Thai Journal of Mathematics
Volume 13 (2015) Number 2 : 259–276

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Convergence of AFEM for Second Order Semi-Linear Elliptic PDEs

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Abstract : We analyze a standard adaptive finite element method (AFEM) for second order semi-linear elliptic partial differential equations (PDEs) with vanishing boundary over a polyhedral domain in $\mathbb{R}^d$, $d \geq 2$. Based on a posteriori error estimates using standard residual technique, we prove the contraction property for the weighted sum of the energy error and the error estimator between two consecutive iterations, which also leads to the convergence of AFEM. The obtained result is based on the assumptions that the initial mesh or triangulation is sufficiently refined and the nonlinear inhomogeneous term $f(x, u(x))$ is Lipschitz in the second variable.

Keywords : contraction property; a posteriori error estimates.
2010 Mathematics Subject Classification : 65N12; 65N30; 65N50.

1 Introduction

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded, polyhedral domain. We consider the second order semi-linear elliptic partial differential equation in gradient form with vanishing boundary condition,

$$-\nabla \cdot (A(x)\nabla u(x)) = f(x, u(x)), \qquad \forall\, x \in \Omega, \tag{1.1}$$

$$u(x) = 0, \qquad \forall\, x \in \partial\Omega, \tag{1.2}$$

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where $f(x, u(x))$ is Lipschitz in the second argument, i.e., there exists a Lipschitz constant L_f such that

$$|f(x, v(x)) - f(x, w(x))| \leq L_f \, |v(x) - w(x)| \, , \qquad \forall \, x \in \Omega,$$

and $A(x)$ is a positive definite matrix for each $x \in \Omega$ with strictly monotonicity property, i.e., there exists a positive constant θ_* such that

$$A(p - q) \cdot (p - q) \geq \theta_* \, |p - q|^2 \, , \qquad \forall \, p, q \in \mathbb{R}^d.$$

We analyzed here a standard AFEM having loops of four procedures:

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}.$$

For a given current triangulation and known data (f, A, Ω), the procedure SOVLE finds the approximate solution; the procedure ESTIMATE computes error estimates in a suitable norm based on a posteriori error estimations; the procedure MARK selects elements according to some marking conditions; the procedure REFINE refines the current mesh to obtain a finer triangulation according to the marked elements. The main purpose is to construct a sequence of triangulations together with approximate solutions that will eventually reduce error in an efficient way in term of degree of freedoms.

In studying convergence of AFEM ones usually concern in how to get the errors to go to zero. For linear elliptic partial differential equations, it started with Dörfler [1], who introduced a crucial marking, and proved strict energy error reduction for the Poisson's equation provided the initial mesh satisfies a fineness assumption. Morin et al [2, 3] proved convergence of the AFEM without restrictions on the initial mesh and introduced the concepts of data oscillation and the interior node property, which later was extended to general second order elliptic partial differential equations by Mekchay and Nochetto [4] . Cascon et al [5] obtained quasi-optimal convergence rate for general standard of AFEMs but without the usages of the local lower bound and interior node property.

For nonlinear elliptic partial differential equations, Dörfler [6] developed a robust strategy for nonlinear Poisson equation; Veeser [7] proved convergence of AFEM for the nonlinear Laplacian; Diening and Kreuzer [8] proved that the AFEM for p-Laplacian is linear convergent; and Garau et al [9] showed that the AFEM for quasi-linear problems converges for Kacanov iterations.

In this paper we organized as follows. In section 2, the standard finite element method and AFEM are formulated. In section 3, the crucial Lemmas required for obtaining the contraction property are given and proved. In the last section the contraction property and the convergence result are presented.

2 Problem and Formulation

Let $H^1(\Omega)$ be the usual Sobolev space of functions in $L^2(\Omega)$ whose first order weak derivatives are also in $L^2(\Omega)$, endowed with the norm

$$\|u\|_1 := \left(\|u\|_0^2 + \|\nabla u\|_0^2\right)^{1/2},$$

where $\|\cdot\|_0$ denotes the standard L^2-norm induced by the standard inner product $\langle\cdot,\cdot\rangle$ in $L^2(\Omega)$. Denoted by $H_0^1(\Omega)$ the space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$. A weak solution of (1.1)-(1.2) is a function $u \in H_0^1(\Omega)$ satisfying

$$\mathcal{B}(u,v) = \mathcal{L}(u;v) \qquad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where the bilinear form $\mathcal{B} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{B}(u,v) = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x)\, dx.$$

The functional $\mathcal{L} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{L}(u;v) = \int_\Omega f(x,u(x))v(x)\, dx.$$

Since A is positive definite, the bilinear $\mathcal{B}$ is *symmetric* and *coercive* on $H_0^1(\Omega)$, i.e.,

$$\mathcal{B}(v,v) \geq c_B \|v\|_1^2, \tag{2.2}$$

for some $c_B > 0$ depending only on A and Ω. It is also *continuous* on $H^1(\Omega)$, i.e.,

$$\mathcal{B}(v,w) \leq C_B \|v\|_1 \|w\|_1, \tag{2.3}$$

for some $C_B > 0$ depending only on A and Ω.

The bilinear form $\mathcal{B}$ induces the *energy norm* on $H_0^1(\Omega)$, defined as

$$|\!|\!| v |\!|\!| := \sqrt{\mathcal{B}(v,v)}, \qquad \forall\, v \in H_0^1(\Omega).$$

The semi-norm on $H^1(\Omega)$ is $|v|_1 := \|\nabla v\|_0$. Note that the norm $\|\cdot\|_1$, the semi-norm $|v|_1$, and the energy norm $|\!|\!|\cdot|\!|\!|$ are all equivalent on $H_0^1(\Omega)$.

Let $\mathcal{T}_0$ be an initial triangulation (mesh) of Ω and $\mathbb{T}$ be the class of all *shape-regular conforming refinements* of $\mathcal{T}_0$. Given any conforming triangulation $\mathcal{T} \in \mathbb{T}$ we define the corresponding finite element space, the space of piecewise polynomial functions of fixed degree $n \geq 1$, by

$$\mathbb{V}(\mathcal{T}) := \left\{ v \in H_0^1(\Omega) : v_{|_T} \in \mathbb{P}_n(T), \forall\, T \in \mathcal{T} \right\},$$

where $\mathbb{P}_n(T)$ denotes the space of all polynomials of degree $\leq n$ defined on the element $T \in \mathcal{T}$. Then there exists the unique approximation of u, called the finite element solution, defined as

$$u_\mathcal{T} \in \mathbb{V}(\mathcal{T}); \qquad \mathcal{B}(u_\mathcal{T}, v) = \mathcal{L}(u_\mathcal{T}; v), \qquad \forall\, v \in \mathbb{V}(\mathcal{T}). \tag{2.4}$$

2.1 L^2−Estimates

For simplicity, we write $f_{\mathcal{T}} := f(x, u_{\mathcal{T}})$, $f_k := f_{\mathcal{T}_k}$ and $f := f(x, u)$. Based on the results obtained by Jumpawai's Thesis [10], the L^2 estimates for the error can be given as follows.

Lemma 2.1. *Let u be a weak solution satisfying* (2.1) *and $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ be the solution of* (2.4). *Then*

$$\|u - u_{\mathcal{T}}\|_0 \le C_1^* \|u - u_{\mathcal{T}}\|_1 \sup_{g \in L^2(\Omega), \|g\|_0 \le 1} \left(\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \right) + C_2^* \|f - f_{\mathcal{T}}\|_0,$$

where C_1^* and C_2^* are constants depending only on data. For a given $g \in L^2(\Omega)$, denoted by $\varphi_g \in H_0^1(\Omega)$ the corresponding unique solution of the linear equation

$$\mathcal{B}(\varphi_g, w) = \langle g, w \rangle, \qquad \forall\, w \in H_0^1(\Omega). \tag{2.5}$$

Proof. Let $w \in L^2(\Omega)$. Then $w \in (L^2(\Omega))^*$, the dual space of $L^2(\Omega)$. Ones can easily show that

$$\|w\|_0 = \sup_{g \in L^2(\Omega), \|g\|_0 \le 1} \langle g, w \rangle . \tag{2.6}$$

From (2.1) and (2.4), we have

$$\mathcal{B}(u - u_{\mathcal{T}}, v) = \langle f - f_{\mathcal{T}}, v \rangle, \qquad \forall\, v \in \mathbb{V}(\mathcal{T}). \tag{2.7}$$

By setting $w := u - u_{\mathcal{T}} \in H_0^1(\Omega)$ in (2.5) and using (2.7), for any $\tilde{v} \in \mathbb{V}(\mathcal{T})$ we have

$$\begin{aligned} \langle g, u - u_{\mathcal{T}} \rangle = \mathcal{B}(\varphi_g, u - u_{\mathcal{T}}) &= \mathcal{B}(\varphi_g - \tilde{v}, u - u_{\mathcal{T}}) + \mathcal{B}(\tilde{v}, u - u_{\mathcal{T}}), \\ &= \mathcal{B}(\varphi_g - \tilde{v}, u - u_{\mathcal{T}},) + \langle f - f_{\mathcal{T}}, \tilde{v} \rangle . \end{aligned} \tag{2.8}$$

Applying continuity of $\mathcal{B}$ and the Cauchy-Schwartz inequality to get

$$\langle g, u - u_{\mathcal{T}} \rangle \le C_B \|u - u_{\mathcal{T}}\|_1 \cdot \|\varphi_g - \tilde{v}\|_1 + \|f - f_{\mathcal{T}}\|_0 \|\tilde{v}\|_0. \tag{2.9}$$

Let $\varphi_{g,\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ be a finite element solution of φ_g in (2.5). By Céa's Lemma [11], p.55,

$$\|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 \le \frac{C_B}{c_B} \inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1. \tag{2.10}$$

Taking $\tilde{v} = \varphi_{g,\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ and using (2.10) in (2.9), it gives

$$\langle g, u - u_{\mathcal{T}} \rangle \le C_B \|u - u_{\mathcal{T}}\|_1 \cdot \|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 + \|f - f_{\mathcal{T}}\|_0 \|\varphi_{g,\mathcal{T}}\|_0,$$

hence,

$$\langle g, u - u_{\mathcal{T}} \rangle \le \frac{C_B^2}{c_B} \|u - u_{\mathcal{T}}\|_1 \left(\inf_{v \in \mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \right) + \|f - f_{\mathcal{T}}\|_0 \|\varphi_{g,\mathcal{T}}\|_0. \tag{2.11}$$

By triangle inequality, the last term of (2.11) becomes to

$$\|\varphi_{g,\mathcal{T}}\|_0 = \|\varphi_{g,\mathcal{T}} - \varphi_g + \varphi_g\|_0 \leq \|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 + \|\varphi_g\|_0. \tag{2.12}$$

By duality technique for linear problem (2.5) on a convex polygonal domain [11], p.92 and regularity theorem [11], p.90, the first term on the right hand side of (2.12) becomes

$$\|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 \leq C_\Omega h \|\varphi_{g,\mathcal{T}} - \varphi_g\|_1 \leq c C_\Omega h^2 \|g\|_0. \tag{2.13}$$

where C_Ω and c are constants depending on the domain Ω and shape-regularity, and $h = \max_{T\in\mathcal{T}} h_T$, $h_T := |T|^{1/d}$, where $|T|$ is the measure of T in $\mathbb{R}^d$. Note that this definition is equivalent to the diameter of T.

Setting $w = \varphi_g$ in (2.12) and applying the coercivity (2.2) and Cauchy-Schwartz inequality, we get

$$c_B\|\varphi_g\|_1^2 \leq \mathcal{B}(\varphi_g, \varphi_g) = \langle g, \varphi_g\rangle \leq \|g\|_0\|\varphi_g\|_0$$

Since $\|v\|_0 \leq \|v\|_1$ for all $v \in H^1(\Omega)$, we get $c_B\|\varphi_g\|_0^2 \leq \|g\|_0\|\varphi_g\|_0$. Therefore,

$$\|\varphi_g\|_0 \leq \frac{1}{c_B}\|g\|_0. \tag{2.14}$$

Combining the previous inequalities into (2.12), we have

$$\begin{aligned}\|\varphi_{g,\mathcal{T}}\|_0 &\leq \|\varphi_{g,\mathcal{T}} - \varphi_g\|_0 + \|\varphi_g\|_0 \\ &\leq c C_\Omega h^2\|g\|_0 + \frac{1}{c_B}\|g\|_0 \\ &= \left(c C_\Omega h^2 + \frac{1}{c_B}\right)\|g\|_0.\end{aligned}$$

The inequality (2.11) becomes

$$\langle g, u - u_\mathcal{T}\rangle \leq \frac{C_B}{c_B}\|u - u_\mathcal{T}\|_1 \inf_{v\in\mathbb{V}(\mathcal{T})}\|\varphi_g - v\|_1 + \left(c C_\Omega h^2 + \frac{1}{c_B}\right)\|f - f_\mathcal{T}\|_0\|g\|_0.$$

By setting $C_1^* = \frac{C_B}{c_B}$ and $C_2^* = c C_\Omega h^2 + \frac{1}{c_B}$ and taking sup over all $\|g\|_0 \leq 1$, we obtain the result

$$\begin{aligned}\|u - u_\mathcal{T}\|_0 &= \sup_{g\in L^2(\Omega), \|g\|_0\leq 1}\langle g, u - u_\mathcal{T}\rangle \\ &\leq C_1^*\|u - u_\mathcal{T}\|_1 \sup_{g\in L^2(\Omega),\|g\|_0\leq 1}\left(\inf_{v\in\mathbb{V}(\mathcal{T})}\|\varphi_g - v\|_1\right) + C_2^*\|f - f_\mathcal{T}\|_0. \square\end{aligned}$$

Corollary 2.2. *Under the hypotheses of Lemma 2.1 and* f *satisfies* $L_f \leq \rho < \frac{1}{C_2^*}$ *for some positive* ρ. *Then*

$$\|u - u_\mathcal{T}\|_0 \leq C_f h\|u - u_\mathcal{T}\|_1$$

where C_f *is a constant depending only on* ρ*, the shape regularity, and the data* (A, Ω).

Proof. By regularity theorem [11], p.90,

$$\inf_{v\in\mathbb{V}(\mathcal{T})} \|\varphi_g - v\|_1 \leq \|\varphi_g - \varphi_{g,\mathcal{T}}\|_1 \leq ch\|g\|_0.$$

Lemma 2.1 becomes

$$\|u - u_{\mathcal{T}}\|_0 \leq cC_1^* h\|u - u_{\mathcal{T}}\|_1 + C_2^*\|f - f_{\mathcal{T}}\|_0.$$

By Lipschitz condition, it follows that $\|f - f_{\mathcal{T}}\|_0 \leq L_f\|u - u_{\mathcal{T}}\|_0$ and by assumption $L_f \leq \rho$, we get $\|f - f_{\mathcal{T}}\|_0 \leq \rho\|u - u_{\mathcal{T}}\|_0$. Hence,

$$\|u - u_{\mathcal{T}}\|_0 \leq cC_1^* h\|u - u_{\mathcal{T}}\|_1 + C_2^*\rho\|u - u_{\mathcal{T}}\|_0.$$

Since $C_2^*\rho < 1$, we can combine terms to get

$$\|u - u_{\mathcal{T}}\|_0 \leq C_f h\|u - u_{\mathcal{T}}\|_1,$$

where $C_f := \frac{cC_1^*}{1-C_2^*\rho}$ is a positive constant. □

2.2 Adaptive Finite Element Method: AFEM

We analyze here a standard adaptive finite element method (AFEM) as a loop of procedures

SOLVE → ESTIMATE → MARK → REFINE.

SOLVE: Given a current triangulation $\mathcal{T}$ and a finite element space $\mathbb{V}(\mathcal{T})$, it produces the finite element solution $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$,

$$u_{\mathcal{T}}\text{=SOLVE}(\mathcal{T}).$$

Since (2.4) is a nonlinear problem, one requires an iterative technique to approximate $u_{\mathcal{T}}$ (for example, see [9] for quasi-linear problem).

ESIMATE: For $\mathcal{T} \in \mathbb{T}$, $T \in \mathcal{T}$ and $v \in H_0^1(\Omega)$ we define the local interior residual

$$\mathcal{R}_T(v) := f(x, v)|_T + \nabla \cdot (A\nabla v)|_T. \tag{2.15}$$

The jump residual on side $S \subset \partial T \cap \Omega$

$$J_S(v) := (A\nabla v)|_S \cdot \vec{n}_T + (A\nabla v)|_S \cdot \vec{n}_{T'}, \tag{2.16}$$

where $\vec{n}_T$ and $\vec{n}_{T'}$ are the outward unit normal vectors on S corresponding to T and T', respectively. The local error indicator $\eta_{\mathcal{T}}(v, T)$ on T is defined via

$$\eta_{\mathcal{T}}^2(v, T) := h_T^2\|\mathcal{R}_T(v)\|_{L^2(T)}^2 + h_T\|J_S(v)\|_{L^2(\partial T\cap\Omega)}^2. \tag{2.17}$$

The global error indicator $\eta_{\mathcal{T}}$ for $\mathcal{T}$ is

$$\eta_{\mathcal{T}}(v) := \left(\sum_{T \in \mathcal{T}} \eta_{\mathcal{T}}^2(v, T) \right)^{1/2},$$

and for any subset $\mathcal{T}' \subset \mathcal{T}$,

$$\eta_{\mathcal{T}}(v, \mathcal{T}') := \left(\sum_{T \in \mathcal{T}'} \eta^2(v, T) \right)^{1/2}.$$

Based on a posteriori error analysis, see [12], Jampawai [10] has obtained the upper bound estimate stated as:

Upper bound: Let u be the weak solution (2.1) of the model problem and u_k =SOLVE($\mathcal{T}_k$). Then

$$|||u - u_k||| \leq C_1 \eta_k(u_k) + C_2 h_k \|f - f_k\|_0 \tag{2.18}$$

where C_1, C_2 depend on the shape regularity and the data (A, Ω), h_k is defined to be the maximum of h_T for T in $\mathcal{T}_k$, and denoting $\eta_k(u_k)$ for $\eta_{\mathcal{T}_k}(u_k)$.

MARK: Given a triangulation $\mathcal{T}$, the set of indicators $\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)\}_{T \in \mathcal{T}}$, and the marking parameter $\theta \in (0, 1]$, the procedure MARK produces a marked subset $\mathcal{M} \subset \mathcal{T}$,

$$\mathcal{M} = \text{MARK}(\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)\}_{T \in \mathcal{T}}, \mathcal{T}, \theta),$$

such that $\mathcal{M}$ satisfies some marking properties in some optimal way. For example, in this paper we use Dörfler Marking [1],

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \geq \theta \eta_{\mathcal{T}}(u_{\mathcal{T}}), \tag{2.19}$$

MARK will find an optimal subset $\mathcal{M}$ satisfying the marking property (2.19).

REFINE: Given a fixed integer $b \geq 1$, for any $\mathcal{T} \in \mathbb{T}$ and $\mathcal{M} \subset \mathcal{T}$ of marked elements, the procedure produces a finer conforming triangulation

$$\mathcal{T}_* = \text{REFINE}(\mathcal{T}, \mathcal{M})$$

by refining all elements $T \in \mathcal{M}$ for b times, and together with a few more elements surrounding to be conforming. Note that $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}_*)$. For $T' \in \mathcal{T}_* \backslash \mathcal{T}$ obtained by refining $T \in \mathcal{T}$, i.e., by using newest vertex bisection method b times, we have

$$|T'| \leq 2^{-b} |T|, \tag{2.20}$$

where $|T|$ is the measure of the element T in $\mathbb{R}^d$. Note that for T', as a child of T,

$$h_{T'} = 2^{-b/d} h_T. \tag{2.21}$$

Adaptive Algorithm.

Given the initial grid $\mathcal{T}_0$, TOL, and marking parameter $0 < \theta \leq 1$, set $k = 0$:

1. $u_k = \text{SOLVE}(\mathcal{T}_k)$;
2. $\{\eta_k(u_k, T)\}_{T\in\mathcal{T}_k} = \text{ESTIMATE}(u_k, \mathcal{T}_k)$; (STOP: if $\eta_k < \text{TOL}$.)
3. $\mathcal{M}_k = \text{MARK}(\{\eta_k(u_k, T)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k, \theta)$;
4. $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{M}_k, \mathcal{T}_k)$, set $k = k+1$, go to Step 1.

Note that from (2.21) the algorithm gives the decreasing sequence $\{h_k\}_{k\geq 0}$, namely, $h_k \leq h_0$ for all k.

3 Lemmas

In this section we prove lemmas required for obtaining the contraction property and the convergence of AFEM stated in the next section. These lemmas are obtained according to the AFEM algorithm, based on the four main procedures, SOLVE, ESTIMATE, MARK, and REFINE.

Lemma 3.1. *Let u be the weak solution of* (2.1)*, $u_k = SOLVE(\mathcal{T}_k)$, and $u_{k+1} = SOLVE(\mathcal{T}_{k+1})$. Then*

$$|||u - u_k|||^2 = |||u - u_{k+1}|||^2 + |||u_{k+1} - u_k|||^2 + 2\left\langle f - f_{k+1}, u_{k+1} - u_k\right\rangle.$$

Proof. By nested property of refinements, we have that $\mathbb{V}_k \subset \mathbb{V}_{k+1} \subset H_0^1(\Omega)$ and $u_{k+1} - u_k \in \mathbb{V}_{k+1} \subseteq H_0^1(\Omega)$. From (2.1) and (2.4), we get

$$\begin{aligned}\left\langle f - f_{k+1}, u_{k+1} - u_k\right\rangle &= \left\langle f, u_{k+1} - u_k\right\rangle - \left\langle f_{k+1}, u_{k+1} - u_k\right\rangle \\ &= \mathcal{B}(u, u_{k+1} - u_k) - \mathcal{B}(u_{k+1}, u_{k+1} - u_k) \\ &= \mathcal{B}(u - u_{k+1}, u_{k+1} - u_k).\end{aligned}$$

By definition of the energy norm, we obtain the followings:

$$\begin{aligned}\mathcal{B}(u - u_{k+1}, u_{k+1} - u_k) &= \mathcal{B}(u - u_{k+1}, u_{k+1} - u + u - u_k) \\ &= \mathcal{B}(u - u_{k+1}, u_{k+1} - u) + \mathcal{B}(u - u_{k+1}, u - u_k) \\ &= -|||u - u_{k+1}|||^2 + \mathcal{B}(u - u_{k+1}, u - u_k),\end{aligned}$$

$$\begin{aligned}\mathcal{B}(u - u_{k+1}, u - u_k) &= \mathcal{B}(u - u_k + u_k - u_{k+1}, u - u_k) \\ &= \mathcal{B}(u - u_k, u - u_k) + \mathcal{B}(u_k - u_{k+1}, u - u_k) \\ &= |||u - u_k|||^2 + \mathcal{B}(u_k - u_{k+1}, u - u_k),\end{aligned}$$

$$\begin{aligned}\mathcal{B}(u_k - u_{k+1}, u - u_k) &= \mathcal{B}(u_k - u_{k+1}, u - u_k + u_{k+1} - u_{k+1}) \\ &= \mathcal{B}(u_k - u_{k+1}, u_{k+1} - u_k) + \mathcal{B}(u_k - u_{k+1}, u - u_{k+1}) \\ &= -|||u_{k+1} - u_k|||^2 - \mathcal{B}(u - u_{k+1}, u_{k+1} - u_k) \\ &= -|||u_{k+1} - u_k|||^2 - \left\langle f - f_{k+1}, u_{k+1} - u_k\right\rangle.\end{aligned}$$

By combining all these terms together, we obtain

$$|||u - u_k|||^2 = |||u - u_{k+1}|||^2 + |||u_{k+1} - u_k|||^2 + 2\left\langle f - f_{k+1}, u_{k+1} - u_k\right\rangle. \qquad \square$$

Lemma 3.2. *Let u satisfies* (2.1)*, $u_k = SOLVE(\mathcal{T}_k)$, and $u_{k+1} = SOLVE(\mathcal{T}_{k+1})$. Given that f satisfying the assumption in Corollary 2.2, then*

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq \frac{3}{2} C_e^2 C_f^2 L_f h^2 \|u - u_{k+1}\|^2 + \frac{1}{2} C_e^2 C_f^2 L_f h^2 \|u - u_k\|^2.$$

Proof. By Cauchy-Schwartz inequality,

$$\begin{aligned}\langle f_{k+1} - f, u_{k+1} - u_k \rangle &= \langle f_{k+1} - f, u_{k+1} - u + u - u_k \rangle \\ &= \langle f_{k+1} - f, u_{k+1} - u \rangle + \langle f_{k+1} - f, u - u_k \rangle \\ &\leq \|f_{k+1} - f\|_0 \|u_{k+1} - u\|_0 + \|f_{k+1} - f\|_0 \|u - u_k\|_0.\end{aligned}$$

Applying the Lipschitz condition for $\|f_{k+1} - f\|_0$, we get

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f \|u_{k+1} - u\|_0^2 + L_f \|u_{k+1} - u\|_0 \|u - u_k\|_0.$$

By Corollary 2.2, we obtain

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f C_f^2 h^2 \|u - u_{k+1}\|_1^2 + L_f C_f^2 h^2 \|u - u_{k+1}\|_1 \|u - u_k\|_1.$$

By the equivalent of norms $\|\cdot\|$ and $\|\cdot\|_1$, i.e., $\|\cdot\|_1 \leq C_e \|\cdot\|$, we obtain

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle \leq L_f C_e^2 C_f^2 h^2 \|u - u_{k+1}\|^2 + L_f C_e^2 C_f^2 h^2 \|u - u_{k+1}\| \|u - u_k\|,$$

and by applying the inequality, $2ab \leq a^2 + b^2$, we get

$$\langle f_{k+1} - f, u_{k+1} - u_k \rangle = \frac{3}{2} C_e^2 C_f^2 L_f h^2 \|u - u_{k+1}\|^2 + \frac{1}{2} C_e^2 C_f^2 L_f h^2 \|u - u_k\|^2. \qquad \square$$

Corollary 3.3. *Under assumption of Lemma 3.2, then*

$$\left(1 - 3C_e^2 C_f^2 L_f h^2\right) \|u - u_{k+1}\|^2 \leq \left(1 + C_e^2 C_f^2 L_f h^2\right) \|u - u_k\|^2 - \|u_{k+1} - u_k\|^2.$$

Proof. By Lemma 3.1, we obtain

$$\|u - u_{k+1}\|^2 = \|u - u_k\|^2 - \|u_{k+1} - u_k\|^2 + 2 \langle f_{k+1} - f, u_{k+1} - u_k \rangle. \tag{3.1}$$

Applying Lemma 3.2 to the last term of (3.1), we get

$$\begin{aligned}\|u - u_{k+1}\|^2 \leq \|u - u_k\|^2 - \|u_{k+1} - u_k\|^2 + 3C_e^2 C_f^2 L_f h^2 \|u - u_{k+1}\|^2 \\ + C_e^2 C_f^2 L_f h^2 \|u - u_k\|^2,\end{aligned}$$

which leads to

$$\left(1 - 3C_e^2 C_f^2 L_f h^2\right) \|u - u_{k+1}\|^2 \leq \left(1 + C_e^2 C_f^2 L_f h^2\right) \|u - u_k\|^2 - \|u_{k+1} - u_k\|^2. \qquad \square$$

Lemma 3.4. *For any $\mathcal{T} \in \mathbb{T}$, there holds for all $v, w \in \mathbb{V}(\mathcal{T})$, and $\delta > 0$,*

$$\eta_{\mathcal{T}}^2(v,T) \leq (1+\delta)\eta_{\mathcal{T}}^2(w,T) + h_T\big(1+\frac{1}{\delta}\big)\|J_S(v-w)\|^2_{L^2(\partial T\cap\Omega)} + 2h_T^2\big(1+\frac{1}{\delta}\big)\Big(\|\nabla\cdot(A\nabla(v-w))\|^2_{L^2(T)} + \|f(v)-f(w)\|^2_{L^2(T)}\Big).$$

Proof. For any $T \in \mathbb{T}$, let $v, w \in \mathbb{V}(\mathcal{T})$. We denote for simplicity for $f(x,v)$ and $f(x,w)$ by $f(v)$ and $f(w)$, respectively. Consider $T \in \mathcal{T}$ and its sides $S \subset \partial T$, by using (2.15) we get,

$$\begin{aligned}\mathcal{R}_T(v) &= \nabla\cdot(A\nabla v) + f(v)\\ &= \nabla\cdot(A\nabla(v-w)) + \nabla\cdot(A\nabla w) + f(w) + f(v) - f(w)\\ &= \mathcal{R}_T(w) + \nabla\cdot(A\nabla(v-w)) + f(v) - f(w).\end{aligned}$$

By linearity of the jump residual (2.16), we have

$$J_S(v) = J_S(v-w) + J_S(w).$$

The local error indicator (2.17) leads to

$$\begin{aligned}\eta_{\mathcal{T}}^2(v,T) &= h_T^2\,\|\mathcal{R}_T(w) + \nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|^2_{L^2(T)}\\ &\quad + h_T\,\|J_S(v-w) + J_S(w)\|^2_{L^2(\partial T\cap\Omega)}.\end{aligned}$$

By triangle inequality,

$$\begin{aligned}\eta_{\mathcal{T}}^2(v,T) &\leq h_T^2\left(\|\mathcal{R}_T(w)\|_{L^2(T)} + \|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)}\right)^2\\ &\quad + h_T\left(\|J_S(v-w)\|_{L^2(\partial T\cap\Omega)} + \|J_S(w)\|_{L^2(\partial T\cap\Omega)}\right)^2.\end{aligned} \tag{3.2}$$

For simplicity, let denote

$$\begin{aligned}a &:= \|\mathcal{R}_T(w)\|_{L^2(T)}\\ p &:= \|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)}\\ q &:= \|J_S(v-w)\|_{L^2(\partial T\cap\Omega)}\\ t &:= \|J_S(w)\|_{L^2(\partial T\cap\Omega)}.\end{aligned}$$

The inequality (3.2) becomes

$$\eta_{\mathcal{T}}^2(v,T) \leq h_T^2\left(a^2 + p^2 + 2ap\right) + h_T\left(q^2 + t^2 + 2qt\right) \tag{3.3}$$

Applying the Young's inequality to ap and qt of (3.3), we obtain, for $\delta > 0$,

$$\begin{aligned}\eta_{\mathcal{T}}^2(v,T) &\leq h_T^2\big(a^2 + p^2 + \delta a^2 + \frac{1}{\delta}p^2\big) + h_T\big(q^2 + t^2 + \delta t^2 + \frac{1}{\delta}q^2\big)\\ &= h_T^2(1+\delta)a^2 + h_T^2\big(1+\frac{1}{\delta}\big)p^2 + h_T\big(1+\frac{1}{\delta}\big)q^2 + h_T(1+\delta)t^2.\end{aligned}$$

Therefore,

$$\eta_{\mathcal{T}}^2(v,T) \le (1+\delta)\big\{h_T^2 a^2 + h_T t^2\big\} + h_T\big(1+\frac{1}{\delta}\big)q^2 + h_T^2\big(1+\frac{1}{\delta}\big)p^2. \qquad (3.4)$$

By (2.17), the first term of the right hand side of (3.4) becomes $\eta_{\mathcal{T}}^2(w,T)$. For the term p^2 we get

$$\begin{aligned} p^2 &= \big(\|\nabla\cdot(A\nabla(v-w)) + f(v) - f(w)\|_{L^2(T)}\big)^2 \\ &\le 2\left(\|\nabla\cdot(A\nabla(v-w))\|_{L^2(T)}^2 + \|f(v)-f(w)\|_{L^2(T)}^2\right). \end{aligned}$$

Finally, the inequality (3.4) becomes

$$\begin{aligned} \eta_{\mathcal{T}}^2(v,T) \le{}& (1+\delta)\eta_{\mathcal{T}}^2(w,T) + h_T\big(1+\frac{1}{\delta}\big)\|J_S(v-w)\|_{L^2(\partial T\cap\Omega)}^2 \\ &+ 2h_T^2\big(1+\frac{1}{\delta}\big)\big(\|\nabla\cdot(A\nabla(v-w))\|_{L^2(T)}^2 + \|f(v)-f(w)\|_{L^2(T)}^2\big). \quad \square \end{aligned}$$

Lemma 3.5. *For $\mathcal{T}_k \in \mathbb{T}$, let $\mathcal{M}_k$ =MARK($\{\eta_k(u_k)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k$), let $\mathcal{T}_{k+1} \in \mathbb{T}$ be defined by $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$ for $\lambda := 1-2^{-b/d} > 0$. Then for $v \in H_0^1(\Omega)$,*

$$\eta_{k+1}^2(v) \le \eta_k^2(v) - \lambda\eta_k^2(v,\mathcal{M}_k).$$

Proof. Let $\overline{\mathcal{M}}_k$ be a set of elements in $\mathcal{T}_k$ that are refined to get $\mathcal{T}_{k+1}$ and $\widetilde{\mathcal{M}}_{k+1}$ be a set of newly obtained elements in $\mathcal{T}_{k+1}$ from the refinement of $\mathcal{T}_k$, i.e., $\widetilde{\mathcal{M}}_{k+1} = \mathcal{T}_{k+1}\backslash(\mathcal{T}_{k+1}\cap\mathcal{T}_k)$. Note that the marked set $\mathcal{M}_k \subseteq \overline{\mathcal{M}}_k \subset \mathcal{T}_k$. It is easy to see that $\bigcup_{T\in\overline{\mathcal{M}}_k} T = \bigcup_{T'\in\widetilde{\mathcal{M}}_{k+1}} T'$ and $\overline{\mathcal{M}}_k \cup (\mathcal{T}_k\cap\mathcal{T}_{k+1}) = \mathcal{T}_k$. Since $\mathcal{T}_{k+1}$ is decomposed into two disjoint subsets $\mathcal{T}_k\cap\mathcal{T}_{k+1}$ and $\widetilde{\mathcal{M}}_{k+1}$, then

$$\eta_{k+1}^2(v) = \sum_{T\in\mathcal{T}_k\cap\mathcal{T}_{k+1}} \eta_{k+1}^2(v,T) + \sum_{T'\in\widetilde{\mathcal{M}}_{k+1}} \eta_{k+1}^2(v,T'). \qquad (3.5)$$

Similarly, $\mathcal{T}_k$ is the disjoint union of $\mathcal{T}_k\cap\mathcal{T}_{k+1}$ and $\overline{\mathcal{M}}_k$, then

$$\eta_k^2(v) = \sum_{T\in\mathcal{T}_k\cap\mathcal{T}_{k+1}} \eta_k^2(v,T) + \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T).$$

From the definition of indicators (2.17), we have that $\eta_k(v,T) = \eta_{k+1}(v,T)$ for all $v \in H_0^1(\Omega)$, $T \in \mathcal{T}_k\cap\mathcal{T}_{k+1}$. Then (3.5) becomes to

$$\eta_{k+1}^2(v) = \eta_k^2(v) + \sum_{T'\in\widetilde{\mathcal{M}}_{k+1}} \eta_{k+1}^2(v,T') - \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T). \qquad (3.6)$$

For a marked element $T \in \mathcal{M}_k$, we set $\mathcal{P}_{k+1}(T) = \{T' \in \mathcal{T}_{k+1} : T' \subset T\} \subset \widetilde{\mathcal{M}}_{k+1}$, the set of all children of T. Thus,

$$\sum_{T\in\mathcal{M}_{k+1}} \eta_{k+1}^2(v,T) = \sum_{T\in\overline{\mathcal{M}}_k}\left(\sum_{T'\in\mathcal{P}_{k+1}(T)} \eta_{k+1}^2(v,T')\right) \le 2^{-b/d}\sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T),$$

where the last term comes from the refinement criteria (2.21). Therefore,

$$\eta_{k+1}^2(v) \leq \eta_k^2(v) + 2^{-b/d} \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T) - \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T). \tag{3.7}$$

By defining $\lambda = 1 - 2^{-b/d} > 0$, (3.7) becomes

$$\eta_{k+1}^2(v) \leq \eta_k^2(v) - \lambda \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T).$$

Since $\mathcal{M}_k \subseteq \overline{\mathcal{M}}_k$, $\sum_{T\in\mathcal{M}_k} \eta_k^2(v,T) \leq \sum_{T\in\overline{\mathcal{M}}_k} \eta_k^2(v,T)$, we finally get

$$\eta_{k+1}^2(v) \leq \eta_k^2(v) - \lambda \sum_{T\in\mathcal{M}_k} \eta_k^2(v,T) = \eta_k^2(v) - \lambda\eta_k^2(v,\mathcal{M}_k).$$ □

Lemma 3.6. *For $\mathcal{T}_k \in \mathbb{T}$ and $\mathcal{M}_k$ =MARK($\{\eta_k(u_k)\}_{T\in\mathcal{T}_k},\mathcal{T}_k$), let $\mathcal{T}_{k+1} \in \mathbb{T}$ defined by $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k,\mathcal{M}_k)$. Then for all $v_k \in \mathbb{V}_k$, $v_{k+1} \in \mathbb{V}_{k+1}$, and $\delta > 0$, there holds*

$$\eta_{k+1}^2(v_{k+1}) \leq (1+\delta)\left\{\eta_k^2(v_k) - \lambda\eta_k^2(v_k,\mathcal{M}_k)\right\} + (1+\frac{1}{\delta})K_k\|v_{k+1}-v_k\|_1^2,$$

where $K_k := C_A + 2L_f^2 h_k^2 + 2C_{AA}(1+h_k)^2$.

Proof. By setting $\mathcal{T} = \mathcal{T}_{k+1}$, $v = v_{k+1}$ and $w = v_k$ in Lemma 3.4, we get

$$\begin{aligned}\eta_{k+1}^2(v_{k+1}) &= \sum_{T\in\mathcal{T}_{k+1}} \eta_{k+1}^2(v_{k+1},T)\\ &\leq (1+\delta)\sum_{T\in\mathcal{T}_{k+1}} \eta_{k+1}^2(v_k,T) + \big(1+\frac{1}{\delta}\big)\sum_{T\in\mathcal{T}_{k+1}} h_T\|J_S(v_{k+1}-v_k)\|_{L^2(\partial T\cap\Omega)}^2\end{aligned}$$
$$+2\big(1+\frac{1}{\delta}\big)\Bigg[\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_{k+1}} h_T^2\|f(v_{k+1})-f(v_k)\|_{L^2(T)}^2\Bigg].$$

To estimate the $\|J_S(v_{k+1}-v_k)\|$ term, we applied the trace theorem and the inverse inequality [13] [P.37, 111], to obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_{k+1}} h_T\|J_S(v_{k+1}-v_k)\|_{L^2(\partial T\cap\Omega)}^2 &\leq 2\sum_{T\in\mathcal{T}_{k+1}} h_T\|(A\nabla(v_{k+1}-v_k))|_T\cdot n\|_{L^2(\partial T\cap\Omega)}^2\\ &\leq C_\partial \sum_{T\in\mathcal{T}_{k+1}} \|A\|_\infty \|\nabla(v_{k+1}-v_k)\|_{L^2(T)}^2,\end{aligned}$$

where C_∂ is a constant depends only on shape-regular parameter, and independent of k, and because A is a bounded matrix. This can be written as

$$\sum_{T\in\mathcal{T}_{k+1}} h_T\|J(v_{k+1}-v_k)\|_{L^2(\partial T\cap\Omega)}^2 \leq C_A\|v_{k+1}-v_k\|_1^2,$$

where $C_A := C_\partial \|A\|_\infty$.

To estimate $\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|$ observe that

$$\nabla\cdot(A\nabla(v_{k+1}-v_k)) = (\nabla\cdot A)\cdot(\nabla(v_{k+1}-v_k)) + A : \nabla^2(v_{k+1}-v_k),$$

where $\nabla^2(v_{k+1}-v_k)$ is Hessian matrix of $v_{k+1}-v_k$ and : denotes the Frobenius inner product (or, component-wise inner product). This leads to an estimate

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|_{L^2(T)}^2 \le \sum_{T\in\mathcal{T}_{k+1}} h_T^2\Big(\|\nabla\cdot A\|_\infty\|\nabla(v_{k+1}-v_k)\|_{L^2(T)} + \|A\|_\infty\|\nabla^2(v_{k+1}-v_k)\|_{L^2(T)}\Big)^2.$$

Applying the inverse estimates [11] [p.85] to the Hessian to get

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_{k+1}-v_k))\|_{L^2(T)}^2 \le \sum_{T\in\mathcal{T}_{k+1}} \big(h_T\|\nabla\cdot A\|_\infty + \|A\|_\infty\big)^2\|\nabla(v_{k+1}-v_k)\|_{L^2(T)}^2.$$

This gives that

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|\nabla\cdot(A\nabla(v_k-v_{k+1}))\|_{L^2(T)}^2 \le C_{AA}(1+h_k)^2\|v_{k+1}-v_k\|_1^2,$$

where $C_{AA} := \max\{\|\nabla\cdot A\|_\infty, \|A\|_\infty\}$.

Finally, by the Lipschitz condition on f, we obtain an estimate

$$\sum_{T\in\mathcal{T}_{k+1}} h_T^2\|f(v_{k+1})-f(v_k)\|_{L^2(T)}^2 \le \sum_{T\in\mathcal{T}_{k+1}} h_T^2L_f^2\|v_{k+1}-v_k\|_{L^2(T)}^2 \le h_k^2L_f^2\|v_{k+1}-v_k\|_1^2.$$

After combining all estimates above and applying Lemma 3.5, we get

$$\eta_{k+1}^2(v_{k+1}) = (1+\delta)\big(\eta_k^2(v_k) - \lambda\eta_k^2(v_k,\mathcal{M}_k)\big) + \left(1+\frac{1}{\delta}\right)K_k\|v_{k+1}-v_k\|_1^2,$$

where $K_k := C_A + 2h_k^2L_f^2 + 2C_{AA}(1+h_k)^2$. □

Note that since $\{h_k\}_{k=0}^\infty$ is decreasing, the constant K_k is bounded above by K^*, which is independent of k, given that, for example, $h_0 < 1$ for the starting triangulation. In this case, $K_k \le C_A + 2L_f^2 + 8C_{AA} =: K^*$ for all k.

4 Contraction Property and Convergence

In this section we prove the contraction property for the weighted sum of the energy error and the error estimator from two consecutive iterations of AFEM. The convergence of AFEM follows directly from the theorem as stated in the corollary.

Theorem 4.1. *Given an initial triangulation $\mathcal{T}_0$ with initial mesh-size h_0, let $\theta \in (0,1]$ and $\{\mathcal{T}_k, \mathbb{V}_k, u_k\}_{k\geq 0}$ be a sequence of triangulations $\mathcal{T}_k$, finite element spaces $\mathbb{V}_k$, and discrete solutions u_k produced by AFEM. Then there exists a constant K depending only on the data, and Lipschitz constants such that if $h_0 < K$, then there exist constants $\alpha, \gamma > 0$ and $0 < \mu < 1$ such that*

$$\gamma\eta_{k+1}^2(u_{k+1}) + \alpha|||u - u_{k+1}|||^2 \leq \mu\left(\gamma\eta_k^2(u_k) + \alpha|||u - u_k|||^2\right).$$

Proof. For simplicity in writing, let denote $\eta_k := \eta_k(u_k)$, $\eta_{k+1} := \eta_{k+1}(u_{k+1})$, $e_{k+1} := |||u - u_{k+1}|||$, and $|||u - u_k||| := e_k$. By setting $v_k = u_k$ and $v_{k+1} = u_{k+1}$ in Lemma 3.6, we get

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(u_k, \mathcal{M}_k)\right\} + (1+\frac{1}{\delta})K_k\|u_{k+1} - u_k\|_1^2. \tag{4.1}$$

Using equivalence of norms and setting $E_k = |||u_{k+1} - u_k|||$, (4.1) becomes

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\eta_k^2(u_k, \mathcal{M}_k)\right\} + (1+\frac{1}{\delta})C_e^2 K_k E_k^2,$$

where C_e is a constant for the equivalence depending on the data A and Ω. Applying Dörfler Marking (2.19), $\eta_k(u_k, \mathcal{M}_k) \geq \theta\eta_k$, we have

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\theta^2\eta_k^2\right\} + (1+\frac{1}{\delta})C_e^2 K_k E_k^2. \tag{4.2}$$

Since $K_k \leq K^*$, (4.2) leads to

$$\eta_{k+1}^2 \leq (1+\delta)\left\{\eta_k^2 - \lambda\theta^2\eta_k^2\right\} + (1+\frac{1}{\delta})C_e^2 K^* E_k^2. \tag{4.3}$$

Multiplying (4.3) by $\gamma := \frac{\delta}{C_e^2 K^*(1+\delta)} > 0$ to obtain

$$\gamma\eta_{k+1}^2 \leq \gamma(1+\delta)\eta_k^2 - \gamma\lambda\theta^2(1+\delta)\eta_k^2 + E_k^2.$$

By Corollary 3.3, if for $h_0 < \frac{1}{\sqrt{3C_e^2 C_f^2 L_f}}$, then

$$\gamma\eta_{k+1}^2 + (1 - 3C_e^2 C_f^2 L_f h_k^2)e_{k+1}^2 \leq \gamma(1+\delta)\eta_k^2 - \gamma\lambda\theta^2(1+\delta)\eta_k^2 + (1 + C_e^2 C_f^2 L_f h_k^2)e_k^2.$$

To balance the η_k term, we can rewrite as, for $\beta > 0$,

$$\begin{aligned}\gamma\eta_{k+1}^2 + \left(1 - 3C_e^2 C_f^2 L_f h_k^2\right)e_{k+1}^2 \leq{}& \gamma(1+\delta)\eta_k^2 + \left(1 + C_e^2 C_f^2 L_f h_k^2\right)e_k^2 \\ &- \beta\gamma\lambda\theta^2(1+\delta)\eta_k^2 - (1-\beta)\gamma\lambda\theta^2(1+\delta)\eta_k^2.\end{aligned} \tag{4.4}$$

Using the upper bound (2.18), the Lipschitz condition on f, the Corollary 2.2, and the equivalence of norms, we get

$$\begin{aligned}e_k &\leq C_1\eta_k + C_2 h_k\|f - f_k\|_0 \\ &\leq C_1\eta_k + C_2 h_k L_f\|u - u_k\|_0 \\ &\leq C_1\eta_k + C_2 L_f C_f h_k^2\|u - u_k\|_1 \\ &\leq C_1\eta_k + C_e C_2 C_f L_f h_k^2 e_k.\end{aligned}$$

If for $h_0 < \frac{1}{\sqrt{C_e C_2 C_f L_f}}$, then we have

$$0 < \left(\frac{1 - C_e C_2 C_f L_f h_k^2}{C_1} \right) e_k \leq \eta_k. \tag{4.5}$$

Combining (4.5) to the right hand side of (4.4), we have

$$\begin{aligned} \gamma\eta_{k+1}^2 + (1-3C_e^2 C_f^2 L_f h_k^2) e_{k+1}^2 \\ &\leq \gamma(1+\delta)\eta_k^2 + \left(1+C_e^2 C_f^2 L_f h_k^2\right) e_k^2 - (1-\beta)\gamma\lambda\theta^2(1+\delta)\eta_k^2 \\ &\quad - \beta\gamma\lambda\theta^2(1+\delta)\left(\frac{1-C_e C_2 C_f L_f h_k^2}{C_1}\right)^2 e_k^2. \end{aligned}$$

For convenience we denote the coefficients as follows;

$$\begin{aligned} \alpha_1 &= 1 - 3C_e^2 C_f^2 L_f h_k^2 > 0, \\ \alpha_2 &= 1 + C_e^2 C_f^2 L_f h_k^2 - \frac{\delta}{C_e^2 K^*}\beta\lambda\theta^2 \left(\frac{1 - C_e C_2 C_f L_f h_k^2}{C_1}\right)^2, \\ \alpha_3 &= (1+\delta)\left(1 - (1-\beta)\lambda\theta^2\right). \end{aligned}$$

This can be written as

$$\gamma\eta_{k+1}^2 + \alpha_1 e_{k+1}^2 \leq \gamma\alpha_3\eta_k^2 + \alpha_2 e_k^2 = \gamma\alpha_3\eta_k^2 + \alpha_1 \left(\frac{\alpha_2}{\alpha_1}\right) e_k^2. \tag{4.6}$$

The result follows by setting $\alpha = \alpha_1$ and showing that $\mu := \max\{\alpha_3, \frac{\alpha_2}{\alpha_1}\} < 1$.

Showing $0 < \alpha_3 < 1$ is equivalent to $0 < (1+\delta)\left(1 - (1-\beta)\lambda\theta^2\right) < 1$. This is the case if we choose $\beta > 0$ such that

$$0 < \beta < 1 - \frac{1}{\lambda\theta^2}\left(\frac{\delta}{1+\delta}\right). \tag{4.7}$$

Since λ and θ are known from AFEM and $\lambda\theta^2 < 1$, then we can choose $\beta > 0$ satisfying (4.7) provided that $\delta > 0$ is pre-selected so that $\frac{1}{\lambda\theta^2} \cdot \frac{\delta}{1+\delta} < 1$, i.e., choosing

$$0 < \delta < \frac{\lambda\theta^2}{1 - \lambda\theta^2}. \tag{4.8}$$

In order to arrive at (4.6) it is required that $h_0 < \min\{\frac{1}{\sqrt{3C_e^2 C_f^2 L_f}}, \frac{1}{\sqrt{C_e C_2 C_f L_f}}\}$, for obtaining (4.4) and (4.5), thus this gives $\alpha_1 > 0$.

We get $\alpha_2 > 0$ by selecting δ satisfying (4.8) and sufficiently small so that

$$\frac{\delta}{C_e^2 K^*}\beta\lambda\theta^2 \left(\frac{1 - C_e C_2 C_f L_f h_k^2}{C_1}\right)^2 < 1.$$

The case $0 < \alpha_2 < \alpha_1$ holds if and only if

$$1 + C_e^2 C_f^2 L_f h_k^2 - \frac{\delta}{C_e^2 K^*}\beta\lambda\theta^2 \left(\frac{1 - C_e C_2 C_f L_f h_k^2}{C_1}\right)^2 < 1 - 3C_e^2 C_f^2 L_f h_k^2.$$

This is equivalent to

$$h_k^2 < \frac{\delta\beta\lambda\theta^2}{4C_e^4 K^* C_f^2 L_f C_1^2}(1 - C_e C_2 C_f L_f h_k^2)^2.$$

For convenience for computation, set $r = C_e C_2 C_f L_f$ and $s = \frac{\delta\beta\lambda\theta^2}{4C_e^4 K^* C_f^2 L_f C_1^2}$. The condition on h_k becomes that

$$h_k^2 < s(1 - 2rh_k^2 + r^2 h_k^4)$$

This is the case if $h_0 < \sqrt{\frac{s}{1+2rs}}$ because $sr^2 h_k^4 \geq 0$.

By selecting $K := \min\left\{\frac{1}{\sqrt{3C_e^2 C_f^2 L_f}}, \frac{1}{\sqrt{C_e C_2 L_f C_f}}, \sqrt{\frac{s}{1+2rs}}\right\} > 0$, the condition $h_0 < K$ will give us the contraction result for (4.6). □

Corollary 4.2 (Convergence)**.** *Under the hypothesis of Theorem 4.1,*

$$\lim_{k\to\infty} \eta_k(u_k) = 0 \text{ and } \lim_{k\to\infty} |||u - u_k||| = 0.$$

Proof. From Theorem 4.1, it is easy to see that

$$\gamma\eta_{k+1}^2(u_{k+1}) + \alpha|||u - u_{k+1}|||^2 \leq \mu^{k+1}\left(\gamma\eta_0^2(u_0) + \alpha|||u - u_0|||^2\right)$$

Since $\lim_{k\to\infty} \mu = 0$ for $\mu \in (0,1)$, and $\gamma, \alpha > 0$, thus

$$\lim_{k\to\infty} \eta_k(u_k) = 0 \text{ and } \lim_{k\to\infty} |||u - u_{k+1}||| = 0.$$ □

5 Examples

In this section, we give some examples of semi-linear elliptic partial differential equations satisfying the assumption of the main Theorem 4.1. The following are examples of nonlinear functions $f(x,u)$ that satisfy the assumptions of the theorem.

Example 5.1. Let $f(x,u) = e^{-\frac{1}{m}u^2}$, where $x \in \Omega := [0,1]^2$ and a constant $m > 0$. It is clear that

$$\left|\frac{\partial f}{\partial u}\right| = \left|\frac{-2u}{me^{\frac{1}{m}u^2}}\right| = \frac{2}{m} \cdot \frac{|u|}{e^{\frac{1}{m}u^2}}, \qquad \forall u \in \mathbb{R}. \tag{5.1}$$

By calculus, $\frac{|u|}{e^{\frac{1}{m}u^2}}$ has absolute maximum $\sqrt{\frac{m}{2e}}$, i.e., $\frac{|u|}{e^{\frac{1}{m}u^2}} \leq \sqrt{\frac{m}{2e}}$, $\forall u \in \mathbb{R}$. Thus,

$$\left|\frac{\partial f}{\partial u}\right| \leq \sqrt{\frac{2}{me}}, \qquad \forall u \in \mathbb{R}.$$

We can choose $L_f = \sqrt{\frac{2}{me}}$. By corollary 2.2, $L_f < \frac{1}{C_2^*}$, if

$$m > \frac{2}{ec_B^2}(cc_BC_B + 1)^2,$$

where $c_B = \theta_*$, $C_B = \|A\|_\infty$ and $c = \frac{(1+c_1)c_2C_B}{c_B}$. In the case where A is identity, we get that $c_B = C_B = 1$, therefore, and we require that $m > \frac{2(c+1)^2}{e}$ in order to satisfy the condition of the Corollary 2.2. Moreover, since $f(x,u)$ is continuous and bounded on Ω,

$$\int_\Omega |f(x,u)|^2 dx = \int_\Omega e^{-\frac{2}{m}u^2}\, dx < \infty.$$

Example 5.2. Let $f((x_1, x_2), u) = x_2 \sin(mx_1u)$, where $(x_1, x_2) \in \Omega := [0,1]^2$ and $m > 0$. It is easy to see that

$$\left|\frac{\partial f}{\partial u}\right| = |mx_1x_2 \cos(mx_1u)| \leq m, \qquad \forall u \in \mathbb{R},\ \forall (x_1, x_2) \in \Omega.$$

Similarly, if we choose $L_f = m.$, then we require that

$$m < \frac{c_B}{cc_BC_B + 1}.$$

In the case where $A = I$ we need that $m < \frac{1}{c+1}$. Moreover, since $f(x,u)$ is continuous and bounded on Ω,

$$\int_\Omega |f(x,u)|^2\, dx = \int_\Omega x_2^2 \sin^2(mx_1u)\, dx < \infty.$$

Acknowledgements : I would like to thank the referees for his comments and suggestions on the manuscript.

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(Received 17 July 2014)
(Accepted 21 March 2015)