are sequences in $[0,1]$ for which $\alpha_n+\beta_n+\gamma_n=a_n+b_n+c_n=d_n+e_n+l_n=1$ for all $n\in\mathbb{N}$. For $x_1\in X$, let $\{x_n\}$ be a sequence defined by

$$\left.\begin{array}{ll} x_{n+1} & = W(S_nx_n, T_n^ny_n, u_n;\alpha_n,\beta_n,\gamma_n) \\ y_n & = W(S_nx_n, T_n^nz_n, v_n; a_n, b_n, c_n) \\ z_n & = W(S_nx_n, T_n^nx_n, w_n; d_n, e_n, l_n) \end{array}\right\} \tag{1.1}$$

where $\{u_n\},\{v_n\},\{w_n\}$ are any sequences in X.

In 2013, Phuengrattana and Suantai [2] introduced a new iterative process for approximating a common fixed point of a finite family $\{T_i\}_{i=1}^N$ of generalized asymptotically quasi$-$nonexpansive mappings in a convex metric space. The following is the iterative process: let C be a convex subset of a convex metric space (X,d,W) and $\{T_i\}_{i=1}^N$ a finite family of generalized asymptotically quasi$-$nonexpansive mappings. Suppose that $\{\alpha_n^{(i)}\}$, for $i=1,2,\ldots,N$, are sequences in $[0,1]$. For $x_1\in C$, let $\{x_n\}$ be a sequence defined by

$$\left.\begin{array}{ll} y_n^{(0)} & = x_n \\ y_n^{(1)} & = W(T_1^ny_n^{(0)}, y_n^{(0)};\alpha_n^{(1)}) \\ y_n^{(2)} & = W(T_2^ny_n^{(1)}, y_n^{(1)};\alpha_n^{(2)}) \\ y_n^{(3)} & = W(T_3^ny_n^{(2)}, y_n^{(2)};\alpha_n^{(3)}) \\ \vdots & \\ y_n^{(N-1)} & = W(T_{N-1}^ny_n^{(N-2)}, y_n^{(N-2)};\alpha_n^{(N-1)}) \\ x_{n+1} & = W(T_N^ny_n^{(N-1)}, y_n^{(N-1)};\alpha_n^{(N)}) \end{array}\right\} \tag{1.2}$$

for all $n\in\mathbb{N}$.

Motivated by [1] and [2], we define a new $n-$steps iterative process to approximate a common fixed point of a finite family $\{T_i\}_{i=1}^N$ of uniformly quasi$-$sup (f_n) Lipschitzian mappings and a finite family $\{S_i\}_{i=1}^N$ of g_n-expansive mappings in convex metric spaces. Our new iterative process is explained as follows: let C be a nonempty convex subset of a convex metric space (X,d,W). Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly quasi$-$sup (f_n) Lipschitzian mappings and $\{S_i\}_{i=1}^N$ a finite family of g_n-expansive mappings of C. Suppose that $\{\alpha_n^{(i)}\},\{\beta_n^{(i)}\},\{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ such that $\alpha_n^{(i)}+\beta_n^{(i)}+\gamma_n^{(i)}=1$ for each $n\in\mathbb{N}$ and $i=1,2,\ldots,N$. For $x_1\in C$, let $\{x_n\}$ be a sequence defined by

$$\left.\begin{array}{ll} y_n^{(0)} & = x_n \\ y_n^{(1)} & = W(S_1^ny_n^{(0)}, T_1^ny_n^{(0)}, u_n^{(0)};\alpha_n^{(1)},\beta_n^{(1)},\gamma_n^{(1)}) \\ y_n^{(2)} & = W(S_2^ny_n^{(1)}, T_2^ny_n^{(1)}, u_n^{(1)};\alpha_n^{(2)},\beta_n^{(2)},\gamma_n^{(2)}) \\ y_n^{(3)} & = W(S_3^ny_n^{(2)}, T_3^ny_n^{(2)}, u_n^{(2)};\alpha_n^{(3)},\beta_n^{(3)},\gamma_n^{(3)}) \\ \vdots & \\ y_n^{(N-1)} & = W(S_{N-1}^ny_n^{(N-2)}, T_{N-1}^ny_n^{(N-2)}, u_n^{(N-2)};\alpha_n^{(N-1)},\beta_n^{(N-1)},\gamma_n^{(N-1)}) \\ x_{n+1}=y_n^{(N)} & = W(S_N^ny_n^{(N-1)}, T_N^ny_n^{(N-1)}, u_n^{(N-1)};\alpha_n^{(N)},\beta_n^{(N)},\gamma_n^{(N)}) \end{array}\right\} \tag{1.3}$$

for all $n \in \mathbb{N}$ and $\{u_n^{(i)}\}$ are any bounded sequences in C.

The purpose of this paper is to extend and improve some results of [1] and [2].

2 Preliminaries

In this section, the definition of mapping which will be used in the paper is presented as follow.

Definition 2.1 ([1, Definition 1.1],[2]). Let C be a nonempty subset of a metric space (X, d), T a self−mapping on C and $f : C \to (0, \infty)$ a function which is bounded above. The set of fixed point of T is denote by $F(T)$, i.e., $F(T) = \{x \in C : Tx = x\}$.
(i) T is *f−expansive* if

$$d(Tx, Ty) \leq \sup_{z \in C} f(z) \cdot d(x, y)$$

for all $x, y \in C$.
(ii) T is *asymptotically f−expansive* if there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} f(x_n) = 1$ satisfying

$$d(T^n x, T^n y) \leq f(x_n) \cdot d(x, y)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.
(iii) T is *asymptotically quasi−f−expansive* if there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} f(x_n) = 1$ satisfying

$$d(T^n x, p) \leq f(x_n) \cdot d(x, p)$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.
(iv) T is *uniformly quasi−sup (f) Lipschitzian* if

$$d(T^n x, p) \leq \sup_{z \in C} f(z) \cdot d(x, p)$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.
(v) T is *uniformly L−Lipschitzian* if there exists contant $L > 0$ such that

$$d(T^n x, T^n y) \leq L \cdot d(x, y)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Some definitions and useful results related to convex structure and convex metric space are recalled next.

Definition 2.2 ([1, Definition 1.2]). Let (X,d) be a metric space. A mapping $W : X^3 \times I^3 \to X$ is said to be a *convex structure* on X if for each $x,y,z \in X$ and $\alpha,\beta,\gamma \in I$ with $\alpha+\beta+\gamma = 1$ satisfy

$$d(W(x,y,z;\alpha,\beta,\gamma),u) \leq \alpha d(x,u) + \beta d(y,u) + \gamma d(z,u)$$

for all $u \in X$. Moreover, a metric space (X,d) with a convex structure W is called a convex metric space which will be denoted by (X,d,W). A nonempty subset C of a convex metric space (X,d,W) is said to be a *convex subset* of (X,d) if $W(x,y,z;\alpha,\beta,\gamma) \in C$ for $(x,y,z) \in C^3$ and $(\alpha,\beta,\gamma) \in I^3$ with $\alpha+\beta+\gamma = 1$.

Definition 2.3 ([2]). Let C be a subset of a metric space (X,d). A finite family of self mappings $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ of C are said to have *Condition A* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ and function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(r) > 0$, respectively for all $r > 0$ such that

$$d(x,T_i x) \geq f(d(x,D)) \quad \textit{and} \quad d(x,S_i x) \geq g(d(x,D))$$

for some i, $1 \leq i \leq N$ and for all $x \in C$, where $d(x,D) = \inf\Big\{d(x,p) : p \in D = \Big(\bigcap\limits_{i=1}^{N} F(T_i)\Big)\bigcap\Big(\bigcap\limits_{i=1}^{N} F(S_i)\Big)\Big\}$.

Definition 2.4 ([2]). Let C be a subset of a metric space (X,d). A mapping T is *semi−compact* if for a sequence $\{x_n\}$ in C with $\lim\limits_{n\to\infty} d(x_n,Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

Lemma 2.5 ([3, Lemma 1.1, Remark 1.3], [4, Lemma 2]). *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of nonnegative real numbers such that $\sum\limits_{n=1}^{\infty} b_n < \infty$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$ and for all $n \in \mathbb{N}$,*

$$a_{n+1} \leq (1+b_n)a_n + c_n.$$

Then,
(*i*) $\lim\limits_{n\to\infty} a_n$ *exists,*
(*ii*) *If* $\liminf\limits_{n\to\infty} a_n = 0$ *then* $\lim\limits_{n\to\infty} a_n = 0$,
(*iii*) *If either* $\liminf\limits_{n\to\infty} a_n = 0$ *or* $\limsup\limits_{n\to\infty} a_n = 0$ *then* $\lim\limits_{n\to\infty} a_n = 0$.

Definition 2.6 ([2, Definition 2.3]). Let $\{x_n\}$ be a sequence in a metric space (X,d) and D a subset of X. We say that $\{x_n\}$ is of *monotone type* I with respect to D if there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $\sum\limits_{n=1}^{\infty} r_n < \infty, \sum\limits_{n=1}^{\infty} s_n < \infty$ and

$$d(x_{n+1},p) \leq (1+r_n)d(x_n,p) + s_n$$

for all $n \in \mathbb{N}$ and $p \in D$. A sequence $\{x_n\}$ is of *monotone type* II with respect to D if for each $p \in D$ there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty} r_n < \infty, \sum_{n=1}^{\infty} s_n < \infty$ and

$$d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + s_n$$

for all $n \in \mathbb{N}$.

Lemma 2.7 ([2, Theorem 2.4])**.** *Let (X, d) be a complete metric space, D a subset of X and $\{x_n\}$ a sequence in X. Then one has the following assertions*:
(i) If $\{x_n\}$ is of monotone type I with respect to D then $\lim_{n\to\infty} d(x_n, D)$ exists.
(ii) If $\{x_n\}$ is of monotone type I with respect to D and $\liminf_{n\to\infty} d(x_n, D) = 0$ then $x_n \to p$ for some $p \in X$ satisfying $d(p, D) = 0$. In particular, if D is closed then $p \in D$.
(iii) If $\{x_n\}$ is of monotone type II with respect to D then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in D$.

3 Main Results

In this section, we let C be a nonempty convex subset of a convex metric space (X, d, W). Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly quasi$-$sup (f_n) Lipschitzian self$-$mappings of C and $\{S_i\}_{i=1}^N$ a finite family of g_n-expansive self$-$mappings of C. Let f_n, g_n be functions which is bounded above such that $U_n = \sup_{x\in C} f_n(x)$ and $E_n = \sup_{x\in C} g_n(x)$. Suppose that $U = \sup_{n\in\mathbb{N}} U_n$ and $E = \sup_{n\in\mathbb{N}} E_n$ are finite and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are sequences in $[0, 1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $n \in \mathbb{N}$. Let $\delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)}$, $\delta_n = \max_{1\leq i\leq N}\{\delta_n^{(i)}\}, \gamma_n = \max_{1\leq i\leq N}\{\gamma_n^{(i)}\}$ and $d(u_n, p) = \max_{1\leq i\leq N}\{d(u_n^{(i-1)}, p)\}$. Let $A = \max\{E, U\}$, sequences $\{\delta_n A\} \subset [1, \infty)$ and $\{\gamma_n d(u_n, p)\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(\delta_n A - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n d(u_n, p) < \infty$. Suppose $D = \Big(\bigcap_{i=1}^{N} F(T_i)\Big) \bigcap \Big(\bigcap_{i=1}^{N} F(S_i)\Big)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be the iteration defined as (1.3). We shall first begin by constructing the following useful inequalities.

Lemma 3.1. *For each $i = 1, 2, \ldots, N-1, n \in \mathbb{N}$ and $p \in D$. The following results hold.*
(i) $d(y_n^{(i)}, p) \leq \delta_n \cdot A \cdot d(y_n^{(i-1)}, p) + \gamma_n d(u_n, p)$.
(ii) $d(y_n^{(i)}, p) \leq \delta_n^i \cdot A^i \cdot d(x_n, p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}]\gamma_n d(u_n, p)$.

$(iii)\quad d(x_{n+1},p) \le \delta_n^i \cdot A^i \cdot d(y_n^{(N-i)},p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}]\gamma_n d(u_n,p).$

Proof. (i) We need to show that

$$d(y_n^{(i)},p) \le \delta_n \cdot A \cdot d(y_n^{(i-1)},p) + \gamma_n d(u_n,p)$$

for any $i = 1,2,\ldots,N-1$, $n \in \mathbb{N}$ and $p \in D$. By Definition 2.2, Definition 2.1$(i),(iv)$, $U_n = \sup_{x\in C} f_n(x)$, $E_n = \sup_{x\in C} g_n(x)$, $U = \sup_{n\in\mathbb{N}} U_n$ and $E = \sup_{n\in\mathbb{N}} E_n$, we have the following inequality,

$$\begin{aligned}
d(y_n^{(i)},p) &= d(W(S_i^n y_n^{(i-1)}, T_i^n y_n^{(i-1)}, u_n^{(i-1)}; \alpha_n^{(i)}, \beta_n^{(i)}, \gamma_n^{(i)}), p)\\
&\le \alpha_n^{(i)} d(S_i^n y_n^{(i-1)},p) + \beta_n^{(i)} d(T_i^n y_n^{(i-1)},p) + \gamma_n^{(i)} d(u_n^{(i-1)},p)\\
&\le \alpha_n^{(i)} g_n(y_n^{(i-1)}) d(y_n^{(i-1)},p) + \beta_n^{(i)} U_n d(y_n^{(i-1)},p) + \gamma_n^{(i)} d(u_n^{(i-1)},p)\\
&\le \alpha_n^{(i)} E_n d(y_n^{(i-1)},p) + \beta_n^{(i)} U_n d(y_n^{(i-1)},p) + \gamma_n^{(i)} d(u_n^{(i-1)},p)\\
&= (\alpha_n^{(i)} E + \beta_n^{(i)} U) d(y_n^{(i-1)},p) + \gamma_n^{(i)} d(u_n^{(i-1)},p).
\end{aligned}$$

Thus by $\delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)}$, $A = \max\{E,U\}$, $\delta_n = \max_{1\le i\le N}\{\delta_n^{(i)}\}, \gamma_n = \max_{1\le i\le N}\{\gamma_n^{(i)}\}$ and $d(u_n,p) = \max_{1\le i\le N}\{d(u_n^{(i-1)},p)\}$, we can rewrite the above inequalities as

$$\begin{aligned}
d(y_n^{(i)},p) &\le \delta_n^{(i)} A d(y_n^{(i-1)},p) + \gamma_n^{(i)} d(u_n^{(i-1)},p)\\
&\le \delta_n A d(y_n^{(i-1)},p) + \gamma_n d(u_n,p).
\end{aligned} \tag{3.1}$$

(ii) We are going to show that

$$d(y_n^{(i)},p) \le \delta_n^i A^i d(x_n,p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}]\gamma_n d(u_n,p)$$

for all $i = 1,2,\ldots,N-1$, by using mathematical induction. Recall inequality (3.1), for any $i = 1,2,\ldots,N-1$, $n \in \mathbb{N}$ and $p \in D$, we know that

$$d(y_n^{(i)},p) \le \delta_n A d(y_n^{(i-1)},p) + \gamma_n d(u_n,p).$$

For $i = 1$,

$$\begin{aligned}
d(y_n^{(1)},p) &\le \delta_n A d(y_n^{(0)},p) + \gamma_n d(u_n,p)\\
&= \delta_n A d(x_n,p) + \gamma_n d(u_n,p)[\sum_{j=1}^{1} \delta_n^{j-1} A^{j-1}].
\end{aligned}$$

Assume that for some $m, 1 \le m \le N-2$,

$$d(y_n^{(m)},p) \le \delta_n^m A^m d(x_n,p) + \gamma_n d(u_n,p)[\sum_{j=1}^{m} \delta_n^{j-1} A^{j-1}].$$

Hence,

$$\begin{aligned} d(y_n^{(m+1)}, p) &\le \delta_n A d(y_n^{(m)}, p) + \gamma_n d(u_n, p) \\ &\le \delta_n A[\delta_n^m A^m d(x_n, p) + \sum_{j=1}^{m} \delta_n^{j-1} A^{j-1} \gamma_n d(u_n, p)] + \gamma_n d(u_n, p) \\ &= \delta_n^{m+1} A^{m+1} d(x_n, p) + [\sum_{j=1}^{m+1} \delta_n^{j-1} A^{j-1}] \gamma_n d(u_n, p). \end{aligned}$$

Therefore, by mathematical induction, we have,

$$d(y_n^{(i)}, p) \le \delta_n^i A^i d(x_n, p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}] \gamma_n d(u_n, p)$$

for all $i = 1, 2, \ldots, N-1$.

(iii) We prove the following inequality, for any $i = 1, 2, \ldots, N-1$,

$$d(x_{n+1}, p) \le \delta_n^i \cdot A^i \cdot d(y_n^{N-i}, p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}] \gamma_n d(u_n, p).$$

Now, for any $i = 1, 2, \ldots, N-1$, $n \in \mathbb{N}$ and $p \in D$,

$$\begin{aligned} d(x_{n+1}, p) &= d(W(S_N^n y_n^{(N-1)}, T_N^n y_n^{(N-1)}, u_n^{(N-1)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)}), p) \\ &\le \alpha_n^{(N)} d(S_N^n y_n^{(N-1)}, p) + \beta_n^{(N)} d(T_N^n y_n^{(N-1)}, p) + \gamma_n^{(N)} d(u_n^{(N-1)}, p) \\ &\le \alpha_n^{(N)} E d(y_n^{(N-1)}, p) + \beta_n^{(N)} U d(y_n^{(N-1)}, p) + \gamma_n^{(N)} d(u_n^{(N-1)}, p) \\ &\le \alpha_n^{(N)} A d(y_n^{(N-1)}, p) + \beta_n^{(N)} A d(y_n^{(N-1)}, p) + \gamma_n^{(N)} d(u_n^{(N-1)}, p). \end{aligned}$$

Together with (i), we have

$$\begin{aligned} d(x_{n+1}, p) &\le \delta_n^{(N)} A d(y_n^{(N-1)}, p) + \gamma_n^{(N)} d(u_n^{(N-1)}, p) \\ &\le \delta_n A d(y_n^{(N-1)}, p) + \gamma_n d(u_n, p) \\ &\le \delta_n A[\delta_n A d(y_n^{(N-2)}, p) + \gamma_n d(u_n, p)] + \gamma_n d(u_n, p) \\ &= \delta_n^2 A^2 d(y_n^{(N-2)}, p) + \delta_n A \gamma_n d(u_n, p) + \gamma_n d(u_n, p) \\ &\le \delta_n^2 A^2[\delta_n A d(y_n^{(N-3)}, p) + \gamma_n d(u_n, p)] + (\delta_n A + 1)\gamma_n d(u_n, p) \\ &= \delta_n^3 A^3 d(y_n^{(N-3)}, p) + \delta_n^2 A^2 \gamma_n d(u_n, p) + (\delta_n A + 1)\gamma_n d(u_n, p) \\ &= \delta_n^3 A^3 d(y_n^{(N-3)}, p) + (\delta_n^2 A^2 + \delta_n A + 1)\gamma_n d(u_n, p) \\ &\vdots \\ &\le \delta_n^i A^i d(y_n^{(N-i)}, p) + [\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}] \gamma_n d(u_n, p) \end{aligned}$$

for all $i = 1, 2, \ldots, N-1$. □

Remark 3.2. *Lemma* 3.1 *generalizes main results of Lemma* 3.1 *in* [2].

Lemma 3.3. *The following results hold.*

(i) *There exist two sequences* $\{\eta_n\}$ *and* $\{\sigma_n\}$ *such that* $\sum_{n=1}^{\infty}\eta_n<\infty$ *and* $\sum_{n=1}^{\infty}\sigma_n<\infty$ *and*

$$d(x_{n+1},p)\le(1+\eta_n)d(x_n,p)+\sigma_n$$

for all $p\in D$ *and* $n\in\mathbb{N}$.
(ii) $\lim_{n\to\infty}d(x_n,p)$ *exists for all* $p\in D$.

Proof. (i) Let $p\in D$ and $n\in\mathbb{N}$. By Lemma 3.1$(ii),(iii)$, we get

$$\begin{aligned}d(x_{n+1},p)&\le\delta_nAd(y_n^{(N-1)},p)+\gamma_nd(u_n,p)\\&\le\delta_nA[\delta_n^{N-1}A^{N-1}d(x_n,p)+[\sum_{j=1}^{N-1}\delta_n^{j-1}A^{j-1}]\gamma_nd(u_n,p)]+\gamma_nd(u_n,p)\\&=\delta_n^NA^Nd(x_n,p)+[\sum_{j=1}^{N}\delta_n^{j-1}A^{j-1}]\gamma_nd(u_n,p)\\&=(1+\eta_n)d(x_n,p)+\sigma_n\end{aligned}$$

where $\eta_n=\sum_{j=1}^{N}\binom{N}{j}(\delta_nA-1)^j$ and $\sigma_n=[\sum_{j=1}^{N}\delta_n^{j-1}A^{j-1}]\gamma_nd(u_n,p)$.

Since $\sum_{n=1}^{\infty}(\delta_nA-1)$ and $\sum_{n=1}^{\infty}\gamma_nd(u_n,p)$ are both finite, it follows that $\sum_{n=1}^{\infty}\eta_n<\infty$ and $\sum_{n=1}^{\infty}\sigma_n<\infty$.

(ii) Let $p\in D$. Then, by the above result, there exist two sequences $\{\eta_n\}$ and $\{\sigma_n\}$ such that $\sum_{n=1}^{\infty}\eta_n<\infty$ and $\sum_{n=1}^{\infty}\sigma_n<\infty$ satisfying

$$d(x_{n+1},p)\le(1+\eta_n)d(x_n,p)+\sigma_n$$

for all $n\in\mathbb{N}$. Then, by Lemma 2.5(i), $\lim_{n\to\infty}d(x_n,p)$ exists for all $p\in D$. ☐

Remark 3.4. *Lemma* 3.3 *generalizes main results of Lemma* 3.2 *in* [2].

Theorem 3.5. *The following results hold when* D *is closed.*
(i) *If* $\{x_n\}$ *converges to a common fixed point in* D *then* $\liminf_{n\to\infty}d(x_n,D)=\limsup_{n\to\infty}d(x_n,D)=0$.
(ii) *If either* $\liminf_{n\to\infty}d(x_n,D)=0$ *or* $\limsup_{n\to\infty}d(x_n,D)=0$ *then* $\{x_n\}$ *converges to a common fixed point in* D.

Proof. (i) Suppose that $\{x_n\}$ converges to a common fixed point p in D. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then

$$d(x_n, p) < \frac{\epsilon}{2}.$$

Taking infimum over $p \in D$, for $n \geq N$ we have

$$d(x_n, D) \leq \frac{\epsilon}{2} < \epsilon.$$

We have $\lim_{n\to\infty} d(x_n, D) = 0$, i.e., $\liminf_{n\to\infty} d(x_n, D) = \limsup_{n\to\infty} d(x_n, D) = 0$.

(ii) Assume that $\liminf_{n\to\infty} d(x_n, D) = 0$ or $\limsup_{n\to\infty} d(x_n, D) = 0$. Using this result together with Lemma 3.3(i) and Lemma 2.5(iii) we have $\lim_{n\to\infty} d(x_n, D) = 0$. Then, by Lemma 2.7$(ii)$, $\lim_{n\to\infty} x_n$ exists that is there exists $q \in X$ such that $x_n \to q$ as $n \to \infty$. Since D is closed, $\{x_n\}$ converges to a common fixed point in D. □

4 Applications

In this section, we let C be a nonempty convex subset of a convex metric space (X, d, W). Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly quasi−sup (f_n) Lipschitzian self−mappings of C and $\{S_i\}_{i=1}^N$ a finite family of g_n−expansive self−mappings of C. Let f_n, g_n be functions which is bounded above such that $U_n = \sup_{x\in C} f_n(x)$ and $E_n = \sup_{x\in C} g_n(x)$. Suppose that $U = \sup_{n\in\mathbb{N}} U_n$ and $E = \sup_{n\in\mathbb{N}} E_n$ are finite and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $n \in \mathbb{N}$. Let $\delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)}$, $\delta_n = \max_{1\leq i\leq N}\{\delta_n^{(i)}\}, \gamma_n = \max_{1\leq i\leq N}\{\gamma_n^{(i)}\}$ and $d(u_n, p) = \max_{1\leq i\leq N}\{d(u_n^{(i-1)}, p)\}$. Let $A = \max\{E, U\}$, sequences $\{\delta_n A\} \subset [1, \infty)$ and $\{\gamma_n d(u_n, p)\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(\delta_n A - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n d(u_n, p) < \infty$. Suppose $D = \Big(\bigcap_{i=1}^{N} F(T_i)\Big) \bigcap \Big(\bigcap_{i=1}^{N} F(S_i)\Big)$ is nonempty and closed. Let $x_1 \in C$ and the sequence $\{x_n\}$ be the iteration defined by (1.3). First, we prove the following lemma.

Lemma 4.1. *A sequence $\{x_n\}$ converges to common fixed point of the families $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, D) = 0$, where $d(x_n, D) = \inf\{d(x_n, p) : p \in D\}$.*

Proof. The necessity condition is obvious. Thus we will only prove the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n, D) = 0$. Then by Lemma 3.3$(i)$, there exist two sequences $\{\eta_n\}$ and $\{\sigma_n\}$ such that $\sum_{n=1}^{\infty}\eta_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$ satisfying

$$d(x_{n+1}, p) \leq (1+\eta_n)d(x_n, p) + \sigma_n$$

for all $p \in D$ and all $n \in \mathbb{N}$. By Definition 2.6, we have $\{x_n\}$ is of monotone type I with respect to D. By Lemma 2.7(ii), we have desired. ☐

Remark 4.2. *Lemma* 4.1 *generalizes main results of Theorem* 3.3 *in* [2].

The following theorem shows that the sequence $\{x_n\}$ converges to common fixed point of the families $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ with two added properties: Condition A (Definition 2.3) and semi−compact (Definition 2.4).

Theorem 4.3. *Let C be a closed convex subset of a complete uniformly convex metric space (X, d, W) with continuous convex structure. Let $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ be finite families of uniformly $L-$Lipschitzian self−mappings of C. Suppose that $\lim_{n\to\infty} d(x_n, T_ix_n) = 0 = \lim_{n\to\infty} d(x_n, S_ix_n)$ for all $i = 1, 2, \ldots, N$. If one of the following is satisfied*:

(i) $\{T_i\}_{i=1}^N$ *and* $\{S_i\}_{i=1}^N$ *satisfy Condition A,*

(ii) *one member of the families* $\{T_i\}_{i=1}^N$ *and* $\{S_i\}_{i=1}^N$ *are semi−compact,*

then $\{x_n\}$ converges to common fixed point of two families $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$.

Proof. (i) Suppose that $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ are satisfy Condition A. Then by Definition 2.3 we have there exists a nondecreasing functions such that
$f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$,
$g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0,\infty)$
with $d(x_n, T_ix_n) \geq f(d(x_n, D))$ and $d(x_n, S_ix_n) \geq g(d(x_n, D))$. Hence,

$$\lim_{n\to\infty} d(x_n, T_ix_n) \geq \lim_{n\to\infty} f(d(x_n, D)),$$
$$\lim_{n\to\infty} d(x_n, S_ix_n) \geq \lim_{n\to\infty} g(d(x_n, D)),$$

for some $1 \leq i \leq N$. By the assumption we know that $\lim_{n\to\infty} d(x_n, T_ix_n) = 0 = \lim_{n\to\infty} d(x_n, S_ix_n)$. It follows that

$$\lim_{n\to\infty} d(x_n, D) = 0.$$

Then by Lemma 4.1, the sequence $\{x_n\}$ converges to common fixed point of the families $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$.

(ii) Suppose that $\{T_i\}_{i=1}^N$ is semi−compact. Then, by Definition 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$. Hence, for each $1 \leq i \leq N$,

$$\begin{aligned} d(p, T_ip) &\leq d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j}) + d(T_ix_{n_j}, T_ip) \\ &\leq d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j}) + Ld(x_{n_j}, p) \\ &\leq (1+L)d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j}) \\ &\to 0. \end{aligned}$$

Thus, we have $T_i p \to p$ for $1 \leq i \leq N$.
The proof in case of $\{S_i\}_{i=1}^N$ being semi−compact is similar to prove the above case. Thus, $p \in D = \Big(\bigcap_{i=1}^{N} F(T_i)\Big) \bigcap \Big(\bigcap_{i=1}^{N} F(S_i)\Big)$. By continuity of $x \mapsto d(x, D)$, we obtain

$$\lim_{j\to\infty} d(x_{n_j}, D) = d(p, D) = 0,$$
$$\lim_{k\to\infty} d(x_{n_k}, D) = d(p, D) = 0.$$

It follows by Lemma 3.3(ii) that $\lim_{n\to\infty} d(x_n, D) = 0$. Hence, by Lemma 4.1, we have $\{x_n\}$ converges to common fixed point of the family $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$. □

Remark 4.4. *Theorem* 4.3 *generalizes main results of Theorem* 3.7 *in* [2].

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