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Intuitionistic Fuzzy Hyperideal Extensions of Semihypergroups

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Abstract : The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, using Atanassov idea, we introduce the concepts of intuitionistic fuzzy hyperideal extension of semihypergroups and intuitionistic fuzzy prime (semiprime) hyperideal of semihypergroup. We discuss the properties of them and study the relationship between prime (semiprime) hyperideals and intuitionistic fuzzy prime (semiprime) hyperideals by means of the extensions of intuitionistic fuzzy hyperideals of semihypergroup.

Keywords : semihypergroup; fuzzy set; intuitionistic fuzzy set; hyperideal; fuzzy hyperideal.
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1 Introduction and Preliminaries

Hyperstructure theory was born in 1934 when Marty [1] defined hypergroups, began to analysis their properties and applied them to groups, rational algebraic functions. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties. In 1965, Zadeh [2]

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introduced the notion of a fuzzy subset of a non-empty set X, as a function from X to $[0, 1]$. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. In 1971, Rosenfeld [3] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. A recent books [4, 5] contains a wealth of applications. In [6], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup, also see [7–10]. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. As an important generalization of the notion of fuzzy sets on a non-empty set X, in 1983, Atanassov introduced in [11] the concept of intuitionistic fuzzy sets on a non-empty set X which give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must be smaller than or equal to 1. The relations between intuitionistic fuzzy sets and algebraic structures have been already considered by many mathematicians. In [12], using Atanassov idea, Davvaz established the intuitionistic fuzzification of the concept of hyperideals in a semihypergroup and investigated some of their properties. Also, see [13–21].

In this paper, using Atanassov idea, we introduce the concepts of intuitionistic fuzzy hyperideal extension of semihypergroups and intuitionistic fuzzy prime (semiprime) hyperideal of semihypergroup. We discuss the properties of them and study the relationship between prime (semiprime) hyperideals and intuitionistic fuzzy prime (semiprime) hyperideals by means of the extensions of intuitionistic fuzzy hyperideals of semihypergroup. Since intuitionistic fuzzy sets theory is a generalization of fuzzy sets theory, the fuzzy sets can be seen as a special situation of the intuitionistic fuzzy sets.

Recall first the basic terms and definitions from the hyperstructure theory.

Definition 1.1. A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called *hyperoperation* or *join operation* on the set H, where H is a non-empty set and $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of H.

Definition 1.2. A *hyperstructure* is called the pair $(H, \circ)$ where $\circ$ is a hyperoperation on the set H.

Definition 1.3. A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are non-empty subsets of H then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \circ B.$$

Definition 1.4. A non-empty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$ and H is called in this case *super-semihypergroup* of B.

Definition 1.5. A non-empty subset I of a semihypergroup H is called a *right* (*left*) *ideal* of H if for all $x \in H$ and $r \in I$,

$$r \circ x \subseteq I (x \circ r \subseteq I).$$

A non-empty subset I of H is called a *hyperideal* (or *two-sided hyperideal*) if it is both a left hyperideal and right hyperideal.

Atanassov introduced in [11,22] the concept of intuitionistic fuzzy sets defined on a non-empty set X as objects having the form:

$$A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \,|x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Obviously, each ordinary fuzzy set may be written as

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \,|x \in X\}.$$

Let A and B be two intuitionistic fuzzy sets on X. The following expressions are defined in [22,23].

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$.
2. $A = B$ if and only if if $A \subseteq B$ and $B \subseteq A$.
3. $A^c = \{\langle x, \lambda_A(x), \mu_A(x)\rangle \,|x \in X\}$.
4. $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}\rangle \,|x \in X\}$.
5. $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}\rangle \,|x \in X\}$.
6. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \,|x \in X\}$.
7. $\Diamond A = \{\langle x, 1 - \lambda_A(x), \lambda_A(x)\rangle \,|x \in X\}$.

For the sake of simplicity, we use the symbol $A = (\mu_A, \lambda_A)$ for intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \,|x \in X\}$. $Im(\mu_A)$ denote the image set of μ_A and similarly, $Im(\lambda_A)$ denote the image set of λ_A.

For a non-empty family $\{A_i = (\mu_{A_i}, \lambda_{A_i}) : i \in I\}$ of intuitionistic fuzzy subsets of X, the $\inf A_i = \{(\inf \mu_{A_i}, \sup \lambda_{A_i}) : i \in I\}$ and the $\sup A_i = \{(\sup \mu_{A_i}, \inf \lambda_{A_i}) : i \in I\}$ are the intuitionistic fuzzy subsets of X defined by:

$\inf A_i : X \to [0,1], x \to \{(\inf \mu_{A_i}(x), \sup \lambda_{A_i}(x)) : i \in I\}$,

$\sup A_i : X \to [0,1], x \to \{(\sup \mu_{A_i}(x), \inf \lambda_{A_i}(x)) : i \in I\}$, where $\inf \mu_{A_i}(x) = \inf\{\mu_{A_i}(x) : i \in I\}$ and $\sup \lambda_{A_i}(x) = \sup\{\lambda_{A_i}(x) : i \in I\}$ and similarly for $\sup \mu_{A_i}(x)$ and $\inf \lambda_{A_i}(x)$.

For any $t \in [0,1]$ and a fuzzy set μ of a semihypergroup H, the set

$$U(\mu;t)=\{x\in H:\mu(x)\geq t\}\text{ (resp. }L(\mu;t)=\{x\in H:\mu(x)\leq t\})$$

is called an upper (resp. lower) t-level cut of μ.

Definition 1.6. Let A, B be two intuitionistic fuzzy sets in a semihypergroup H. Then,

$$A\subseteq B \text{ if and only if } (\forall x\in H), (\mu_A(x)\leq\mu_B(x)\ \&\ \lambda_A(x)\leq\lambda_B(x)),$$

$$A\cap B=\{\langle x,\min\{\mu_A(x),\mu_B(x)\},\max\{\lambda_A(x),\lambda_B(x)\}\rangle\,|x\in H\},$$

$$A*B=\{\langle x,\mu_{A*B}(x),\lambda_{A*B}\rangle\,|x\in H\},$$

where

$$\mu_{A*B}(x)=\begin{cases}\sup\limits_{x\in y\circ z}\{\min\{\mu_A(y),\mu_B(z)\}\} & \text{if } x\in y\circ z\\ 0 & \text{otherwise,}\end{cases}$$

$$\lambda_{A*B}(x)=\begin{cases}\inf\limits_{x\in y\circ z}\{\max\{\lambda_A(y),\lambda_B(z)\}\} & \text{if } x\in y\circ z\\ 1 & \text{otherwise.}\end{cases}$$

Definition 1.7. Let H be a semihypergroup. An intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in H is called an *intuitionistic fuzzy semihypergroup* in H if for all $x,y\in H$,

$$\inf_{z\in x\circ y}\{\mu_A(z)\}\geq\min\{\mu_A(x),\mu_A(y)\}\text{ and }\sup_{z\in x\circ y}\{\lambda_A(z)\}\leq\max\{\lambda_A(x),\lambda_A(y)\}.$$

Definition 1.8. Let H be a semihypergroup. An intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in H is called a *left (resp. right) intuitionistic fuzzy hyperideal* of H if for all $x,y\in H$,

1. $\mu_A(y)\leq\inf\limits_{z\in x\circ y}\{\mu_A(z)\}$ (resp. $\mu_A(x)\leq\inf\limits_{z\in x\circ y}\{\mu_A(z)\}$).
2. $\sup\limits_{z\in x\circ y}\{\lambda_A(y)\}\leq\lambda_A(y)$ (resp. $\sup\limits_{z\in x\circ y}\{\lambda_A(z)\}\leq\lambda_A(x)$).

An instuitionistic fuzzy set A in H is called an intuitionistic fuzzy two-sided hyperideal of H if it is both an intuitionistic fuzzy left and an intuitionistic right hyperideal of H.

Let H be a semihypergroup. For $x\in H$, we define

$$X_x=\{(y,z)\in H\times H:x\in y\circ z\}.$$

Definition 1.9. Let H be a semihypergroup. An intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in H is called an *intuitionistic fuzzy prime hyperideal* if for all $x,y\in H$,

$$\inf_{t\in x\circ y}\mu_A(t)=\mu_A(x)\vee\mu_A(y)$$

and

$$\sup_{t\in x\circ y} \lambda_A(t) = \lambda_A(x) \vee \lambda_A(y).$$

Definition 1.10. Let H be a semihypergroup. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in H is called an *intuitionistic fuzzy semiprime hyperideal* if for all $x, y \in H$,

$$\inf_{t\in x\circ x} \mu_A(t) \leq \mu_A(x)$$

and

$$\sup_{t\in x\circ x} \lambda_A(t) \geq \lambda_A(x).$$

Definition 1.11. Let H be a semihypergroup. Then, a hyperideal I of H is called

1. *prime* if for the hyperideals A, B of H, $A \circ B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.
2. *semiprime* if for a hyperideal A of H, $A \circ A \subseteq I$ implies that $A \subseteq I$.

Proposition 1.12. *Let H be a semihypergroup and $\emptyset \neq I \subseteq H$. Then, I is a prime hyperideal of H if and only if $X = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy prime hyperideal of H, where $X = (\Phi_I, \Psi_I)$ is the characteristic function of I.*

Proof. Let $x, y \in H$. If $x \circ y \subseteq I$, then $x \in I$ or $y \in I$. Thus $\Phi_I(x) = 1$ or $\Phi_I(y) = 1$. Thus we have

$$\inf_{t\in x\circ y} \Phi_I(t) = 1 = \Phi_I(x) \vee \Phi_I(y)$$

and

$$\sup_{t\in x\circ y} \Psi_I(t) = 0 = \Psi_I(x) \wedge \Psi_I(y).$$

If $x \circ y \notin I$, then $x \notin I$ and $y \notin I$. Thus $\Phi_I(x) = 0$ and $\Phi_I(y) = 0$. Thus we have

$$\inf_{t\in x\circ y} \Phi_I(t) = 0 = \Phi_I(x) \vee \Phi_I(y)$$

and

$$\sup_{t\in x\circ y} \Psi_I(t) = 1 = \Psi_I(x) \wedge \Psi_I(y).$$

Conversely, suppose that $X = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy prime hyperideal of H and $x \circ y \subseteq I$. It follows that

$$1 = \inf_{t\in x\circ y} \Phi_I(t) = \Phi_I(x) \vee \Phi_I(y).$$

Hence, $\Phi_I(x) = 1$ or $\Phi_I(y) = 1$, i.e., $x \in I$ or $y \in I$. Thus, I is prime hyperideal. □

In similiar way it can be proved the following proposition:

Proposition 1.13. *Let H be a semihypergroup and $\emptyset \neq I \subseteq H$. Then, I is a semiprime hyperideal of H if and only if $X = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy semiprime hyperideal of H, where $X = (\Phi_I, \Psi_I)$ is the characteristic function of I.*

2 Intuitionistic Fuzzy Hyperideal Extensions

Definition 2.1. Let H be a semihypergroup, $A = (\mu_A, \lambda_A)$ an intuitionistic fuzzy subset of H and $x \in H$. Then,

$$\langle x, A\rangle (y) = \{(y, \langle x, \mu_A\rangle (y), \langle x, \lambda_A\rangle (y)) : x \in H\}$$

is the intuitionistic fuzzy subset of H, where the function $\langle x, \mu_A\rangle : H \to [0, 1]$ and $\langle x, \lambda_A\rangle : H \to [0, 1]$ defined by $\langle x, \mu_A\rangle (y) = \inf\limits_{t\in x\circ y} \mu_A(t)$ and $\langle x, \lambda_A\rangle (y) = \sup\limits_{t\in x\circ y} \lambda_A(t)$ is called the extension of A by x.

Proposition 2.2. *Let H be a commutative semihypergroup, $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy hyperideal of H and $x \in H$. Then, $\langle x, A\rangle$ is an intuitionistic fuzzy hyperideal of H.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy hyperideal of commutative semihypergroup H and $p, q \in H$. Then,

$$\inf_{t\in p\circ q} \langle x, \mu_A\rangle (t) \geq \inf_{s\in x\circ p\circ q} \mu_A(s) \geq \inf_{k\in x\circ p} \mu_A(k) = \langle x, \mu_A\rangle (p)$$

and

$$\sup_{t\in p\circ q} \langle x, \lambda_A\rangle (t) \leq \sup_{s\in x\circ p\circ q} \lambda_A(s) \leq \sup_{k\in x\circ p} \lambda_A(k) = \langle x, \lambda_A\rangle (p).$$

Thus $\langle x, A\rangle$ is an intuitionistic fuzzy right hyperideal of H. Since H is a commutative semihypergroup, we have $\langle x, A\rangle$ is an intuitionistic fuzzy hyperideal of H. □

Proposition 2.3. *Let H be a commutative semihypergroup, $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy prime hyperideal of H. Then, $\langle x, A\rangle$ is an intuitionistic fuzzy prime hyperideal of H for all $x \in H$.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy prime hyperideal of H. By Proposition 2.2, $\langle x, A\rangle$ is an intuitionistic fuzzy hyperideal of H. Let $y, z \in H$.

Then,

$$\begin{aligned}
\inf_{t\in y\circ z} \langle x,\mu_A\rangle(t) &= \inf_{t\in y\circ z}\inf_{s\in x\circ t}\mu_A(s)\\
&= \inf_{t\in y\circ z}\{\mu_A(x)\vee \inf_{t\in y\circ z}\mu_A(t)\}\\
&= \mu_A(x)\vee \inf_{t\in y\circ z}\mu_A(t)\\
&= \mu_A(x)\vee\mu_A(y)\vee\mu_A(z)\\
&= (\mu_A(z)\vee\mu_A(y))\vee(\mu_A(x)\vee\mu_A(z))\\
&= \inf_{k\in x\circ y}\mu_A(k)\vee \inf_{l\in x\circ z}\mu_A(l)\\
&= \langle x,\mu_A\rangle(y)\vee\langle x,\mu_A\rangle(z)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t\in y\circ z} \langle x,\lambda_A\rangle(t) &= \sup_{t\in y\circ z}\sup_{s\in x\circ t}\lambda_A(s)\\
&= \sup_{t\in y\circ z}\{\lambda_A(x)\wedge \sup_{t\in y\circ z}\lambda_A(t)\}\\
&= \lambda_A(x)\wedge \sup_{t\in y\circ z}\lambda_A(t)\\
&= \lambda_A(x)\wedge\lambda_A(y)\wedge\lambda_A(z)\\
&= (\lambda_A(z)\wedge\lambda_A(y))\wedge(\lambda_A(x)\wedge\lambda_A(z))\\
&= \sup_{k\in x\circ y}\lambda_A(k)\wedge \sup_{l\in x\circ z}\lambda_A(l)\\
&= \langle x,\lambda_A\rangle(y)\wedge\langle x,\lambda_A\rangle(z).
\end{aligned}$$

Hence, by Definition 1.9, $\langle x,A\rangle$ is an intuitionistic fuzzy prime hyperideal of H. □

Definition 2.4. Let H be a semihypergroup and $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy subset of H. Then, we define *supp* $\mu_A=\{x\in H:\mu_A(x)>0\}$ and $inff\ \lambda_A=\{x\in H:\lambda_A(x)<1\}$.

For the sake of simplicity, we denote for all $x\in H$, $x^n=x\circ x\circ x\circ\ldots.\circ x$ (n-copies).

Proposition 2.5. *Let H be a semihypergroup, $A=(\mu_A,\lambda_A)$ an intuitionistic fuzzy hyperideal of H and $x\in H$. The following statements hold true:*

1. $A\subseteq\langle x,A\rangle$.
2. $\langle x^{n+1},A\rangle\subseteq\langle x^{n+2},A\rangle,\forall n\in N$.
3. *If $\mu_A(x)>0$ and $\lambda_A(x)<1$, then supp* $\langle x,\mu_A\rangle=H$ *and* $inff\ \langle x,\lambda_A\rangle=H$.

Proof. (1). Let $y\in H$. Then, since A is an intuitionistic fuzzy hyperideal of H,

$$\langle x,\mu_A\rangle(y)=\inf_{t\in x\circ y}\mu_A(t)\geq\mu_A(y)$$

and

$$\langle x, \lambda_A \rangle (y) = \sup_{t \in x \circ y} \lambda_A(t) \geq \lambda_A(y).$$

Hence, $A \subseteq \langle x, A \rangle$.

(2). We have to prove that

$$\begin{aligned} (\langle x^{n+1}, \mu_A \rangle) &\leq (\langle x^{n+2}, \mu_A \rangle) \text{ and} \\ (\langle x^{n+1}, \lambda_A \rangle) &\geq (\langle x^{n+2}, \lambda_A \rangle). \end{aligned}$$

Now

$$\begin{aligned} (\langle x^{n+2}, \mu_A \rangle)(y) &= \inf_{t \in x^{n+2} \circ y} \mu_A(t) \\ &= \inf_{t \in x \circ x^{n+1} \circ y} \mu_A(t) \\ &\geq \inf_{s \in x^{n+1} \circ y} \mu_A(s) \\ &= \langle x^{n+1}, \mu_A \rangle (y) \end{aligned}$$

and

$$\begin{aligned} (\langle x^{n+2}, \lambda_A \rangle)(y) &= \sup_{t \in x^{n+2} \circ y} \lambda_A(t) \\ &= \sup_{t \in x \circ x^{n+1} \circ y} \lambda_A(t) \\ &\leq \sup_{s \in x^{n+1} \circ y} \lambda_A(s) \\ &= \langle x^{n+1}, \lambda_A \rangle (y). \end{aligned}$$

Hence, $\langle x^{n+1}, A \rangle \subseteq \langle x^{n+2}, A \rangle, \forall n \in N$.

(3). Since $\langle x, A \rangle$ is an intuitionistic fuzzy subset of H, by the definition, $supp\ \langle x, A \rangle \subseteq H$. Let $y \in H$. Since A is an intuitionistic fuzzy hyperideal of H, we have

$$\langle x, \mu_A \rangle (y) = \inf_{t \in x \circ y} \mu_A(t) \geq \mu_A(x) > 0$$

and

$$\langle x, \mu_A \rangle (y) = \inf_{t \in x \circ y} \mu_A(t) \geq \mu_A(x) > 0$$

and also

$$\langle x, \lambda_A \rangle (y) = \sup_{t \in x \circ y} \lambda_A(t) \leq \lambda_A(x) < 1.$$

Then, $\langle x, \mu_A \rangle (y) > 0$ and $\langle x, \lambda_A \rangle (y) < 1$. So $y \in supp\ \langle x, \mu_A \rangle$ and $y \in inff\ \langle x, \lambda_A \rangle$. □

Definition 2.6. Let H be a semihypergroup, $M \subseteq H$ and $x \in H$. We define $\langle x, M\rangle = \{y \in H : x \circ y \subseteq M\}$.

Proposition 2.7. *Let H be a semihypergroup, $\emptyset \neq M \subseteq H$. Then, $\langle x, \Phi_M\rangle = \Phi_{\langle x,M\rangle}$ and $\langle x, \Psi_M\rangle = \Psi_{\langle x,M\rangle}$ for every $x \in H$, where (Φ_M, Ψ_M) denotes the characteristic function of M, where*

$$\Phi_M(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$$

$$\Psi_M(x) = \begin{cases} 0 & \text{if } x \in M \\ 1 & \text{if } x \notin M. \end{cases}$$

Proof. Let $x, y \in H$. We have the two following cases:

Case 1. $y \in \langle x, M\rangle$. Then, $x \circ y \subseteq M$. This means $\Phi_M(t) = 1, \forall t \in x \circ y$ and $\Psi_M(t) = 0, \forall t \in x \circ y$. Hence, $\inf\limits_{t \in x \circ y} \Phi_M(t) = 1$ and $\sup\limits_{t \in x \circ y} \Psi_M(t) = 0$ whence $\langle x, \Phi_M\rangle = 1$ and $\langle x, \Psi_M\rangle = 0$. Also $\Phi_{\langle x,M\rangle} = 1$ and $\Psi_{\langle x,M\rangle} = 0$.

Case 2. $y \notin \langle x, M\rangle$. Then, $x \circ y \nsubseteq M$. So there exists $t \in x \circ y, \Phi_M(t) = 0$ and $\Psi_M(t) = 1$. Hence, $\inf\limits_{s \in x \circ y} \Phi_M(s) = 0$ and $\sup\limits_{s \in x \circ y} \Psi_M(s) = 1$. Thus $\langle x, \Phi_M\rangle = 0$ and $\langle x, \Psi_M\rangle = 1$. Again $\Phi_{\langle x,M\rangle} = 0$ and $\Psi_{\langle x,M\rangle} = 1$. Thus we conclude $\langle x, \Phi_M\rangle = \Phi_{\langle x,M\rangle}$ and $\langle x, \Psi_M\rangle = \Psi_{\langle x,M\rangle}$. □

Proposition 2.8. *Let H be a semihypergroup and $A = (\mu_A, \lambda_A)$ be a non-empty intuitionistic fuzzy subset of H. Then, for every $t \in [0, 1]$, $\langle x, A_t\rangle = \langle x, A\rangle_t$ for all $x \in H$, where A_t denotes $U(\mu_A : t)$ and $L(\lambda_A : t)$.*

Proof. Let $y \in \langle x, A\rangle_t$. This means $y \in U(\langle x, \mu_A\rangle : t)$ and $y \in L(\langle x, \lambda_A\rangle : t)$. Then, $\langle x, \mu_A\rangle (y) \geq t$ and $\langle x, \lambda_A\rangle (y) \leq t$. Hence, $\inf\limits_{s \in x \circ y} \mu_A(s) \geq t$ and $\sup\limits_{s \in x \circ y} (s) \leq t$. This gives $\mu_A(s) \geq t$ and $\lambda_A(s) \leq t$ for all $s \in x \circ y$ and hence $x \circ y \subseteq U(\mu_A : t)$ and $x \circ y \subseteq L(\lambda_A : t)$. Consequently, $y \in \langle x, U(\mu_A : t)\rangle$ and $y \in \langle x, L(\lambda_A : t)\rangle$, i.e. $y \in \langle x, A_t\rangle$. It follows that $\langle x, A\rangle_t \subseteq \langle x, A_t\rangle$. Reversing the above argument we can deduce that $\langle x, A_t\rangle \subseteq \langle x, A\rangle_t$. □

Proposition 2.9. *Let H be a commutative semihypergroup and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of H such that $\langle x, A\rangle = A$ for every $x \in H$. Then, $A = (\mu_A, \lambda_A)$ is a constant function.*

Proof. Let $x, y \in H$. Then, by hypothesis we have

$$\begin{aligned} \mu_A(x) &= \langle y, \mu_A\rangle (x) \\ &= \inf_{t \in y \circ x} \mu_A(t) \\ &= \inf_{t \in x \circ y} \mu_A(t) \\ &= \langle x, \mu_A\rangle (y) = \mu_A(y) \end{aligned}$$

and

$$\begin{aligned}\lambda_A(x) &= \langle y, \lambda_A\rangle(x)\\ &= \sup_{t\in y\circ x} \lambda_A(t)\\ &= \sup_{t\in x\circ y} \lambda_A(t)\\ &= \langle x, \lambda_A\rangle(y) = \lambda_A(y)\end{aligned}$$

Hence, $A = (\mu_A, \lambda_A)$ is a constant function. □

Corollary 2.10. *Let H be a commutative semihypergroup and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy prime hyperideal of H. If $A = (\mu_A, \lambda_A)$ is not constant, then $A = (\mu_A, \lambda_A)$ is not a maximal intuitionistic fuzzy prime hyperideal of H.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy prime hyperideal of H. Then, by Proposition 2.3, for every $x \in H$, $\langle x, A\rangle$ is an intuitionistic fuzzy prime hyperideal of H. By Proposition 2.5(1), $A \subseteq \langle x, A\rangle$ for all $x \in H$. If $\langle x, A\rangle = A$ for all $x \in H$, then by Proposition 2.9, A is constant which is not the case by hypothesis. Hence, there exists $x \in H$ such that $A \subset \langle x, A\rangle$. This completes the proof. □

Proposition 2.11. *Let H be a commutative semihypergroup. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy semiprime hyperideal of H, then $\langle x, A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H for every $x \in H$.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy semiprime hyperideal of H and $x, y \in H$. Then, $\inf\limits_{t\in y\circ y} \mu_A(t) = \inf\limits_{t\in y\circ y} \inf\limits_{s\in x\circ t} \mu_A(s) \leq \inf\limits_{t\in y\circ y} \inf\limits_{s\in x\circ t} \inf\limits_{k\in s\circ x} \mu_A(k) = \inf\limits_{t\in y\circ x} \inf\limits_{s\in x\circ t} \inf\limits_{k\in s\circ y} \mu_A(k) = \langle x, \mu_A\rangle(y)$. Also, $\sup\limits_{t\in y\circ y} \lambda_A(t) = \sup\limits_{t\in y\circ y} \sup\limits_{s\in x\circ t} \lambda_A(s) \geq$
$\geq \sup\limits_{t\in y\circ y} \sup\limits_{s\in x\circ t} \sup\limits_{k\in s\circ x} \lambda_A(k) = \sup\limits_{t\in y\circ x} \sup\limits_{s\in x\circ t} \sup\limits_{k\in s\circ y} \lambda_A(k) = \langle x, \lambda_A\rangle(y)$.

By Proposition 2.2, $\langle x, A\rangle$ is an intuitionistic fuzzy hyperideal of H. Consequently, $\langle x, A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H for all $x \in H$. □

Corollary 2.12. *Let H be a commutative semihypergroup, $\{A_i\}_{i\in I}$ be a non-empty family of intuitionistic fuzzy semiprime hyperideals of H and let $A = (\mu_A, \lambda_A) = (\inf \mu_{A_i}, \sup \lambda_{A_i})_{i\in I}$. Then, for every $x \in H$, $\langle x, A\rangle$ is an intuitionistic fuzzy semiprime hyperideals.*

Proof. Since each $A_i = (\mu_{A_i}, \lambda_{A_i})(i \in I)$ is an intuitionistic fuzzy hyperideal, $\mu_{A_i}(0) \neq 0$ and $\lambda_{A_i}(0) \neq 1, \forall i \in I$. (Each μ_{A_i} and λ_{A_i} are non-empty, so there exists $x_i \in H$ such that $\mu_{A_i}(x_i) \neq 0$ and $\lambda_{A_i}(x_i) \neq 1, \forall i \in I$. Also $\mu_{A_i}(0) = \inf\limits_{t\in 0\circ x_i} (t) \geq \mu_{A_i}(x_i$ and $\lambda_{A_i}(0) = \sup\limits_{t\in 0\circ x_i} (t) \leq \lambda_{A_i}(x_i, \forall i \in I$. Hence, $\forall i \in I, \mu_{A_i}(0) \neq 0$ and $\lambda_{A_i} \neq 1)$. Consequently, $\mu_A \neq 0$ and $\lambda_A \neq 1$. Thus A is non-empty. Let

$x, y \in H$. The we have

$$\begin{aligned} \inf_{t \in x \circ y} \mu_A(t) &= \inf_{t \in x \circ y} \{\mu_{A_i} : i \in I\}(t) \\ &= \inf\{\inf_{t \in x \circ y} \mu_{A_i}(t) : i \in I\} \\ &\geq \inf\{\mu_{A_i} : i \in I\} \\ &= \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} \sup_{t \in x \circ y} \lambda_A(t) &= \sup_{t \in x \circ y} \{\lambda_{A_i} : i \in I\}(t) \\ &= \sup\{\sup_{t \in x \circ y} \lambda_{A_i}(t) : i \in I\} \\ &\leq \sup\{\lambda_{A_i} : i \in I\} \\ &= \lambda_A(x). \end{aligned}$$

Hence, H is a commutative semihypergroup, A is an intuitionistic fuzzy hyperideal of H.

If $a \in H$, then we have

$$\begin{aligned} \mu_A(a) &= \inf\{\mu_{A_i} : i \in I\}(a) \\ &= \inf\{\mu_{A_i}(a) : i \in I\} \\ &\geq \inf\{\inf_{t \in a \circ a} \mu_{A_i}(t) : i \in I\} \\ &= \inf_{t \in a \circ a} \{\inf\{\mu_{A_i}(t) : i \in I\}\} \\ &= \inf_{t \in a \circ a} \{\inf\{\mu_{A_i} : i \in I\}(t)\} \\ &= \inf_{t \in a \circ a} \mu_A(t) \end{aligned}$$

and

$$\begin{aligned} \lambda_A(a) &= \sup\{\lambda_{A_i} : i \in I\}(a) \\ &= \sup\{\lambda_{A_i}(a) : i \in I\} \\ &\leq \sup\{\sup_{t \in a \circ a} \lambda_{A_i}(t) : i \in I\} \\ &= \sup_{t \in a \circ a} \{\sup\{\lambda_{A_i}(t) : i \in I\}\} \\ &= \sup_{t \in a \circ a} \{\sup\{\lambda_{A_i} : i \in I\}(t)\} \\ &= \sup_{t \in a \circ a} \lambda_A(t). \end{aligned}$$

This means that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy semiprime hyperideal of H. Hence, by Proposition 2.11, for every $x \in H, \langle x, A \rangle$ is an intuitionistic fuzzy semiprime hyperideal of H. □

Proposition 2.13. *Let H be a commutative semihypergroup, $\{H_i\}_{i\in I}$ be a non-empty family of semiprime hyperideal of H and $A = \bigcap_{i\in I} H_i \neq \emptyset$. Then, $\langle x, X_A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H for all $x \in H$, where $X_A = (\Phi_A, \Psi_A)$ is the characteristic function of A.*

Proof. Since $A \neq \emptyset$, for any hyperideal P of H, $P \circ P \subseteq A$ implies $P \circ P \subseteq H_i, \forall i \in I$. Since each H_i is a semiprime hyperideal of H, $P \subseteq H_i, \forall i \in I$. So $P \subseteq \bigcap_{i\in I} H_i = A$. Hence, A is a semiprime hyperideal of H. So the characteristic function $X_A = (\Phi_A, \Psi_A)$ of A is an intuitionistic fuzzy semiprime hyperideal of H. By Proposition 2.12, for all $x \in H$, $\langle x, X_A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H. This completes the proof.

In the following we present another alternative proof of the above proposition:

Since $A = \bigcap_{i\in I} H_i \neq \emptyset$, $X_A \neq \emptyset$ and $\Psi_A \neq \emptyset$. Let $x \in H$. Then, $x \in A$ or $x \notin A$. If $x \in A$, then $\Phi_A(x) = 1$, $\Psi_A(x) = 0$ and $x \in H_i, \forall i \in I$. Then, we have

$$\inf\{\Phi_{H_i} : i \in I\}(x) = \inf\{\Phi_{H_i}(x) : i \in I\} = 1 = \Phi_A(x)$$

and

$$\sup\{\Psi_{H_i} : i \in I\}(x) = \sup\{\Psi_{H_i}(x) : i \in I\} = 0 = \Psi_A(x).$$

If $x \notin A$, then $\Phi_A(x) = 0, \Psi_A(x) = 1$ and there exists $i \in I, x \notin H_i$. It follows that $\Phi_{H_i}(x) = 0, \Psi_{H_i}(x) = 1$. Thus we have

$$\inf\{\Phi_{H_i} : i \in I\}(x) = \inf\{\Phi_{H_i}(x) : i \in I\} = 0 = \Phi_A(x)$$

and

$$\sup\{\Psi_{H_i} : i \in I\}(x) = \sup\{\Psi_{H_i}(x) : i \in I\} = 1 = \Psi_A(x).$$

So we have that $\Phi_A = \inf\{\Phi_{H_i} : i \in I\}$ and $\Psi_A = \sup\{\Psi_{H_i} : i \in I\}$. Therefore, for all $i \in I$, $X_{H_i} = (\Phi_{H_i}, \Psi_{H_i})$ is an intuitionistic fuzzy semiprime hyperideal of H. Consequently, by Corollary 2.12, for all $x \in H, \langle x, X_A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H. □

Theorem 2.14. *Let H be a semihypergroup. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy prime hyperideal of H and $x \in H$ such that $A(x) = \left(\inf_{y\in H} \mu_A(y), \sup_{y\in H} \lambda_A(y)\right)$, then $\langle x, A\rangle = A$. Conversely, if $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy hyperideal of H such that for all $y \in H$, $\langle y, A\rangle = A$ with $A(y)$ not maximal in $A(H)$, then $A = (\mu_A, \lambda_A)$ is prime.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy prime hyperideal of H and $x \in H$ be such that $\mu_A(x) = \inf_{y\in H} \mu_A(y)$ and $\lambda_A(x) = \sup_{y\in H} \lambda_A(y)$. Let $z \in H$. Then, $\mu_A(x) \leq \mu_A(z)$ and $\lambda_A(x) \geq \lambda_A(z)$. Hence,

$$\mu_A(x) \vee \mu_A(z) = \mu_A(z) \qquad (1)$$

and

$$\lambda_A(x) \wedge \lambda_A(z) = \lambda_A(z) \qquad (2)$$

Now $\langle x, \mu_A \rangle (x) = \inf_{t \in x \circ z} \mu_A(t)$ and $\langle x, \lambda_A \rangle (x) = \sup_{t \in x \circ z} \lambda_A(t)$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy prime hyperideal of H, we have $\inf_{t \in x \circ z} \mu_A(t) = \mu_A(x) \vee \mu_A(z)$ and $\sup_{t \in x \circ z} \lambda_A(t) = \lambda_A(x) \wedge \lambda_A(z)$. Using (1) and (2), it follows $\inf_{t \in x \circ z} \mu_A(t) = \mu_A(z)$ and $\sup_{t \in x \circ z} \lambda_A(t) = \lambda_A(z)$. Hence, $\langle x, \mu_A \rangle (x) = \mu_A(z)$ and $\langle x, \lambda_A \rangle (x) = \lambda_A(z)$. Consequently, $\langle x, A \rangle = A$.

Conversely, let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy hyperideal of H such that for all $y \in H$, $\langle y, A \rangle = A$ with $A(y)$ not maximal in $A(H)$ and let $x_1, x_2 \in H$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy hyperideal of H, we have

$$\inf_{t \in x_1 \circ x_2} \mu_A(t) \geq \mu_A(x_1) \text{ and } \sup_{t \in x_1 \circ x_2} \lambda_A(t) \leq \lambda_A(x_1) \ (3)$$

and

$$\inf_{t \in x_1 \circ x_2} \mu_A(t) \geq \mu_A(x_2) \text{ and } \sup_{t \in x_1 \circ x_2} \lambda_A(t) \leq \lambda_A(x_2) \ (4).$$

We have the two following cases:

Case 1. Either $(\mu_A(x_1), \lambda_A(x_1))$ or $(\mu_A(x_2), \lambda_A(x_2))$ is maximal in $A(H)$. Suppose that $(\mu_A(x_1), \lambda_A(x_1))$ is maximal in $A(H)$. Then, $\inf_{t \in x_1 \circ x_2} \mu_A(t) \leq \mu_A(x_1)$ and $\sup_{t \in x_1 \circ x_2} \lambda_A(t) \leq \lambda_A(x_1)$. Consequently, $\inf_{t \in x_1 \circ x_2} \mu_A(t) = \mu_A(x_1) = \{\mu_A(x_1) \vee \mu_A(x_2)\}$ and $\sup_{t \in x_1 \circ x_2} \lambda_A(t) = \lambda_A(x_1) = \lambda_A(x_1) \wedge \lambda_A(x_2)$.

Case 2. Neither $(\mu_A(x_1), \lambda_A(x_1))$ nor $(\mu_A(x_2), \lambda_A(x_2))$ is maximal in $A(H)$. By hypothesis $\langle x_1, A \rangle = A$ i.e. $\langle x_1, \mu_A \rangle = \mu_A$, $\langle x_1, \lambda_A \rangle = \lambda_A$, $\langle x_2, \mu_A \rangle = \mu_A$ and $\langle x_2, \lambda_A \rangle = \lambda_A$. Hence, $\langle x_1, \mu_A \rangle (x_2) = \mu_A(x_2)$ and $\langle x_1, \lambda_A \rangle (x_2) = \lambda_A(x_2)$. This implies $\inf_{t \in x_1 \circ x_2} \mu_A(t) = \mu_A(x_2)$ and $\sup_{t \in x_1 \circ x_2} \lambda_A(t) = \lambda_A(x_2)$. So, using (3) and (4), we have $\inf_{t \in x_1 \circ x_2} \mu_A(t) = \mu_A(x_1) \vee \mu_A(x_2)$ and $\sup_{t \in x_1 \circ x_2} \lambda_A(t) = \lambda_A(x_1) \wedge \lambda_A(x_2)$. Therefore, we have that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy prime hyperideal of H. □

Corollary 2.15. *Let H be a semihypergroup and I a hyperideal of H. Then, I is prime if and only if for $x \in H$ such that $x \notin I$, $\langle x, X_I \rangle = X_I$, where $X_I = (\Phi_I, \Psi_I)$ is the characteristic function of I.*

Proof. Let I be a prime hyperideal of H. Then, by Proposition 1.12, $X_I = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy prime hyperideal of H. For $x \in H$ such that $x \notin I$, we have

$$\Phi_I(x) = 0 = \inf_{y \in H} \Phi_I(y)$$

and

$$\Psi_I(x) = 1 = \inf_{y \in H} \Psi_I(y)$$

Then, by Theorem 2.14, $\langle x, X_I \rangle = X_I$.

Conversely, let $\langle x, X_I \rangle = X_I$ for all $x \in H$ such that $x \notin I$. Let $y \in H$ be such that $X_I(y)$ is not maximal in $X_I(H)$. Then, $\Phi_I(y) = 0$ and $\Psi_I(y) = 1$, so $y \notin I$. Thus $\langle y, X_I \rangle = X_I$. By Theorem 2.14, X_I is an intuitionistic fuzzy prime hyperideal of H. So I is prime hyperideal of H. □

References

[1] F. Marty, Sur une generalization de la notion de group, 8th Congres Math. Scandinaves, Stockholm. (1934) 45–49.

[2] L. A. Zadeh, Fuzzy Sets, Inform. and Control. 8 (1965) 338–353.

[3] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512-517.

[4] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht. 2003.

[5] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, International Academic Press, USA. 2007.

[6] B. Davvaz, Fuzzy hyperideals in semihypergroups, Italian J. Pure Appl. Math. 8 (2000) 67-74.

[7] B. Davvaz, Fuzzy H_v-groups, Fuzzy Sets and Systems 101 (1999) 191-195.

[8] B. Davvaz, Characterizations of sub-semihypergroups by various triangular norms, Czechoslovak Mathematical Journal 55(4) (2005) 923-932.

[9] B. Davvaz, Extensions of fuzzy hyperideals in H_v-semigroups, Soft Computing 11(9) (2007) 829-837.

[10] X.-Y. Xie, Fuzzy ideal extensions of semigroups, Soochow Journal of Mathematics 27(2) (2001) 125-138.

[11] K. Atanassov, Intuitionistic fuzzy sets, Central Tech. Library, Bulgarian Academy Science, Sofia, Bulgaria, Rep. No. 1697/84, 1983.

[12] B. Davvaz, Intuitionistic hyperideals of semihypergroups, Bull. Malays. Math. Sci. Soc. (2) 29(1) (2006) 203-207.

[13] S. Chauhan, B.D. Pant, S. Radenovic, Common fixed point theorems for R-weakly commuting mappings with common limit in the range property, Journal of the Indian Mathematical Society (2014), article in press.

[14] S. Chauhan, M. Imdad and B. Samet, Coincidence and common fixed point theorems in modifed intuitionistic fuzzy metric spaces, Mathematical and Computer Modelling, vol. 58, Article ID 892900, 2013.

[15] S. Chauhana, W. Shatanawi, S. Kumar, S. Radenovic, Existence and uniqueness of fixed points in modifed intuitionistic fuzzy metric spaces, J. Nonlinear Sci. Appl. 7 (2014) 28-41.

[16] I. Cristea, Hyperstructures and fuzzy sets endowed with two membership functions, Fuzzy Sets and Systems 160 (2009) 1114-1124.

[17] W.A. Dudek, B. Davvaz, Y. B. Jun, On intuitionistic fuzzy subhyperquasigroups of hyperquasigroups, Information Sciences 170 (2005) 251-262.

[18] M. Imdad, S. Chauhan, S. Dalal, Unified fixed point theorems via common limit range property in modified intuitionistic fuzzy metric spaces, Abstract and Applied Analysis 2013, Art. ID 413473, 11 pp.

[19] B. D. Pant, S. Kumar, S. Chauhan, Common fixed point of weakly compatible maps on intuitionistic fuzzy metric spaces, Journal of Advanced Studies in Topology 1(1) (2010) 41-49.

[20] W. Sintunavarat, S. Chauhan, P. Kumam, Some fixed point results in modofied intuitionistic fuzzy metric spaces, Afrika Matematika 25(2) (2014) 461-473.

[21] P. Corsini, M. Shabir, T. Mahmood, Semisipmle semihypergroups in terms of hyperideals and fuzzy hyperideals, Iranian Journal of Fuzzy Systems, 8(1) (2011) 95-111.

[22] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.

[23] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61 (1994) 137-142.

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