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The Distributional Products by the Laurent Series

C. K. Li and M. A. Aguirre

Abstract : Applying the following formulas

$$(x - i0)^{-k} = x^{-k} + i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x)$$

and

$$\lim_{\lambda \to -s} \frac{x_-^\lambda}{\Gamma(\lambda+1)} = (-1)^{s-1} \delta^{(s-1)}(x)$$

due to Gel'fand, we evaluate the distributional product $H(x) \cdot \delta^{(k)}(x)$ for $k = 0, 1, 2, \ldots$ and hence we are able to derive $\delta^{(m)}(x) \cdot \delta^{(l)}(x)$ by induction.

Furthermore, using the Laurent series of x_+^λ and r^λ, we directly compute the products $x_+^{-k} \cdot \delta^{(p)}(x)$ of one variable and $r^{-n-2m} \cdot \delta^{(2s)}(r)$ of n variables. Finally, we imply $x_+^{-m} \cdot x_+^{-l} = x_+^{-m-l}$ by Fisher's result, where

$$x_+^{-m} = \lim_{\lambda \to -m} \frac{\partial}{\partial \lambda} [(\lambda + m) x_+^\lambda],$$

the regular part of the Laurent expansion of x_+^λ about $\lambda = -m$.

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1 Introduction

One of the well-known problems in the theory of generalized functions is the lack of definitions for products and convolutions of distributions in general, although they are in great demand for quantum field theory. In elementary particle physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions [1]. The sequential method (Antosik, Mikusiński, and Sikorski 1972 [2]) and complex analysis approach (Bremermann 1965 [3]), including non-standard analysis, have been the main tools in dealing with products, powers and convolutions of distributions. Fisher, with his collaborators [4-8], has actively used Jone's δ-sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, where $\rho(x)$ is a fixed infinitely differentiable function on $\mathbb{R}$ with four properties :

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)dx = 1$,

and Van der Corput's neutrix limit (in order to abandon unwanted infinite quantities from asymptotic expansions) to deduce numerous products, powers, convolutions and compositions of distributions on $\mathbb{R}$ since 1969. One of Fisher's definitions for multiplication is given as follows [5] :

Definition 1.1 Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty}(fg_n, \phi) = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [9]) having domain $N' = \{1, 2, \cdots\}$ and range the real numbers, with negligible functions that are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \;\; \ln^r n \quad (\lambda > 0, \; r = 1, 2, \cdots)$$

and all functions of n that converge to zero in the normal sense as n tends to infinity.

The following theorem due to Fisher [5] is easily proved from the above definition.

Theorem 1.1 Let f and g be distributions in $\mathcal{D}'$ and suppose that the non-commutative neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist. Then the product $f' \circ g$ (or $f \circ g'$) exists and

$$(f \circ g)' = f' \circ g + f \circ g'.$$

It is obvious to see that the product of definition 1 is not symmetric and hence $f \circ g \neq g \circ f$ in general. Furthermore, such products are dependent on the choice of function $\rho(x)$, which, unfortunately, does not seem a property of distributional products [8].

To extend multiplications from one-dimensional to m-dimensional, Li [10-11] constructed several workable δ-sequences on R^m for non-commutative neutrix products such as $r^{-k} \circ \nabla\delta$ as well as $r^{-k} \circ \triangle^l \delta$, where $\triangle$ denotes the Laplacian. Aguirre [12] used the Laurent series expansion of r^λ and derived a more general (natural) product $r^{-k} \cdot \nabla(\triangle^l \delta)$ by calculating the residue of r^λ. His approach is an interesting example of using complex analysis to obtain products of distribution on R^m [13-16].

The objective of this paper is to obtain natural products of distributions with the help of Laurent series and some of Gel'fand's identities [17]. In particular, we will employ the following two expansions.

$$x_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!(\lambda+k)}\delta^{(k-1)}(x) + x_+^{-k} + (\lambda+k)x_+^{-k}\ln x_+ + \cdots$$

and

$$r^\lambda = \frac{a_{-1}}{\lambda+n+2m} + a_0 + a_1(\lambda+n+2m) + \cdots .$$

This approach is not only simpler than the sequential method but also without recourse to Van der Corput's neutrix limit nor to a delta sequence (an idea that requires a bit more machinery).

2 The Product $\delta^{(m)}(x) \cdot \delta^{(l)}(x)$

In the following, let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. We define the locally summable functions x_+^λ and x_-^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad x_-^\lambda = \begin{cases} |x|^\lambda & \text{if } x < 0 \\ 0 & \text{if } x > 0. \end{cases}$$

The distributions x_+^λ and x_-^λ are then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1} \quad \text{and} \quad (x_-^\lambda)' = -\lambda x_-^{\lambda-1}.$$

It follows that if r is a positive integer and $-r-1 < \lambda < -r$, then

$$(x_+^\lambda,\ \phi(x)) = \int_0^\infty x^\lambda \left[\phi(x) - \sum_{i=0}^{r-1} \frac{\phi^{(i)}(0)}{i!}x^i\right] dx \quad \text{and}$$
$$(x_-^\lambda,\ \phi(x)) = \int_{-\infty}^0 |x|^\lambda \left[\phi(x) - \sum_{i=0}^{r-1} \frac{\phi^{(i)}(0)}{i!}x^i\right] dx.$$

Lemma 2.1 *The products* $x_+^\lambda \cdot x_-^{-k-\lambda}$ *and* $x_-^\lambda \cdot x_+^{-k-\lambda}$ *exist and*

$$x_+^\lambda \cdot x_-^{-k-\lambda} = -\frac{\pi\ cosec\lambda\pi}{2(k-1)!}\,\delta^{(k-1)}(x), \tag{2.1}$$

$$x_-^\lambda \cdot x_+^{-k-\lambda} = \frac{(-1)^k\ \pi\ cosec\lambda\pi}{2(k-1)!}\,\delta^{(k-1)}(x) \tag{2.2}$$

where $\lambda \neq 0, \pm1, \pm2, \ldots$ *and* $k = 1, 2, \ldots$.

Proof. The following two formulas can be found in [17]

$$(x - i0)^{-k} = x^{-k} + i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x)$$

and

$$(x - i0)^{\lambda} = x_+^{\lambda} + e^{-\lambda \pi i} x_-^{\lambda}.$$

Furthermore, $(x - i0)^{\lambda}$ is an entire function of λ.

Using the following Gel'fand's identities [17]

$$\begin{aligned} x_+^{\lambda} &= \frac{(-1)^{n-1}}{(n-1)!(\lambda+n)} \delta^{(n-1)}(x) + F_{-n}(x_+, \lambda), \\ x_-^{\lambda} &= \frac{1}{(n-1)!(\lambda+n)} \delta^{(n-1)}(x) + F_{-n}(x_-, \lambda), \\ e^{\pm i\lambda\pi} &= (-1)^n[1 \pm (\lambda+n)\pi + \cdots] \end{aligned}$$

where $F_{-n}(x_+, \lambda)$ and $F_{-n}(x_-, \lambda)$ are the regular parts of the Laurent expansions of x_+^{λ} and x_-^{λ} respectively, we arrive at

$$\begin{aligned} \lim_{\lambda \to -k} (x - i0)^{\lambda} &= \lim_{\lambda \to -k} (x_+^{\lambda} + e^{-\lambda \pi i} x_-^{\lambda}) \\ &= x^{-k} + i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x) \\ &= (x - i0)^{-k}. \end{aligned}$$

It follows that

$$(x - i0)^{\lambda} \cdot (x - i0)^{\mu} = (x - i0)^{\lambda+\mu}.$$

In particular, we have

$$(x - i0)^{\lambda} \cdot (x - i0)^{-\lambda-k} = (x - i0)^{-k}$$

by letting $\mu \to -\lambda - k$.

Hence, we come to

$$\begin{aligned} x^{-k} + i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x) &= (x - i0)^{-k} \\ &= (x - i0)^{\lambda} (x - i0)^{-\lambda-k} \\ &= (x_+^{\lambda} + e^{-\lambda\pi i} x_-^{\lambda})(x_+^{-\lambda-k} + e^{(\lambda+k)\pi i} x_-^{-\lambda-k}) \\ &= [x_+^{-k} + (-1)^k x_-^{-k}] + [(-1)^k x_+^{\lambda} \cdot x_-^{-\lambda-k} + x_-^{\lambda} \cdot x_+^{-\lambda-k}] \cos \lambda\pi \\ &\quad + i[(-1)^k x_+^{\lambda} \cdot x_-^{-\lambda-k} - x_-^{\lambda} \cdot x_+^{-\lambda-k}] \sin \lambda\pi \end{aligned}$$

which clearly implies

$$x^{-k} = x_+^{-k} + (-1)^k x_-^{-k},$$

$$(-1)^k x_+^\lambda \cdot x_-^{-\lambda-k} + x_-^\lambda \cdot x_+^{-\lambda-k} = 0,$$

and

$$[(-1)^k x_+^\lambda \cdot x_-^{-\lambda-k} - x_-^\lambda \cdot x_+^{-\lambda-k}] \sin \lambda\pi = \pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x).$$

Therefore, we obtain

$$2(-1)^k x_+^\lambda \cdot x_-^{-\lambda-k} \sin \lambda\pi = \pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(x).$$

This completes the proof of equation (2.1), and equation (2.2) follows from

$$x_-^\lambda \cdot x_+^{-\lambda-k} = (-1)^{k+1} x_+^\lambda \cdot x_-^{-\lambda-k} = \frac{(-1)^k \, \pi \, \mathrm{cosec}\lambda\pi}{2(k-1)!} \, \delta^{(k-1)}(x).$$

□

Remark : Equation (1) was first obtained by Fisher in [5] with a much longer and more complex proof by Definition 1.1. His work was based on the δ-sequence and the neutrix limit.

Lemma 2.2 *The products $x_+^r \cdot \delta^{(r+k-1)}(x)$ and $\delta^{(r+k-1)}(x) \cdot x_+^r$ exist and*

$$x_+^r \cdot \delta^{(r+k-1)}(x) = \delta^{(r+k-1)}(x) \cdot x_+^r = \frac{(-1)^r (r+k-1)!}{2(k-1)!} \delta^{(k-1)}(x)$$

for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$. In particular, we have

$$H(x) \cdot \delta^{(k-1)}(x) = \frac{\delta^{(k-1)}(x)}{2}.$$

Proof. Let s be a positive integer. By the following two identities

$$\frac{\pi}{\sin \lambda\pi} = \frac{\Gamma(1-\lambda)\,\Gamma(1+\lambda)}{\lambda}$$

if λ is near $-s$ and $\lambda \neq -s$,

$$\lim_{\lambda \to -s} \frac{x_-^\lambda}{\Gamma(\lambda+1)} = (-1)^{s-1} \delta^{(s-1)}(x)$$

as well as equation (2), we have

$$(-1)^{s-1} \delta^{(s-1)}(x) \cdot x_+^{s-k} = \frac{(-1)^{k+1}\,(s-1)!}{2(k-1)!} \delta^{(k-1)}(x)$$

It follows from setting $r = s - k$ that

$$\delta^{(r+k-1)}(x) \cdot x_+^r = \frac{(-1)^r (r+k-1)!}{2(k-1)!} \delta^{(k-1)}(x).$$

With a very similar argument, we can show that

$$x_+^r \cdot \delta^{(r+k-1)}(x) = \frac{(-1)^r (r+k-1)!}{2(k-1)!}\delta^{(k-1)}(x).$$

This completes the proof of Lemma 2.2. □

Theorem 2.1 *The product $\delta^{(m)}(x) \cdot \delta^{(l)}(x)$ exists and*

$$\delta^{(m)}(x) \cdot \delta^{(l)}(x) = 0$$

for m, $l = 0, 1, 2, \ldots$.

Proof. From Lemma 2.2, we have $H(x) \cdot \delta(x) = \delta(x)/2$ and differentiate both sides to get

$$\delta(x) \cdot \delta(x) + H(x) \cdot \delta'(x) = \frac{1}{2}\delta'(x)$$

which shows that $\delta^2(x) = 0$. Similarly we can derive $\delta(x) \cdot \delta'(x) = \delta'(x) \cdot \delta(x) = 0$ by noting that

$$\begin{aligned}
&\delta'(x) \cdot \delta(x) + \delta(x) \cdot \delta'(x) = 0, \\
&H(x) \cdot \delta'(x) = \frac{1}{2}\delta'(x), \\
&\delta(x) \cdot \delta'(x) + H(x) \cdot \delta''(x) = \frac{1}{2}\delta''(x).
\end{aligned}$$

The theorem obviously follows by induction. □

Remark 1 : Li used a different approach [18] to show that $\delta^2(x) = 0$ for $x \in R$ by applying the Hilbert transform

$$\phi(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\phi(t)}{t-z}\, dt,$$

where $\phi \in \mathcal{D}(R)$ and Im $z > 0$.

From Cauchy's representation of distribution, we have

$$\begin{aligned}
(\delta^2(x),\ \phi(x)) &= \lim_{\epsilon \to 0^+} \mathrm{Re}(\delta^2(z - i\epsilon), \phi(z)) \\
&\triangleq \lim_{\epsilon \to 0^+} \mathrm{Re}\frac{1}{(2\pi i)^2}\oint_{|z-i\epsilon|=\frac{\epsilon}{2}} \frac{\phi(z)}{(z-i\epsilon)^2} dz.
\end{aligned}$$

By Cauchy's integral formula, we come to

$$(\delta^2(x),\ \phi(x)) = \lim_{\epsilon \to 0^+} \mathrm{Re}\frac{1}{2\pi i}\frac{\phi'(i\epsilon)}{(2-1)!} = \mathrm{Re}\frac{1}{2\pi i}\phi'(0) = 0.$$

Therefore $\delta^2(x) = 0$.

Remark 2 : Using the normalization, Güttinger [20] obtained the product of improper operators

$$\delta^{(r)} \cdot H(x) = \sum_{i=0}^{r} b_i \delta^{(r-i)}(x),$$

where $b_0, b_1, \ldots, b_r$ are arbitrary constants. Furthermore, he derived a similar type of product of $\delta^{(m)}(x) \cdot \delta^{(l)}(x)$ under certain conditions.

3 The Product $x_+^{-k} \cdot \delta^{(p)}(x)$

Theorem 3.1 *The product $x_+^{-k} \cdot \delta^{(p)}(x)$ exists and*

$$x_+^{-k} \cdot \delta^{(p)}(x) = \frac{(-1)^k p!}{2(p+k)!} \delta^{(k+p)}(x)$$

for $p = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$. In particular, we have $H(x) \cdot \delta^{(p)}(x) = \delta^{(p)}(x)/2$ again by letting $k \to 0$.

Proof. From the Laurent series of x_+^λ

$$x_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!(\lambda+k)} \delta^{(k-1)}(x) + x_+^{-k} + (\lambda+k) x_+^{-k} \ln x_+ + \cdots,$$

we have

$$\delta^{(p-1)}(x) = (-1)^{p-1}(p-1)! \lim_{\mu \to -p} (\mu+p)\, x_+^\mu.$$

We define distribution x_+^{-k} as the regular part of the Laurent expansion of x_+^λ about $\lambda = -k$ by

$$x_+^{-k} = \lim_{\lambda \to -k} \frac{\partial}{\partial \lambda} [(\lambda+k)\, x_+^\lambda].$$

Hence, we get

$$x_+^{-k} \cdot \delta^{(p-1)}(x) = (-1)^{p-1}(p-1)! \lim_{\lambda \to -k} \lim_{\mu \to -p} \frac{\partial}{\partial \lambda} \Big[(\lambda+k)\,(\mu+p)\, x_+^{\lambda+\mu} \Big],$$

due to the facts x_+^λ, x_+^μ and $x_+^{\lambda+\mu}$ are analytic for $\lambda,\ \mu,\ \lambda+\mu \neq -1, -2, \ldots$ and

$$x_+^{\lambda+\mu} = x_+^\lambda \cdot x_+^\mu.$$

Obviously,

$$(\lambda+k)\,(\mu+p) = \frac{1}{2}[(\lambda+k+\mu+p)^2 - (\lambda+k)^2 - (\mu+p)^2].$$

It follows that

$$\begin{aligned} x_+^{-k}\cdot\delta^{(p-1)}(x) &= \frac{(-1)^{p-1}(p-1)!}{2}\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\Big[(\lambda+k+\mu+p)^2x_+^{\lambda+\mu}\Big] \\ &\quad -\frac{(-1)^{p-1}(p-1)!}{2}\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\Big[(\lambda+k)^2x_+^{\lambda+\mu}\Big] \\ &\quad -\frac{(-1)^{p-1}(p-1)!}{2}\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\Big[(\mu+p)^2x_+^{\lambda+\mu}\Big] \\ &= I_1+I_2+I_3. \end{aligned}$$

Since

$$x_+^{\lambda+\mu}=\frac{(-1)^{k+p-1}}{(k+p-1)!\,(\lambda+\mu+k+p)}\,\delta^{(k+p-1)}(x)+x_+^{-k-p}+\cdots,$$

we have

$$\begin{aligned} I_1 &= \frac{(-1)^{p-1}(p-1)!}{2}\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\Big[(\lambda+k+\mu+p)^2 \\ &\quad \cdot\frac{(-1)^{k+p-1}}{(k+p-1)!\,(\lambda+\mu+k+p)}\,\delta^{(k+p-1)}(x)+\cdots\Big] \\ &= \frac{(-1)^k(p-1)!}{2(p+k-1)!}\,\delta^{(k+p-1)}(x). \end{aligned}$$

As for I_2, we arrive at

$$\begin{aligned} I_2 &= -\frac{(-1)^{p-1}(p-1)!}{2}\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\Big[(\lambda+k)^2 \\ &\quad \cdot\frac{(-1)^{k+p-1}}{(k+p-1)!\,(\lambda+\mu+k+p)}\,\delta^{(k+p-1)}(x)+\cdots\Big] \\ &= -\frac{(-1)^k(p-1)!}{2(p+k-1)!}\,\delta^{(k+p-1)}(x). \end{aligned}$$

due to the following fact

$$\lim_{\lambda\to -k}\lim_{\mu\to -p}\frac{\partial}{\partial\lambda}\left\{\frac{(\lambda+k)^2}{\lambda+\mu+k+p}\right\}=1.$$

With a very similar argument, we can show that

$$I_3=\frac{(-1)^k(p-1)!}{2(p+k-1)!}\,\delta^{(k+p-1)}(x).$$

Replacing $p-1$ by p, we complete the proof of Theorem 3.1. □

Remark 1 : It is easy to derive the products $x_-^{-k}\cdot\delta^{(p)}(x)$ and $x^{-k}\cdot\delta^{(p)}(x)$ from our result, where $x^{-k}=x_+^{-k}+(-1)^k x_-^{-k}$. We leave them for interested readers.

Remark 2 : One may attempt to construct the following entire function

$$f(x_+^\lambda) = \begin{cases} \frac{x_+^\lambda}{\Gamma(\lambda+1)}, & \lambda \neq -1, -2, \dots, \\ \delta^{(-\lambda-1)}, & \lambda = -1, -2, \dots. \end{cases}$$

Then

$$f(x_+^\lambda)f(x_+^\mu) = \frac{\Gamma(\lambda+\mu+1)f(x_+^{\lambda+\mu})}{\Gamma(\lambda+1)\Gamma(\mu+1)},$$

which can be used to show Theorem 2.1. Indeed, we have

$$\begin{aligned} \delta^{(m)}(x) \cdot \delta^{(l)}(x) &= \lim_{\lambda \to -m-1} \frac{x_+^\lambda}{\Gamma(\lambda+1)} \cdot \lim_{\mu \to -l-1} \frac{x_+^\mu}{\Gamma(\mu+1)} \\ &= \lim_{\lambda \to -m-1} \lim_{\mu \to -l-1} \frac{x_+^\lambda}{\Gamma(\lambda+1)} \cdot \frac{x_+^\mu}{\Gamma(\mu+1)} \\ &= \lim_{\lambda \to -m-1} \lim_{\mu \to -l-1} \frac{\Gamma(\lambda+\mu+1)x_+^{\lambda+\mu}}{\Gamma(\mu+1)\Gamma(\lambda+1)\Gamma(\lambda+\mu+1)} = 0, \end{aligned}$$

since

$$\lim_{\mu \to -l-1} \frac{1}{\Gamma(\mu+1)} = 0$$

and the limit of other terms exists by using the formula

$$\frac{\Gamma(z)}{\Gamma(z-n)} = \frac{(-1)^n\Gamma(-z+n+1)}{\Gamma(1-z)}.$$

However, it is impossible to obtain Lemma 2.2 and Theorem 3.1 along the same line because it does not produce a term x_+^r or x_+^{-k} at all.

4 The Products $r^{-2m-n} \cdot \delta^{(2s)}(r)$ and $r^{-2m} \cdot \delta^{(2s)}(r)$

Theorem 4.1 *The product* $r^{-2m-n} \cdot \delta^{(2s)}(r)$ *exists and*

$$r^{-2m-n} \cdot \delta^{(2s)}(r) = 0$$

for $s,\ m = 0, 1, 2, \dots$ *and* $n = 1, 2, \dots$.

Proof. From the Laurent series of r^λ

$$r^\lambda = \frac{a_{-1}}{\lambda+n+2m} + a_0 + a_1(\lambda+n+2m) + \cdots$$

where $a_{-1} = \dfrac{\Omega_n\delta^{(2m)}(r)}{(2m)!}$, $a_0 = \Omega_n r^{-2m-n}$ and $a_1 = \Omega_n r^{-2m-n}\ln r$ (Ω_n is the hypersurface area of the unit sphere).

The distribution r^{-2m-n} as the regular part of the Laurent expansion of r^{λ} about $\lambda = -n-2m$ is defined by

$$r^{-2m-n} = \frac{1}{\Omega_n} \lim_{\lambda \to -n-2m} \frac{\partial}{\partial \lambda} \left[(\lambda+n+2m)\, r^{\lambda} \right]. \tag{4.1}$$

Clearly, for $s = 0, 1, \ldots$, we have

$$\delta^{(2s)}(r) = \frac{(2m)!}{\Omega_n} \lim_{\mu \to -n-2s} \left[(\mu+n+2s)\, r^{\mu} \right] \tag{4.2}$$

from the the Laurent series of r^{μ}.
It follows that

$$\begin{aligned} & r^{-2m-n} \cdot \delta^{(2s)}(r) \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left[(\lambda+n+2m)\, r^{\lambda} \cdot (\mu+n+2s)\, r^{\mu} \right] \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left[(\lambda+n+2m)(\mu+n+2s)\, r^{\lambda+\mu} \right]. \end{aligned}$$

Applying the following two identities

$$\begin{aligned} & (\lambda+n+2m)(\mu+n+2s) \\ &= \frac{1}{2} \left\{ (\lambda+\mu+n+2m+n+2s)^2 - (\lambda+n+2m)^2 - (\mu+n+2s)^2 \right\}, \\ & r^{\lambda+\mu} = \frac{b_{-1}}{\lambda+\mu+n+2m+2s} + b_0 + b_1(\lambda+\mu+n+2m+2s) + \cdots, \end{aligned}$$

we come to

$$\begin{aligned} & r^{-2m-n} \cdot \delta^{(2s)}(r) \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda+\mu+n+2m+n+2s)^2}{2(\lambda+\mu+n+2m+2s)}\, b_{-1} + \cdots \right\} \\ & -\frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda+n+2m)^2}{2(\lambda+\mu+n+2m+2s)}\, b_{-1} + \cdots \right\} \\ & -\frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\mu+n+2s)^2}{2(\lambda+\mu+n+2m+2s)}\, b_{-1} + \cdots \right\} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By direct computation, we obtain

$$\lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda+\mu+n+2m+n+2s)^2}{2(\lambda+\mu+n+2m+2s)}\, b_{-1} \right\} = 0 \tag{4.3}$$

and the rest in I_1 is zero since there is only one n in the denominators after taking the partial derivative, which will never vanish after the two limits, while all numerators disappear.

Similarly, we get $I_2 = I_3 = 0$. This completes the proof of theorem 4.1. □

Theorem 4.2 *The product* $r^{-2m} \cdot \delta^{(2s)}(r)$ *exists and*

$$r^{-2m} \cdot \delta^{(2s)}(r) = \frac{(2m)!}{2\Omega_n(2m+2s)!}\,\delta^{(2m+2s)}(r)$$

for $m,\ s = 0, 1, 2, \ldots$ *and* $n = 1, 2, \ldots$.

Proof. By equations (4.1) and (4.2), we have

$$\begin{aligned} &r^{-2m} \cdot \delta^{(2s)}(r) = r^n r^{-2m-n} \cdot \delta^{(2s)}(r) \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} r^n \frac{\partial}{\partial \lambda} \left[(\lambda + n + 2m)\, r^{\lambda} \cdot (\mu + n + 2s)\, r^{\mu} \right] \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left[(\lambda + n + 2m)(\mu + n + 2s)\, r^{\lambda+\mu+n} \right]. \end{aligned}$$

Using the following two identities

$$\begin{aligned} &(\lambda + n + 2m)(\mu + n + 2s) \\ &\qquad = \frac{1}{2}\left\{ (\lambda + \mu + n + 2m + n + 2s)^2 - (\lambda + n + 2m)^2 - (\mu + n + 2s)^2 \right\}, \end{aligned}$$

$$r^{\lambda+\mu+n} = \frac{c_{-1}}{\lambda + \mu + n + n + 2m + 2s} + c_0 + c_1(\lambda + \mu + n + n + 2m + 2s) + \cdots,$$

where

$$c_{-1} = \frac{\Omega_n \delta^{(2m+2s)}(r)}{(2m+2s)!} \quad \text{and } c_0 = \Omega_n r^{-n-2m-2s}.$$

We infer

$$\begin{aligned} &r^{-2m} \cdot \delta^{(2s)}(r) \\ &= \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda + \mu + n + 2m + n + 2s)^2}{2(\lambda + \mu + n + n + 2m + 2s)}\, c_{-1} + \cdots \right\} \\ &- \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda + n + 2m)^2}{2(\lambda + \mu + n + n + 2m + 2s)}\, c_{-1} + \cdots \right\} \\ &- \frac{(2m)!}{\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\mu + n + 2s)^2}{2(\lambda + \mu + n + n + 2m + 2s)}\, c_{-1} + \cdots \right\} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By direct computation, we obtain

$$\begin{aligned} &\frac{(2m)!}{2\Omega_n^2} \lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda + \mu + n + 2m + n + 2s)^2}{(\lambda + \mu + n + 2m + n + 2s)}\, c_{-1} \right\} \qquad (4.4) \\ &= \frac{(2m)!}{2\Omega_n^2}\, c_{-1} \end{aligned}$$

and the rest in I_1 is zero. Notice that there is a $2n$ term in the denominator of equation (6), while there is only one n in the denominator of equation (5).

As for I_2, we can see that

$$\lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\lambda + n + 2m)^2}{\lambda + \mu + n + 2m + n + 2s} \right\} = 1$$

which clearly allows us to deduce

$$I_2 = -\frac{(2m)!}{2\Omega_n^2} c_{-1}$$

as the other terms in I_2 are zero after the partial derivative and the two limits.

Similarly, we have

$$\lim_{\lambda \to -n-2m} \lim_{\mu \to -n-2s} \frac{\partial}{\partial \lambda} \left\{ \frac{(\mu + n + 2s)^2}{\lambda + \mu + n + 2m + n + 2s} \right\} = -1$$

which claims

$$I_3 = -\frac{(2m)!}{2\Omega_n^2} (-c_{-1}) = \frac{(2m)!}{2\Omega_n^2} c_{-1}.$$

This completes the proof of theorem 4.2. □

5 The Product $x_+^{-m} \cdot x_+^{-l}$

In 2004, Fisher and Taş [19] proved the following theorem by definition 1.1 and theorem 1.1.

Theorem 5.1 *The non-commutative neutrix products of $x_+^\lambda \ln^p x_+$ and $x_+^\mu \ln^q x_+$ and of $x_-^\lambda \ln^p x_-$ and $x_-^\mu \ln^q x_-$ exist and*

$$x_+^\lambda \ln^p x_+ \circ x_+^\mu \ln^q x_+ = x_+^{\lambda+\mu} \ln^{p+q} x_+$$
$$x_-^\lambda \ln^p x_- \circ x_-^\mu \ln^q x_- = x_-^{\lambda+\mu} \ln^{p+q} x_-$$

for $\lambda + \mu < -1$ and $\lambda,\ \mu,\ \lambda + \mu \neq -1, -2, -3, \cdots$.

The key step to show the theorem in their work is to derive

$$x_+^\lambda \circ x_+^\mu = x_+^{\lambda+\mu} \tag{5.1}$$

for $\lambda + \mu < -1$ and $\lambda,\ \mu,\ \lambda + \mu \neq -1, -2, -3, \cdots$, based on the δ-sequence and the neutrix limit. However, it seems infeasible to compute the product x_+^{-m} and x_+^{-l} along the same line because of the singularity of x_+^{-m} and x_+^{-l}. In this section, we make use of the Laurent series of $x_+^{\lambda+\mu}$ to show the following theorem.

Theorem 5.2 *The product $x_+^{-m} \cdot x_+^{-l}$ exists and $x_+^{-m} \cdot x_+^{-l} = x_+^{-m-l}$ for $m,\ l = 1, 2, \ldots$.*

Proof. From the definition, we have

$$x_+^{-m} = \lim_{\lambda \to -m} \frac{\partial}{\partial \lambda} \Big[(\lambda + m) x_+^{\lambda}\Big], \quad x_+^{-l} = \lim_{\mu \to -l} \frac{\partial}{\partial \mu} \Big[(\mu + l) x_+^{\mu}\Big].$$

By equation (7)

$$x_+^{-m} \cdot x_+^{-l} = \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial}{\partial \lambda} \left[\frac{\partial}{\partial \mu} (\lambda + m)(\mu + l) x_+^{\lambda + \mu} \right].$$

Using the Laurent series

$$\begin{aligned} x_+^{\lambda+\mu} &= \frac{(-1)^{m+l-1}}{(m+l-1)!\,(\lambda+\mu+m+l)}\, \delta^{(m+l-1)}(x) + x_+^{-m-l} + \cdots, \\ &= A_{-1} + A_0 + \cdots, \end{aligned}$$

and the identity

$$(\lambda + m)\,(\mu + l) = \frac{1}{2}[(\lambda + m + \mu + l)^2 - (\lambda + m)^2 - (\mu + l)^2],$$

we come to

$$\begin{aligned} & x_+^{-m} \cdot x_+^{-l} = \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial}{\partial \lambda} \left[\frac{\partial}{\partial \mu} (\lambda + m)(\mu + l) x_+^{\lambda + \mu} \right] \\ & = \frac{1}{2} \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial^2}{\partial \lambda \partial \mu} \Big[(\lambda + m + \mu + l)^2 (A_{-1} + A_0 + \cdots) \Big] \\ & -\frac{1}{2} \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial^2}{\partial \lambda \partial \mu} \Big[(\lambda + m)^2 (A_{-1} + A_0 + \cdots) \Big] \\ & -\frac{1}{2} \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial^2}{\partial \lambda \partial \mu} \Big[(\mu + l)^2 (A_{-1} + A_0 + \cdots) \Big] \\ & = I_1 + I_2 + I_3. \end{aligned}$$

Obviously,

$$\lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial^2}{\partial \lambda \partial \mu} \left[(\lambda + m + \mu + l)^2 \frac{(-1)^{m+l-1}}{(m+l-1)!\,(\lambda + \mu + m + l)} \right] = 0,$$

and

$$\frac{1}{2} \lim_{\lambda \to -m} \lim_{\mu \to -l} \frac{\partial^2}{\partial \lambda \partial \mu} (\lambda + m + \mu + l)^2 = 1.$$

We can show that

$$I_1 = x_+^{-m-l}.$$

With a very similar technique as above, we can reach that

$$I_2 = I_3 = 0.$$

This completes the proof of theorem 5.2. □

To end this paper, we would like to mention that one can derive more natural products of distribution of one or n variables by following our approach, which is simpler than the sequential method.

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C. K. Li
Department of Mathematics and Computer Science
Brandon University
Brandon, Manitoba, Canada R7A 6A9.
e-mail : lic@brandonu.ca

Manuel A. Aguirre T.
Núcleo Consolidado Matemática Pura y Aplicada
Facultad de Ciencias Exactas
Paraje Arroyo Seco,Campus Universitario
Universidad nacional del Centro
Tandil, Argentina.
e-mail : maguirre@exa.unicen.edu.ar