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Invertibility of the Related Spaces of a Fuzzy Topological Space

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Abstract : In this paper we examine whether the invertibility of a topological or fuzzy topological space is transferrable to its associated space. We further investigate the effect of invertibility on quotient spaces. Finally we study the invertibility of product spaces and establish that the invertibility of a component space is sufficient for the invertibility of the product space, but it is not necessary.

Keywords : homeomorphism; invertible fts; associated space; quotient space; product space.
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1 Introduction

Ever since the introduction of the concept in 1961 by Doyle and Hocking [1], many have contributed to the development of invertible spaces. The pioneering authors themselves continued the investigation of invertibility and came up with the concept of continuously invertible spaces [2]. Gray [3] has proved that if an invertible space possesses a nonempty open subspace which is metrizable, then the space is metrizable. The effect of invertibility on separation axioms, investigated in [1], has been further explored by Levine [4] obtaining some local properties

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which are necessarily global in invertible spaces. Simultaneously, Wong undertook a detailed study on invertible spaces and improved certain results by weakening the conditions and obtained simpler proofs in [5]. In [6], Umen exhibited some of the properties of orbits in invertible spaces. Naimpally opened a quick glimpse on the function spaces of invertible spaces in [7]. The concept of generalized invertible spaces was introduced and studied by Hong in [8] and [9]. Ryeburn defined invertibility analogously for uniform spaces and showed that the whole space possesses certain properties holding for subspaces [10]. Hildebrand and Poe examined separation axioms for invertible spaces in [11]. The concept of semi-invertible spaces was introduced by Crossley and Hildebrand in [12]. Invertibility in infinite dimensional spaces was thoroughly investigated by Tseng and Wong [13].

In [14], Mathew extended the concept of invertibility to fuzzy topological spaces and examined the basic nature of such spaces. In [15], the authors discussed some properties of invertible fuzzy topological spaces. Later, we studied the relationship between invertibility and separation axioms in [16]. In continuation of this, Rao discussed the effect of invertibility on separation axioms of fuzzy topological spaces in [17]. In [18] we further explored the properties of invertible as well as completely invertible fuzzy topological spaces. Based on the structure of inverting pairs, we introduced different types of invertible fuzzy topological spaces and derived certain characterizing properties in [19]. We also examined the relationship between homogeneity and invertibility in [20].

A related problem here is to investigate the invertibility of the associated spaces. In [18] it has been proved that the associated topological space of a completely invertible fuzzy topological space need not be invertible. Here we prove that the associated fuzzy topological space of a given topological space is always type 1 invertible. Also the associated fuzzy topological space (X, F) of a topological space (X, τ) is type 2 invertible if and only if (X, τ) is not invertible.

In general, the invertibility of a quotient space doesn't depend on the invertibility of the fuzzy topological space and vice-versa. It is proved that quotient space of a topologically generated fuzzy topological space is always type 1 invertible. Also the quotient space of a completely invertible fuzzy topological space need not be invertible. We also obtain some conditions under which the quotient space of a completely invertible fuzzy topological space is completely invertible.

Finally we investigate the product of a family of invertible fuzzy topological spaces and prove that the product space is invertible if at least one of the co-ordinate spaces is invertible. Further, the product space is type 2 completely invertible if and only if each co-ordinate space is type 2 completely invertible.

2 Preliminaries

In this section we include certain definitions and known results needed for the subsequent development of the study. Throughout this paper X stands for a non empty set with at least two elements and I stands for the unit interval $[0, 1]$. For any fuzzy subset g of X, by $\not{C}(g)$ we mean the complement of g in X. A fuzzy subset with constant degree of membership α is denoted by $\underline{\alpha}$. The identity map on X is denoted by e. A fuzzy subset g of X is said to be proper if $g \neq \underline{0}, \underline{1}$.

Definition 2.1. [21] Let X and Y be two sets and let $\theta : X \to Y$ be a function. Then for any fuzzy subset g in X, $\theta(g)$ is a fuzzy subset in Y defined by

$$\theta(g)(y) \quad = \quad \begin{cases} \sup\{g(x) : \ x \in X, \ \theta(x) = y\}; & \theta^{-1}(y) \neq \phi \\ 0; & \theta^{-1}(y) = \phi \end{cases}$$

For a fuzzy subset h in Y, we define $\theta^{-1}(h)(x) = h(\theta(x)), \forall x \in X$. Obviously $\theta^{-1}(h)$ is a fuzzy subset in X.

Definition 2.2. [14] An fts (X, F) is said to be *invertible with respect to a proper open fuzzy subset* g if there is a homeomorphism θ of (X, F) such that $\theta(\not{C}(g)) \leq g$. This homeomorphism θ is called an *inverting map* for g and g is said to be an *inverting fuzzy subset* of (X, F).

If an fts (X, F) is invertible, then there exists an inverting fuzzy subset g and an inverting map θ of (X, F). This g and θ together called an inverting pair of (X, F). Clearly there can be different inverting pairs for an invertible fts.

Definition 2.3. [14] An fts (X, F) is said to be *completely invertible* if for every $g \neq \underline{0}, \underline{1}, \in F$, there is a homeomorphism θ of (X, F) such that $\theta(\not{C}(g)) \leq g$.

It should be noted that for a completely invertible fts every proper open fuzzy subset is an inverting fuzzy subset.

Definition 2.4. [22] Let (X, F) be an fts. A subfamily $\mathcal{B}$ of F is a *base* for F if each member of F can be expressed as the join of some members of $\mathcal{B}$.

Theorem 2.5. [18] *Let (X, F) be an fts with $\mathcal{B}$ as a base. Then (X, F) is completely invertible if and only if (X, F) is invertible with respect to all members of $\mathcal{B}$.*

Definition 2.6. [19] An invertible fts (X, F) is said to be *type 1* if identity is an inverting map.

Definition 2.7. [19] An invertible fts (X, F) is said to be *type 2* if identity is an inverting map for all the inverting fuzzy subsets.

Theorem 2.8. [19] *Let (X, F) be an invertible fts. Then (g, e) is an inverting pair of (X, F) if and only if $\underline{\frac{1}{2}} \leq g$.*

Theorem 2.9. [19] *An fts (X, F) is type 2 completely invertible if and only if $\underline{\frac{1}{2}} \leq g$ for every $g \neq \underline{0}, \underline{1}, \in F$.*

Theorem 2.10. [18] *Let (X, F) be an fts invertible with respect to g where X is finite. Then $|X| \leq 2|supp\ g|$.*

Definition 2.11. [22] The fuzzy subset x_λ of X, with $x \in X$ and $0 < \lambda \leq 1$ defined by

$$x_\lambda(y) \quad = \quad \begin{cases} \lambda; & y = x \\ 0; & \text{otherwise} \end{cases}$$

is called a *fuzzy point* in X with support x and value λ.

Definition 2.12 ([26])**.** A fuzzy point x_λ is called *weak* or *strong according* as $\lambda \leq \frac{1}{2}$ or $\lambda > \frac{1}{2}$.

Definition 2.13. [25] A fuzzy subset g of X is said to be *strong* if for any $x \in X$, g contains a strong fuzzy point x_λ.

Definition 2.14. [27] An fts (X, F) is called *fully stratified space* iff $\underline{\alpha} \in F$, $\forall \alpha \in I$. An fts (X, F) is called *purely stratified* if whenever $f \in F$ then $f = \underline{\alpha}$ for some $\alpha \in [0, 1]$.

Theorem 2.15. [19] *Every fully stratified fts is type 1 invertible.*

Theorem 2.16. [18] *A fully stratified fts cannot be completely invertible.*

Theorem 2.17. [19] *A purely stratified invertible fts is type 2 invertible.*

Notations 2.18. Let (X, F) be an fts. Then for $g \in F$ and $\lambda \in I$, we define
$S_\lambda(g) = \{x \in X;\ g(x) = \lambda\}$.

3 Associated Spaces

Definition 3.1. [24] For a topology τ on X let $\omega(\tau)$ be the set of all lower semi-continuous functions from (X, τ) to $[0, 1]$. Then $\omega(\tau)$ turns out to be a fuzzy topology on X called the *associated fuzzy topology* of (X, τ). A fuzzy topology of the form $\omega(\tau)$ is called *topologically generated*. For a fuzzy topology F on X, $i(F)$ is the topology on X induced by all functions $f : X \to I_r$, where $f \in F$ and $I_r = [0, 1]$ with subspace topology of right ray topology on R.

Theorem 3.2. [23] *If (X, F) is topologically generated, then the group of homeomorphisms of (X, F) and the group of homeomorphisms of $(X, i(F))$ are the same.*

Remark 3.3. [18] *The associated topological space of a completely invertible fts need not be invertible. Conversely, the complete invertibility of the associated topological space of an fts need not imply the invertibility of the fts.*

Theorem 3.4. *Any topologically generated fts is type 1 invertible but not completely invertible.*

Proof. Every topologically generated fts is fully stratified. Hence the result follows from Theorem 2.15 and Theorem 2.16. □

Remark 3.5. *From Theorem 3.4 it follows that the associated fts of a given topological space (X, τ) is always invertible irrespective of the invertibility of (X, τ). But the following theorem derives that the associated fts is type 2 invertible iff (X, τ) is not invertible.*

Theorem 3.6. *The associated fts (X, F) of a topological space (X, τ) is type 2 invertible iff (X, τ) is not invertible.*

Proof. Suppose (X, F) is type 2 invertible. If possible, assume that (X, τ) is invertible. Let (A, θ) be an inverting pair of (X, τ), then by Theorem 3.2, θ is a homeomorphism of (X, F). Let $\alpha \in (\frac{1}{2}, 1)$ and consider $g \in I^X$ defined by

$$g(x) \quad = \quad \begin{cases} \alpha; & x \in A \\ 1 - \alpha; & \text{otherwise} \end{cases}$$

Then clearly (g, θ) is an inverting pair of (X, F). Also here e cannot be an inverting map for g, so that (X, F) is not type 2 invertible, a contradiction.
Conversely suppose (X, τ) is not invertible. Since (X, F) is topologically generated, it is type 1 invertible. If possible let f be an inverting fuzzy subset of (X, F) for which e is not an inverting map. Then by Theorem 2.8, $f(y) < \frac{1}{2}$ for some $y \in X$. Now by Theorem 3.2, any inverting map θ of f is also a homeomorphism of (X, τ). Let $f(y) = \alpha$ and let $A = f^{-1}(\alpha, 1]$, then $A \neq \phi, \subset X$. Clearly (A, θ) is an inverting pair of (X, τ) , a contradiction. Consequently (X, F) is type 2 invertible. $\square$

Remark 3.7. *From the above proof it follows that every inverting map of a topological space (X, τ) is always an inverting map of the associated fts (X, F). But conversely an inverting map of (X, F) need not be an inverting map of (X, τ). For, e is an inverting map for a type 1 invertible fts, but it cannot be an inverting map for the associated topological space. But from the converse part of the proof of the above theorem it is clear that an inverting map other than identity of an fts (X, F) is also an inverting map of the associated topological space.*

Theorem 3.8. *If the associated topological space $(X, i(F))$, where X is finite, is completely invertible then for each $g \in F$, there exists some $\alpha \in I$ such that $g \leq \underline{\alpha}$ with $|S_\alpha(g)| \geq \frac{|X|}{2}$.*

Proof. Suppose the associated topological space $(X, i(F))$ of an fts (X, F) is completely invertible. Let g be an open fuzzy subset of X. If possible assume that for any $\alpha \in I$, $|S_\alpha(g)| < \frac{|X|}{2}$. Let $\lambda = \max_{x \in X} g(x)$ and $\beta = \max_{x \notin S_\lambda(g)} g(x)$. Now consider $A = f^{-1}(\beta, 1]$. Clearly $A = S_\lambda(g) \neq \phi, \in i(F)$ and by Theorem 2.10, $(X, i(F))$ is not invertible with respect to A, a contradiction. Thus for any $g \in F$ there is an $\alpha \in I$ with $|S_\alpha(g) \geq \frac{|X|}{2}$. Choose α_0 be the maximum of all such $\alpha' s$. Now if possible assume that $g \nleq \underline{\alpha_0}$. Then there is a $\gamma > \alpha$ such that $\gamma = g(x)$ for some $x \in X$. Now consider $g^{-1}\overline{(\alpha_0}, 1] = B$. Clearly $B \neq \phi, \in i(F)$ and $(X, i(F))$ is not invertible with respect to B, a contradiction. $\square$

Corollary 3.9. *If the associated topological space $(X, i(F))$, where $|X|$ is odd, is completely invertible then for each $g \in F$, there exists a unique $\alpha \in I$ such that $g \leq \underline{\alpha}$ with $|S_\alpha(g)| \geq \frac{|X|}{2}$.*

Proof. Follows from the proof of Theorem 3.8. □

Theorem 3.10. *If the associated topological space $(X, i(F))$, where X is infinite, is completely invertible then for each $g \in F$, there exists some $\alpha \in I$ such that $S_\alpha(g)$ is infinite and $g \leq \underline{\alpha}$.*

Proof. Proof is similar to that of Theorem 3.8. □

Remark 3.11. *Converse of Theorem 3.8 is not true. For, let $X = \{a, b, c, d\}$. Consider $g \in I^X$ defined by $g(a) = \frac{1}{2}$, $g(b) = \frac{2}{3}$, $g(c) = \frac{2}{3}$, $g(d) = \frac{2}{3}$. Then (X, F) is an fts where $F = \{\underline{0}, \underline{1}, g\}$. Here $g \leq \underline{\frac{2}{3}}$ and $S_{\frac{2}{3}}(g) \geq \frac{|X|}{2}$. Clearly $(X, i(F))$ where $i(F) = \{X, \phi, \{b, c, d\}\}$ is not invertible.*

4 Quotient Spaces

Definition 4.1 ([27])**.** Let (X, F) be an fts and R be an equivalence relation on X. Let X/R be the quotient set and let $p : X \to X/R$ be the quotient map. Let G be the family of fuzzy subsets in X/R defined by $G = \{g : p^{-1}(g) \in F\}$. Then G is called the *quotient fuzzy topology* for X/R and $(X/R, G)$ is called the *quotient fts*.

Theorem 4.2. *If an fts (X, F) is type 2 completely invertible then any non-trivial quotient space of (X, F) is type 2 completely invertible.*

Proof. Let (X, F) be a type 2 completely invertible fts. Then by Theorem 2.9, $\underline{\frac{1}{2}} \leq f$ for every $f \neq \underline{0}, \underline{1}, \in F$. Consider any quotient space $(X/R, G)$ of (X, F). Then for any $g \neq \underline{0}, \in G$, $g \geq \underline{\frac{1}{2}}$ so that $(X/R, G)$ is type 2 completely invertible again by Theorem 2.9. □

Remark 4.3. *Converse of the above theorem is not true, in general. For example let $X = [2, 3]$ and consider $f \in I^X$ defined by $f(x) = \frac{1}{x}$, $\forall x$. Then (X, F) is an fts where $F = \{\underline{0},\ \underline{1},\ f,\ \underline{\frac{2}{3}}\}$. Clearly f is not an inverting fuzzy subset of (X, F). Let $(X/R, G)$ be any non-trivial quotient space of (X, F). Then clearly $G = \{\underline{0},\ \underline{1},\ \underline{\frac{2}{3}}\}$ so that $(X/R, G)$ is type 2 completely invertible*

Corollary 4.4. *Let (X, F) be an fts in which every $f \neq \underline{0}, \in F$ is a strong fuzzy subset. Then any non-trivial quotient space of (X, F) is type 2 completely invertible.*

Proof. Since (X, F) is an fts in which every $f \neq \underline{0}, \in F$ is a strong fuzzy subset, it is type 2 completely invertible. Hence the result follows from Theorem 4.2. □

Remark 4.5. *The quotient space of a completely invertible fts need not be invertible. Let X be the set of all natural numbers and let $Y = \{2x : x \in X\}$. For each $m \in Y$, define $f_m, g_m \in I^X$ by*

$$f_m(x) = \begin{cases} \frac{1}{3}; & 1 \leq x \leq m \\ \frac{2}{3}; & otherwise \end{cases}$$

$$g_m(x) = \begin{cases} \frac{1}{3}; & m < x \leq 2m \\ \frac{2}{3}; & otherwise \end{cases}$$

Let F be the fuzzy topology on X generated by $\{f_m,\ g_m;\ m \in Y\}$.
Now consider for each $m \in Y$, $\theta_m : X \to X$ defined by

$$\theta_m(x) = \begin{cases} x + m; & 1 \leq x \leq m \\ x - m; & m < x \leq 2m \\ x; & otherwise. \end{cases}$$

Clearly θ_m is a homeomorphism of (X, F). Also for each $m \in Y$, (f_m, θ_m) and (g_m, θ_m) are inverting pairs of (X, F) so that (X, F) is completely invertible. Now consider the equivalence relation R on X such that $X/R = \{A, B, C\}$ where $A = \{1\}$, $B = \{2\}$ and $C = \{x \in X : x > 2\}$. Consider $f \in I^{X/R}$ defined by $f(A) = \frac{1}{3}$, $f(B) = \frac{1}{3}$, $f(C) = \frac{2}{3}$. Then the quotient space $(X/R, G)$ of (X, F) is given by $G = \{\underline{0}, \underline{1}, f\}$ and is not invertible.

The following example shows that the quotient space of an fts may be completely invertible, even if the fts is not invertible.

Example 4.6. Let X be the set of all non-zero real numbers. For each $\alpha \in (\frac{2}{3}, 1]$, define $f_\alpha, g_\alpha \in I^X$ by

$$f_{\alpha(x)} = \begin{cases} \alpha; & x > o \\ 1 - \alpha; & \text{otherwise,} \end{cases}$$

$$g_{\alpha(x)} = \begin{cases} 1 - \alpha; & x > o \\ \alpha; & \text{otherwise,} \end{cases}$$

Also consider $h \in I^X$ defined by

$$h(x) = \begin{cases} x; & 0 < x < 1 \\ 0; & -1 < x < 0, \\ \frac{1}{|x|}; & \text{otherwise} \end{cases}$$

Let F be the fuzzy topology on X generated by $\{h,\ f_\alpha,\ g_\alpha \in I^X,\ \alpha \in (\frac{2}{3}, 1]\}$. Clearly the fts (X, F) is not invertible. Consider the equivalence relation R on X defined by xRy iff $xy > 0$. Then the quotient space $(X/R, G)$ is completely invertible.

Theorem 4.7. *An fts with a type 1 invertible quotient space is type 1 invertible.*

Proof. Let $(X/R, G)$ be a quotient fts of (X, F) and suppose it is type 1 invertible. Let (g, e) be an inverting pair of $(X/R, G)$. Then by Theorem 2.8, $\underline{\frac{1}{2}} \leq g$. Now consider $f \in I^X$ defined by $f = p^{-1}(g)$ where p is the quotient map. Then clearly $f \in F$ such that $\underline{\frac{1}{2}} \leq f$ and (f, e) is an inverting pair of (X, F). Hence (X, F) is type 1 invertible. $\square$

Remark 4.8. *There are type 1 invertible fts for which no non-trivial quotient space is invertible. For, let X be the set of all natural numbers. Consider $f, g, h \in I^X$ defined by $f(x) = \frac{x}{x+1}$, $\forall x \in X$,*

$$\begin{aligned} g(x) &= \begin{cases} \frac{1}{3}; & x \text{ is even} \\ \frac{1}{5}; & x \text{ is odd} \end{cases} \\ h(x) &= \begin{cases} \frac{1}{5}; & x \text{ is even} \\ \frac{1}{3}; & x \text{ is odd} \end{cases} \end{aligned}$$

Then (X, F) is an fts where $F = \{\underline{0}, \underline{1}, f, g, h, \underline{\frac{1}{3}}, \underline{\frac{1}{5}}\}$. Clearly (f, e) is an inverting pair of (X, F) so that (X, F) is type 1 invertible. Let $(X/R, G)$ be any non-trivial quotient space of (X, F). Clearly $m < \underline{\frac{1}{2}}$, $\forall m \in G$ so that $(X/R, G)$ is not invertible.

Theorem 4.9. *An fts (X, F) is invertible with $\underline{\alpha}$; $\alpha \in [\frac{1}{2}, 1)$ as an inverting fuzzy subset iff every quotient space of (X, F) is type 1 invertible.*

Proof. Let $(X/R, G)$ be any quotient space of (X, F). Consider the fuzzy subset g of X/R defined by $g(D) = \alpha$, $\forall D \in X/R$. Clearly $g \in G$ and (g, e) is an inverting pair of $(X/R, G)$.
Conversely suppose every quotient space of (X, F) is type 1 invertible. Let R be an equivalence relation on X such that $X/R = \{X\}$. Consider the quotient space $(X/R, G)$. Since it is type 1 invertible, there exists an inverting fuzzy subset $f \in G$ such that $f(X) = \alpha$ for some $\alpha \in [\frac{1}{2}, 1)$. Let $g = p^{-1}(f)$, where p is the quotient map, then $g = \underline{\alpha}$ and $g \in F$ so that (g, e) is an inverting pair of (X, F). $\square$

Corollary 4.10. *Any quotient space of a fully stratified fts is type 1 invertible.*

Proof. Let (X, F) be a fully stratified fts. Then $\forall \alpha \in [\frac{1}{2}, 1]$, $\underline{\alpha} \in F$. Hence by Theorem 4.9, every quotient space of (X, F) is type 1 invertible. $\square$

Corollary 4.11. *Any quotient space of a topologically generated fts is type 1 invertible.*

Proof. A topologically generated fts is fully stratified. Hence the result follows from Corollary 4.10. $\square$

Theorem 4.12. *An fts (X, F) is purely stratified and invertible iff every quotient space of it is purely stratified and invertible.*

Proof. Straightforward. $\square$

Corollary 4.13. *Any quotient space of a purely stratified invertible fts is type 2 invertible.*

Proof. A purely stratified invertible fts is type 2 invertible. Hence the result follows from the above theorem. □

5 Product Spaces

Definition 5.1 ([27])**.** Let $(X_i, F_i),\ i \in J$ be a family of fts. Let X be the cartesian product of $\{X_i,\ i \in J\}$ and let P_i be the projection of the product X into the i^{th} coordinate set X_i. An $x \in X$ is of the form $(x_i, i \in J)$ where x_i is the i^{th} component of x. Let $Q(J)$ denote the family of all finite subsets of J. Putting $\mathcal{B} = \{\bigcap_{i \in K} P_i^{-1}(U_i) : U_i \in F_i, K \in Q(J)\}$, we call the fuzzy topology F which takes $\mathcal{B}$ as a base, the product fuzzy topology for X, and $\mathcal{B}$ the defining base for the product fuzzy topology. The pair (X, F) is called the *product space* of the fts $(X_i, F_i),\ i \in J$.

Theorem 5.2. *The product fuzzy topology (X, F) of a family of fts $(X_i, F_i),\ i \in J$ is invertible if (X_j, F_j) is invertible for some $j \in J$.*

Proof. Let P_i be the projection of the product X into the i^{th} coordinate set X_i and $\mathcal{B}$ be the defining base for (X, F). Suppose (X_j, F_j) is invertible for some $j \in J$. Let (g_j, θ_j) be an inverting pair of (X_j, F_j). Now consider $f = P_j^{-1}(g_j)$, clearly $f \in \mathcal{B}$ so that $f \in F$. Define $\theta : X \to X$ by $\theta(x) = y$ where $y = (y_i)$ such that $y_i = x_i,\ i \neq j,\ y_j = \theta_j(x_j)$. Clearly θ is a homeomorphism of (X, F). Let $x \in X$ and $\theta^{-1}(x) = z$. Then $\theta(\not{C}(f))(x) = \not{C}(f)(z) = \not{C}(P_j^{-1}(g_j))(z) = \not{C}(g_j)(z_j) = \not{C}(g_j)(\theta_j^{-1}(x_j)) = \theta_j(\not{C}(g_j))(x_j) \leq g_j(x_j) = P_j^{-1}(g)(x) = f(x)$ so that (f, θ) is an inverting pair of (X, F). □

Corollary 5.3. *The product of a family of invertible fuzzy topological spaces is invertible.*

Proof. Follows from Theorem 5.2 □

Theorem 5.4. *The product fuzzy topology (X, F) of a family of fts $(X_i, F_i),\ i \in J$ is type 1 invertible if (X_j, F_j) is type 1 invertible for some $j \in J$.*

Proof. Let P_i be the projection of the product X into the i^{th} coordinate set X_i and $\mathcal{B}$ be the defining base for (X, F). Suppose (X_j, F_j) is type 1 invertible for some $j \in J$. Let (g_j, e) be an inverting pair of (X_j, F_j). Then by Theorem 2.8, $g_j \geq \underline{\frac{1}{2}}$. Now consider $P_j^{-1}(g_j) = f$, clearly $f \in F$ and $f \geq \underline{\frac{1}{2}}$ so that e is an inverting map for f. Hence (X, F) is type 1 invertible. □

Corollary 5.5. *The product of a family of type 1 invertible fuzzy topological spaces is type 1 invertible.*

Proof. Follows from Theorem 5.4. □

Remark 5.6. *Following example shows that the converses of Theorem 5.2 and Theorem 5.4 are not true. For each* $\alpha \in [0, \frac{1}{2})$*, let* $X_\alpha = \{a, b\}$ *and* $f_\alpha(a) = \frac{1}{2},\ f_\alpha(b) = \alpha$*. Then* (X_α, F_α) *where* $F_\alpha = \{\underline{0}, \underline{1}, f_\alpha\}$*, is an fts for each* $\alpha \in [0, \frac{1}{2})$*. Let* X *be the cartesian product of* X_α *for* $\alpha \in [0, \frac{1}{2})$*. Let* F *be the product fuzzy topology on* X *and* P_α *be the projection of the product* X *into the* α^{th} *coordinate set* X_α*. Then* $g_\alpha = P_\alpha^{-1}(f_\alpha) \in F$ *for each* $\alpha \in [0, \frac{1}{2})$ *so that* $\bigvee\limits_{\alpha} g_\alpha = \underline{\frac{1}{2}} \in F$*. Note that here* $\underline{\frac{1}{2}}$ *is the only inverting fuzzy subset. Hence* (X, F) *is type 1 invertible.*

Also here (X, F) *is type 2 invertible even if none of the co-ordinate spaces are so. Conversely the type 2 invertibility of all the co-ordinate spaces need not imply the type 2 invertibility of the product space as shown in the following example.*

Example 5.7. Let $X_1 = \{a, b\}$ and $X_2 = \{c, d\}$. Define $g_1,\ g_2 \in I^{X_2}$ by $g_1(c) = \frac{2}{3},\ g_1(d) = \frac{1}{4}$ and $g_2(c) = \frac{1}{4},\ g_2(d) = \frac{2}{3}$. Then (X_1, F_1) and (X_2, F_2) are fuzzy topological spaces where $F_1 = \{\underline{0}, \underline{1}, \underline{\frac{2}{3}}, \underline{\frac{1}{3}}\}$ and $F_2 = \{\underline{0}, \underline{1}, g_1, g_2, \underline{\frac{1}{4}}, \underline{\frac{2}{3}}\}$. Clearly (X_1, F_1) and (X_2, F_2) are type 2 invertible. Also let (X, F) be the product fuzzy topology of (X_1, F_1) and (X_2, F_2) and let P_1 be the projection of the product X into the coordinate set X_1 and P_2 be the projection of the product X into the coordinate set X_2. Consider $f \in I^X$ defined by $f(a, c) = \frac{1}{3},\ f(a, d) = \frac{2}{3},\ f(b, c) = \frac{1}{3},\ f(b, d) = \frac{2}{3}$. Clearly $f = P_1^{-1}(\underline{\frac{1}{3}}) \vee P_2^{-1}(g_2) \in F$. Now define $\theta : X \to X$ by $\theta(a, c) = (a, d),\ \theta(a, d) = (a, c),\ \theta(b, c) = (b, d),\ \theta(b, d) = (b, c)$. Clearly θ is a homeomorphism of (X, F) and is an inverting map for f. Also identity is not an inverting map for f. Hence (X, F) is not type 2 invertible.

Theorem 5.8. *If the product fuzzy topology* (X, F) *of a family of invertible fts* $(X_i, F_i),\ i \in J$ *is type 2 invertible, then each* (X_i, F_i) *is type 2 invertible.*

Proof. Suppose (X, F) be type 2 invertible. If possible assume that (X_j, F_j) is not type 2 invertible for some j. Then there exists an inverting fuzzy subset g_j of (X_j, F_j) such that e is not an inverting map for g_i. Now define $\theta : X \to X$ by $\theta(x) = y$ where $y = (y_i)$ such that $y_i = x_i,\ i \neq j,\ y_j = \theta_j(x_j)$. Clearly $\theta \neq e$ is a homeomorphism of (X, F). Now consider $f = P_j^{-1}(g_j)$, clearly (f, θ) is an inverting pair of (X, F) and e is not an inverting map for f, a contradiction. Hence (X_j, F_j) is type 2 invertible. □

In [15], it is claimed that the product of two completely invertible fuzzy topological spaces is completely invertible. But this is not true as shown in the following example.

Example 5.9. For, let $X_1 = \{a, b, c, d\}$ and $X_2 = \{p, q, r, s\}$. Consider $f_1,\ g_1 \in I^{X_1}$ and $f_2,\ g_2 \in I^{X_2}$ defined by $f_1(a) = 1,\ f_1(b) = 0,\ f_1(c) = 1,\ f_1(d) = 0,\ g_1(a) = 0,\ g_1(b) = 1,\ g_1(c) = 0,\ g_1(d) = 1,\ f_2(p) = 1,\ f_2(q) = 1,\ f_2(r) = 0,\ f_2(s) = 0,\ g_2(p) = 0,\ g_2(q) = 0,\ g_2(r) = 1,\ g_2(s) = 1$. Then (X_1, F_1) and (X_2, F_2) are fuzzy topological spaces where $F_1 = \{\underline{0},\ \underline{1},\ f_1,\ g_1\}$ and $F_2 =$

$\{\underline{0},\ \underline{1},\ f_2,\ g_2\}$. Let X be the cartesian product of X_1 and X_2. Let F be the product fuzzy topology on X and let P_1 be the projection of the product X into the coordinate set X_1 and P_2 be the projection of the product X into the coordinate set X_2. Now consider $f \in I^X$ defined by $f = P_1^{-1}(f_1) \wedge P_2^{-1}(f_2)$. Clearly $f \neq \underline{0}, \in F$ and $2|supp\ f| < |X|$. Then by Theorem 2.10, (X, F) is not invertible with respect to f so that it is not completely invertible.

Theorem 5.10. *The product fuzzy topology (X, F) of a family of fts (X_i, F_i), $i \in J$ is type 2 completely invertible if and only if (X_j, F_j) is type 2 completely invertible for each $j \in J$.*

Proof. Let (X_i, F_i), $i \in J$ be a family of type 2 completely invertible fts. Let X be the cartesian product $\{X_i,\ i \in J\}$ and let P_i be the projection of the product X into the i^{th} coordinate set X_i. Let $\mathcal{B}$ be the defining base for the product fuzzy topology (X, F). Let $f \neq \underline{0},\ \underline{1}, \in \mathcal{B}$, then $f = \bigcap_{i \in K} P_i^{-1}(g_i); g_i \in F_i$ where K is a finite subset of J. Since $g_i \geq \underline{\frac{1}{2}}$, $\forall g_i \neq \underline{0}, \in F_i$, $i \in J$, we have $f \geq \underline{\frac{1}{2}}$ so that (f, e) is an inverting fuzzy pair of (X, F). Then by Theorem 2.5, (X, F) is type 2 completely invertible.
Conversely suppose that (X, F) is type 2 completely invertible. If possible assume that (X_j, F_j) is not type 2 completely invertible for some $j \in J$. Then by Theorem 2.9, there exists an $f_j \neq \underline{0}, \in F_j$ such that $f_j(x_j) < \frac{1}{2}$ for some $x_j \in X_j$. Now consider $f \in I^X$ defined by $f = P_j^{-1}(f_j)$. Clearly $f \in F$ and (f, e) is not an inverting pair of (X, F) so that (X, F) is not type 2 completely invertible, a contradiction. Hence for each $i \in J$, (X_i, F_i) is type 2 completely invertible. □

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