Concircular Curvature Tensor of a Semi-Symmetric Metric Connection in a Kenmotsu Manifold

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Abstract : The object of the present paper is to study a Kenmotsu manifold admitting a semi-symmetric metric connection whose concircular curvature tensor satisfies certain curvature conditions.

1 Introduction

The product of an almost contact manifold M and the real line R carries a natural almost complex structure. However if one takes M to be an almost contact metric manifold and suppose that the product metric G on $M \times R$ is Kaehlerian, then the structure on M is cosymplectic [1] and not Sasakian. On the other hand Oubina [2] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinate on R, is Kaehlerian, then M is Sasakian and conversely.

In [3], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, M is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M is the product of a line or a circle with a Kaehler manifold of constant holomorphic

sectional curvature. If $c < 0$, M is a warped product space $R \times_f C^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [4]. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by Jun et al. [5], Özgür and De [6], De and Pathak [7], Yıldız et al. [8] and others.

Hayden [9] introduced semi-symmetric linear connection on a Riemannian manifold and this was further developed by Yano [10], Amur and Pujar [11], Prvanović [12], De and Biswas [13], Sharfuddin and Hussain [14], Binh [15], Ö. Zengin et al. [16], Chaubey and Ojha [17, 18], Yılmaz [19] and others.

Let M^n be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and D be the Levi-Civita connection on (M^n, g).

A linear connection ∇ defined on (M^n, g) is said to be semi-symmetric [20] if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form and ξ is a vector field given by

$$\eta(X) = g(X, \xi), \tag{1.2}$$

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

A semi-symmetric connection ∇ is called a semi-symmetric metric connection [9] if it further satisfies

$$\nabla g = 0. \tag{1.3}$$

A relation between the semi-symmetric metric connection ∇ and the Levi-Civita connection D on (M^n, g) has been obtained by Yano [10] which is given by

$$\nabla_X Y = D_X Y + \eta(Y)X - g(X, Y)\xi. \tag{1.4}$$

We also have

$$(\nabla_X \eta)(Y) = (D_X \eta)(Y) - \eta(X)\eta(Y) + \eta(\xi)g(X, Y). \tag{1.5}$$

Further, a relation between the curvature tensor R of the semi-symmetric metric connection ∇ and the curvature tensor K of the Levi-Civita connection D is given by

$$\begin{aligned} R(X, Y)W = K(X, Y)W + \alpha(X, W)Y - \alpha(Y, W)X + g(X, W)QY \\ - g(Y, W)QX, \end{aligned} \tag{1.6}$$

where α is a tensor field of type (0,2) and Q is a tensor field of type (1,1) which is given by

$$\alpha(Y, W) = g(QY, W) = (D_Y \eta)(W) - \eta(Y)\eta(W) + \frac{1}{2}\eta(\xi)g(Y, W). \tag{1.7}$$

From (1.6) and (1.7), we obtain

$$\tilde{R}(X,Y,W,U) = \tilde{K}(X,Y,W,U) - \alpha(Y,W)g(X,U) + \alpha(X,W)g(Y,U)$$
$$- g(Y,W)\alpha(X,U) + g(X,W)\alpha(Y,U), \tag{1.8}$$

where

$$\tilde{R}(X,Y,W,U) = g(R(X,Y)W,U), \quad \tilde{K}(X,Y,W,U) = g(K(X,Y)W,U). \tag{1.9}$$

A transformation of a $(2n+1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([21, 22]). A concircular transformation is always a conformal transformation [21]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [23]). An interesing invariant of a concircular transformation is the concircular curvature tensor $\mathcal{Z}$. It is defined by ([22, 24])

$$\mathcal{Z}(X,Y)W = R(X,Y)W - \frac{r}{2n(2n+1)}[g(Y,W)X - g(X,W)Y]. \tag{1.10}$$

From (1.10), it follows that

$$\tilde{\mathcal{Z}}(X,Y,W,U) = \tilde{R}(X,Y,W,U) - \frac{r}{2n(2n+1)}[g(Y,W)g(X,U) - g(X,W)g(Y,U)], \tag{1.11}$$

and

$$\tilde{\mathcal{Z}}(X,Y,W,U) = g(\mathcal{Z}(X,Y)W,U), \tag{1.12}$$

where X, Y, W, $U \in \chi(M)$ and $\mathcal{Z}$ is the concircular curvature tensor and r is the scalar curvature with respect to the semi-symmetric metric connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In this paper we study the concircular curvature tensor on Kenmotsu manifolds with respect to the semi-symmetric metric connection. The paper is organized as follows: After introduction in Section 2, we give a brief account of the Kenmotsu manifolds. Section 3 is devoted to study ϕ-concircularly flat Kenmotsu manifolds with respect to the semi-symmetric metric connection and also determine the ϕ-sectional curvature of the plane by two vectors. Section 4 deals with $\mathcal{Z} \cdot S = 0$ in a Kenmotsu manifold with respect to the semi-symmetric metric connection. Finally, we study $\mathcal{Z} \cdot \mathcal{Z} = 0$ in a Kenmotsu manifold with respect to the semi-symmetric metric connection and we prove that the manifold is an η-Einstein manifold provided the scalar curvature of the manifold is not equal to $2n(6n-1)$.

2 Preliminaries

Let M be an $(2n + 1)$-dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying [25]

$$\phi^2(X) = -X + \eta(X)\xi, \ \ g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1, \ \ \phi(\xi) = 0, \ \ \eta(\phi(X)) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X, Y on M. If an almost contact metric manifold satisfies

$$(D_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

then M is called a Kenmotsu manifold [4]. From the above relations, it follows that

$$D_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(D_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.6}$$

Moreover the curvature tensor K and the Ricci tensor $\tilde{S}$ of the Kenmotsu manifold with respect to the Levi-Civita connection satisfies

$$K(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$K(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$K(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$\tilde{S}(\phi X, \phi Y) = \tilde{S}(X, Y) + 2n\eta(X)\eta(Y), \tag{2.10}$$

$$\tilde{S}(X, \xi) = -2n\eta(X). \tag{2.11}$$

We state the following lemma which will be used in the next section:

Lemma 2.1 ([4]). *Let M be an η-Einstein Kenmotsu manifold of the form $\tilde{S}(X, Y)$ $= ag(X, Y) + b\eta(X)\eta(Y)$. If $b = constant(or, a = constant)$, then M is an Einstein one.*

3 ϕ-Concircularly Flat Kenmotsu Manifolds with respect to the Semi-Symmetric Metric Connection

Let C be the Weyl conformal curvature tensor of a $(2n + 1)$-dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into the direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is an 1-dimensional linear subspace of $\chi_p(M)$ generated by ξ_p. Then we have a map:

$$C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

(1) $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow L(\xi_p)$, i.e, the projection of the image of C in $\phi(\chi_p(M))$ is zero.

(2) $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M))$, i.e, the projection of the image of C in $L(\xi_p)$ is zero.

$$C(X,Y)\xi = 0. \tag{3.1}$$

(3) $C : \phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M)) \longrightarrow L(\xi_p)$, i.e, when C is restricted to $\phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M))$, the projection of the image of C in $\phi(\chi_p(M))$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi W = 0. \tag{3.2}$$

The cases (1) and (2) were considered in [26] and [27] respectively. The case (3) was considered in [28] for the case M is a K-contact manifold. Furthermore in [29], the authors studied contact metric manifolds satisfying (3). Analogous to the definition of ξ-conformally flat and ϕ-conformally flat, we give the folowing definitions:

Definition 3.1. A Kenmotsu manifold is said to be ϕ-concircuiarly flat with respect to semi-symmetric metric connection if

$$g(\mathcal{Z}(\phi X, \phi Y)\phi W, \phi U) = 0, \tag{3.3}$$

where $X, Y, W, U \in \chi(M)$.

Definition 3.2. A Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor $\tilde{S}$ of the Levi-Civita connection is of the form

$$\tilde{S}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.4}$$

where a and b are smooth functions on the manifold.

Using (1.7), (2.2) and (2.6) in (1.6), we obtain

$$R(X,Y)W = K(X,Y)W - 3g(Y,W)X + 3g(X,W)Y + 2\eta(Y)\eta(W)X$$
$$- 2\eta(X)\eta(W)Y + 2g(Y,W)\eta(X)\xi - 2g(X,W)\eta(Y)\xi. \tag{3.5}$$

Using (1.9) in (3.5), we get

$$\tilde{R}(X,Y,W,U) = \tilde{K}(X,Y,W,U) - 3g(Y,W)g(X,U) + 3g(X,W)g(Y,U)$$
$$+ 2\eta(Y)\eta(W)g(X,U) - 2\eta(X)\eta(W)g(Y,U)$$
$$+ 2g(Y,W)\eta(X)\eta(U) - 2g(X,W)\eta(Y)\eta(U). \tag{3.6}$$

Let $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $X = U = e_i;\ 1 \leq i \leq 2n + 1$, in (3.6) and also using (2.1) and (2.2), we have

$$S(Y,W) = \tilde{S}(Y,W) - 2(3n - 1)g(Y,W) + 2(2n - 1)\eta(Y)\eta(W). \tag{3.7}$$

Putting $W = \xi$ in (3.7) and using (2.1), (2.2) and (2.11), we obtain

$$S(Y,\xi) = -4n\eta(Y). \tag{3.8}$$

Let $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $Y = W = e_i$; $1 \leq i \leq 2n + 1$, in (3.7) and also using (2.2), it follows that

$$r = \tilde{r} - 2n(6n - 1). \tag{3.9}$$

where S and $\tilde{r}$ are the Ricci tensor and the scalar curvature with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively.

Putting $X = \phi X$, $Y = \phi Y$, $W = \phi W$ and $U = \phi U$ in (1.11) and using (1.12), we get

$$g(\mathcal{Z}(\phi X, \phi Y)\phi W, \phi U) = \tilde{R}(\phi X, \phi Y, \phi W, \phi U) - \frac{r}{2n(2n+1)}[g(\phi Y, \phi W)g(\phi X, \phi U)$$
$$- g(\phi X, \phi W)g(\phi Y, \phi U)]. \tag{3.10}$$

Using (3.6) and (3.9) in (3.10), we obtain

$$g(\mathcal{Z}(\phi X, \phi Y)\phi W, \phi U) = \tilde{K}(\phi X, \phi Y, \phi W, \phi U) - \left[3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right]$$
$$\times [g(\phi Y, \phi W)g(\phi X, \phi U) - g(\phi X, \phi W)g(\phi Y, \phi U)]. \tag{3.11}$$

Again using (3.3) in (3.11), we have

$$\tilde{K}(\phi X, \phi Y, \phi W, \phi U) = \left[3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right][g(\phi Y, \phi W)g(\phi X, \phi U)$$
$$- g(\phi X, \phi W)g(\phi Y, \phi U)]. \tag{3.12}$$

Let $\{e_1, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, \ldots, \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (3.12) and summing over $i = 1$ to $2n$, it follows that

$$\sum_{i=1}^{2n} \tilde{K}(\phi e_i, \phi Y, \phi W, \phi e_i) = \sum_{i=1}^{2n}\left[3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right][g(\phi Y, \phi W)g(\phi e_i, \phi e_i)$$
$$- g(\phi e_i, \phi W)g(\phi Y, \phi e_i)]. \tag{3.13}$$

From (3.13) yields

$$\tilde{S}(\phi Y, \phi W) = (2n - 1)\left[3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right]g(\phi Y, \phi W). \tag{3.14}$$

Using (2.3) and (2.10) in (3.14), we obtain

$$\tilde{S}(Y, W) = (2n - 1)\left(3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right)g(Y, W)$$
$$- \left[(2n - 1)\left(3 + \frac{\tilde{r} - 2n(6n - 1)}{2n(2n+1)}\right) + 2n\right]\eta(Y)\eta(W). \tag{3.15}$$

Therefore, $\tilde{S}(Y,W) = ag(Y,W) + b\eta(Y)\eta(W)$, where $a = (2n-1)\left(3 + \frac{\tilde{r} - 2n(6n-1)}{2n(2n+1)}\right)$, $b = -\left[(2n-1)\left(3 + \frac{\tilde{r} - 2n(6n-1)}{2n(2n+1)}\right) + 2n\right]$. This result shows that the manifold is an η-Einstein manifold.

Hence we can state the following theorem:

Theorem 3.3. *If a Kenmotsu manifold is ϕ-concircularly flat with respect to semi-symmetric metric connection, then the manifold is an η-Einstein manifold.*

Definition 3.4. A plane section in $\chi(M)$ is called a ϕ-section if there exists a unit vector X in $\chi(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is orthogonal basis of the plane section. Then the sectional curvature $'K(X, \phi X)$ is called a ϕ-sectional curvature.

Let $\xi^{\perp}$ denoted the $(2n+1)$ dimensional distribution orthogonal to ξ in a Kenmotsu manifold admitting a semi-symmetric metric connection whose curvature tensor vanishes. Then for any $X \in \xi^{\perp}$, $g(X, \xi) = 0$. Now we shall determine the ϕ-sectional curvature $'K$ at the plane determine by the vectors $X, \phi X \in \xi^{\perp}$. Putting $Y = \phi X$, $W = \phi X$ and $U = X$ in (3.6), we get

$$\tilde{K}(X, \phi X, \phi X, X) = 3[g(X,X)g(\phi X, \phi X) - g(X, \phi X)g(X, \phi X)]. \qquad (3.16)$$

Then

$$'K(X, \phi X) = \frac{\tilde{K}(X, \phi X, \phi X, X)}{g(X,X)g(\phi X, \phi X) - g(X, \phi X)^2} = 3. \qquad (3.17)$$

Summing up we can state the following theorem :

Theorem 3.5. *If a Kenmotsu manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, then the ϕ-sectional curvature of the plane determined by two vectors $X, \phi X \in \xi^{\perp}$ is 3.*

4   Kenmotsu Manifolds with respect to the Semi-Symmetric Metric Connection Satisfying $\mathcal{Z} \cdot S = 0$

In this section, we consider Kenmotsu manifold with respect to the semi-symmetric metric connection M^{2n+1} satisfying condition

$$(\mathcal{Z}(U,Y) \cdot S)(W, X) = 0.$$

Then we have

$$S(\mathcal{Z}(U,Y)W, X) + S(W, \mathcal{Z}(U,Y)X) = 0. \qquad (4.1)$$

Putting $U = \xi$ in (4.1), it follows that

$$S(\mathcal{Z}(\xi, Y)W, X) + S(W, \mathcal{Z}(\xi, Y)X) = 0. \tag{4.2}$$

Putting $X = \xi$ and using (2.1) in (1.10), we get

$$\mathcal{Z}(\xi, Y)W = R(\xi, Y)W - \frac{r}{2n(2n+1)}[g(Y, W)\xi - \eta(W)Y]. \tag{4.3}$$

Again putting $X = \xi$ in (3.5) and using (2.8), we obtain

$$R(\xi, Y)W = 2[\eta(W)Y - g(Y, W)\xi]. \tag{4.4}$$

Using (3.9), (4.3) and (4.4) in (4.2), we have

$$\left[\frac{\tilde{r} + 6n - 4n^2}{2n(2n+1)}\right][\eta(W)S(X, Y) - g(Y, W)S(X, \xi)$$
$$+ \eta(X)S(W, Y) - g(Y, X)S(W, \xi)] = 0. \tag{4.5}$$

Now we consider

 Case (I): $\tilde{r} \neq 4n^2 - 6n$ and

$$\eta(W)S(X, Y) - g(Y, W)S(X, \xi) + \eta(X)S(W, Y) - g(Y, X)S(W, \xi) = 0. \tag{4.6}$$

 Case (II): $\tilde{r} = 4n^2 - 6n$ and

$$\eta(W)S(X, Y) - g(Y, W)S(X, \xi) + \eta(X)S(W, Y) - g(Y, X)S(W, \xi) = 0. \tag{4.7}$$

 Case (III): $\tilde{r} = 4n^2 - 6n$ and

$$\eta(W)S(X, Y) - g(Y, W)S(X, \xi) + \eta(X)S(W, Y) - g(Y, X)S(W, \xi) \neq 0. \tag{4.8}$$

From Case (I), we obtain

$$\eta(W)S(X, Y) - g(Y, W)S(X, \xi) + \eta(X)S(W, Y) - g(Y, X)S(W, \xi) = 0. \tag{4.9}$$

Putting $W = \xi$ in (4.9) and using (3.7) and (3.8), we get

$$\tilde{S}(X, Y) = (2n - 2)g(X, Y) - 2(2n - 1)\eta(X)\eta(Y). \tag{4.10}$$

Let $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $X = Y = e_i$; $1 \leq i \leq 2n + 1$, in (4.10) and also using (2.2), it follows that

$$\tilde{r} = 4n^2 - 6n,$$

which is contradictory.

Then from Case (II), we have

$$\eta(W)S(X,Y) - g(Y,W)S(X,\xi) + \eta(X)S(W,Y) - g(Y,X)S(W,\xi) = 0. \quad (4.11)$$

Putting $W = \xi$ in (4.11) and using (3.7) and (3.8), we obtain

$$\tilde{S}(X,Y) = (2n-2)g(X,Y) - 2(2n-1)\eta(X)\eta(Y). \quad (4.12)$$

Therefore, $\tilde{S}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a = (2n-2)$ and $b = -2(2n-1)$. This result shows that the manifold is an η-Einstein manifold.

From Case (III), we get

$$\eta(W)S(X,Y) - g(Y,W)S(X,\xi) + \eta(X)S(W,Y) - g(Y,X)S(W,\xi) \neq 0. \quad (4.13)$$

Putting $W = \xi$ in (4.13) and using (3.7) and (3.8), we have

$$\tilde{S}(X,Y) \neq (2n-2)g(X,Y) - 2(2n-1)\eta(X)\eta(Y). \quad (4.14)$$

Let $\{e_1, \ldots, e_{2n}, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Putting $X = Y = e_i$; $1 \leq i \leq 2n+1$, in (4.14) and also using (2.2), it follows that

$$\tilde{r} \neq 4n^2 - 6n,$$

which is also contradictory.

From the above discussions we can state the following theorem:

Theorem 4.1. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $\mathcal{Z} \cdot S = 0$, then the manifold is an η-Einstein manifold provided the relation (4.7) holds.*

Since a and b are both constant, by Lemma 2.1, we get the following:

Corollary 4.2. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $\mathcal{Z} \cdot S = 0$, then the manifold is an Einstein manifold provided the relation (4.7) holds.*

5 Kenmotsu Manifolds with respect to the Semi-Symmetric Metric Connection Satisfying $\mathcal{Z} \cdot \mathcal{Z} = 0$

In this section we consider Kenmotsu manifold with respect to the semi-symmetric metric connection M^{2n+1} satisfying condition

$$(\mathcal{Z}(X,Y).\mathcal{Z})(U,V)W = 0.$$

Then we have

$$(\mathcal{Z}(X,Y)\mathcal{Z})(U,V)W - \mathcal{Z}(\mathcal{Z}(X,Y)U,V)W - \mathcal{Z}(U,\mathcal{Z}(X,Y)V)W$$
$$- \mathcal{Z}(U,V)\mathcal{Z}(X,Y)W = 0. \qquad (5.1)$$

Putting $X = \xi$ in (5.1), it follows that

$$(\mathcal{Z}(\xi,Y)\mathcal{Z})(U,V)W - \mathcal{Z}(\mathcal{Z}(\xi,Y)U,V)W - \mathcal{Z}(U,\mathcal{Z}(\xi,Y)V)W$$
$$- \mathcal{Z}(U,V)\mathcal{Z}(\xi,Y)W = 0. \qquad (5.2)$$

Putting $U = \xi$ in (5.2), we obtain

$$(\mathcal{Z}(\xi,Y)\mathcal{Z})(\xi,V)W - \mathcal{Z}(\mathcal{Z}(\xi,Y)\xi,V)W - \mathcal{Z}(\xi,\mathcal{Z}(\xi,Y)V)W$$
$$- \mathcal{Z}(\xi,V)\mathcal{Z}(\xi,Y)W = 0. \qquad (5.3)$$

From (3.9), (4.3) and (4.4), we get

$$\mathcal{Z}(\xi,Y)W = \left[\frac{\tilde{r} - 4n^2 + 6n}{4n^2 + 2n}\right] [\eta(W)Y - g(Y,W)\xi]. \qquad (5.4)$$

Putting $W = \xi$ in (5.4) and using (2.1) and (2.2), we have

$$\mathcal{Z}(\xi,Y)\xi = \left[\frac{\tilde{r} - 4n^2 + 6n}{4n^2 + 2n}\right] [Y - \eta(Y)\xi]. \qquad (5.5)$$

Now using (5.4) and (5.5) in (5.3), it follows that

$$3\left(\frac{\tilde{r} - 12n^2 + 2n}{2n(2n+1)}\right) g(V,W)Y - \left(\frac{\tilde{r} - 12n^2 + 2n}{2n(2n+1)}\right) K(Y,V)W$$
$$- 3\left(\frac{\tilde{r} - 12n^2 + 2n}{2n(2n+1)}\right) g(Y,W)V - 2\eta(V)\eta(W)Y + 2\eta(Y)\eta(W)V$$
$$- 2g(V,W)\eta(Y)\xi + 2g(Y,W)\eta(V)\xi = 0. \qquad (5.6)$$

Contracting Y in (5.6), we obtain

$$\tilde{S}(V,W) = \left[\frac{\tilde{r} - 12n^2 + 2n}{2n+1} + \frac{36n^2(2n+1)}{\tilde{r} - 12n^2 + 2n} + 3n - \frac{4n(2n+1)}{\tilde{r} - 12n^2 + 2n}\right] g(V,W)$$
$$+ \left[\frac{2n(4n-2)(2n+1)}{\tilde{r} - 12n^2 + 2n}\right] \eta(V)\eta(W). \qquad (5.7)$$

Therefore, $\tilde{S}(V,W) = ag(V,W) + b\eta(V)\eta(W)$, where $a = [\frac{\tilde{r}-12n^2+2n}{2n+1} + \frac{36n^2(2n+1)}{\tilde{r}-12n^2+2n} + 3n - \frac{4n(2n+1)}{\tilde{r}-12n^2+2n}]$ and $b = [\frac{2n(4n-2)(2n+1)}{\tilde{r}-12n^2+2n}]$.

We can state the following theorem:

Theorem 5.1. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $\mathcal{Z} \cdot \mathcal{Z} = 0$, then the manifold is an η-Einstein manifold provided the scalar curvature of the manifold is not equal to $2n(6n-1)$.*

Since a and b are both constant, by Lemma 2.1, we get the following:

Corollary 5.2. *If a Kenmotsu manifold with respect to the semi-symmetric metric connection satisfying $\mathcal{Z} \cdot \mathcal{Z} = 0$, then the manifold is an Einstein manifold provided the scalar curvature of the manifold is not equal to $2n(6n-1)$.*

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