Thai Journal of Mathematics
Volume 13 (2015) Number 1 : 227–236
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Periodic Behavior of Solutions of a Certain Piecewise Linear System of Difference Equations

Wirot Tikjha [†,♭,1] **Somsri Jintanasonti**[†] **and Yongwimon Lenbury**[‡,♭]

[†]Faculty of Science and Technology, Pibulsongkram Rajabhat University,
Phitsanuloke, Thailand
e-mail : wirottik@psru.ac.th (W. Tikjha)
e-mail : srijin@gmail.com (S. Jintanasonti)
[‡]Department of Mathematics, Mahidol University
Rama 6 Road, Bangkok, Thailand
e-mail : scylb@mahidol.ac.th (Y. Lenbury)
[♭]Centre of Excellence in Mathematics, PERDO, CHE, Thailand

Abstract : In this paper, we study the behavior of solutions of piecewise linear system of difference equations $x_{n+1} = |x_n| - y_n - 2$ and $y_{n+1} = x_n + |y_n|$ with initial condition that (x_0, y_0) is in $R^2 - \{(x, y) : x < 0 \text{ and } y < 0\}$. After we observe via a computer program and some direct computations, we found that the system has an equilibrium point and periodic solutions. We also show that the solution of the system is eventually periodic with prime period 3 by finding the pattern of solutions and formulating the statements that involve the natural numbers and then proving by mathematical induction.

Keywords : Difference equations; Periodic solutions; System of piecewise linear difference equations.
2010 Mathematics Subject Classification : 39A10; 39A11.

1 Introduction

Difference equations usually describe the evolution of a certain phenomenon over the course of time. In mathematics, a difference equation is a sequence of

[1]Corresponding author.

numbers that are the functions of previous numbers. The following definitions [1] are used in this paper. A *difference equation of order* $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), n = 0, 1, \ldots \tag{1.1}$$

where f is a continuous function which maps some set J^{k+1} into J. The set J is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of integers.

A *solution* of Eq.(1.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies Eq.(1.1) for all $n \geq 0$. If we prescribe a set of $(k+1)$ *initial conditions*

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in J$$

then

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-k})$$
$$x_2 = f(x_1, x_0, \ldots, x_{-k+1})$$
$$\vdots$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) exists for all $n \geq -k$ and is uniquely determined by the initial conditions.

A solution of Eq.(1.1) which is constant for all $n \geq -k$ is called an *equilibrium solution* of Eq.(1.1). If

$$x_n = \bar{x} \text{ for all } n \geq -k$$

is an equilibrium solution of Eq.(1.1), then $\bar{x}$ is called an *equilibrium point*, or simply an *equilibrium*, of Eq.(1.1).

A solution$\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *periodic with period* p (or a *period* p *solution*) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \text{ for all } n \geq -k \tag{1.2}$$

We say that the solution is *periodic with prime period* p if p is the smallest positive integer for which (1.2) holds. In this case, a p-tuple

$$(x_{n+1}, x_{n+2}, \cdots, x_{n+p})$$

of any p consecutive values of the solution is called a p - *cycle* of Eq.(1.1).

A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.1) is called *eventually periodic with period* p if there exists an integer $N \geq -k$ such that $\{x_n\}_{n=-k}^{\infty}$ is periodic with period p; that is,

$$x_{n+p} = x_n \text{ for all } n \geq N$$

Difference equations have rich and far reaching applications, especially in the field of Biology and Economics [2, 3, 4].

Devaney [5, 6] investigated the piecewise linear difference equation, known as the gingerbreadman map,

$$x_{n+1} = |x_n| - x_{n-1} + 1, n = 1, 2, 3, \ldots \tag{1.3}$$

which has been shown to be chaotic in certain regions and stable in others. The name of this equation is due to the fact that solutions in the plane look like a gingerbreadman" [5] when graphed. Equation(1.3) is equivalent to the piecewise linear system,

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n \end{cases}, n = 0, 1, \ldots \tag{1.4}$$

Gerasimos Ladas made significant contributions to the generalized gingerbreadman map as an open problem in the form of 81 piecewise linear systems:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0, 1, \ldots \tag{1.5}$$

where the initial conditions x_0, and y_0 are arbitrary real numbers and the parameters a, b, c, and d are integers between -1 and 1, inclusively.

Grove and his team [7] found that the behavior of solution of the piecewise linear system of difference equation,

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n + y_n \end{cases}, n = 0, 1, \ldots \tag{1.6}$$

is eventually periodic with period 3 for every initial condition $(x_0, y_0) \in \mathbf{R}^2$. We would like to study a generalization of system (1.6), which is the system

$$\begin{cases} x_{n+1} = |x_n| - y_n - b \\ y_{n+1} = x_n + y_n \end{cases}, n = 0, 1, \ldots \tag{1.7}$$

with initial condition $(x_0, y_0) \in \mathbf{R}^2$ and b is positive integer. So, we investigate system(1.7) by substituting b with 2. In this paper we consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n - 2 \\ y_{n+1} = x_n + |y_n| \end{cases}, n = 0, 1, \ldots \tag{1.8}$$

with initial condition (x_0, y_0) being some points in $\mathbf{R}^2$. The results of this problem will help us to predict the behavior of generalization of system (1.6) and (1.8) and we believe that the investigating of the system will give us the germ of generality that is required to understand systems with more complicated behavior. We let $\begin{pmatrix} a, & b \\ c, & d \\ e, & f \end{pmatrix}$ be a 3-cycle that consists of 3 points, i.e. $(a, b), (c, d), (e, f)$.

2 Main Results

The solutions of System(1.8) are has the unique equilibrium point

$$(\bar{x}, \bar{y}) = \left(-\frac{4}{5}, -\frac{2}{5}\right)$$

and two 3-cycles

$$P_3^1 = \begin{pmatrix} 0, & -2 \\ 0, & 2 \\ -4, & 2 \end{pmatrix} \qquad \text{and} \qquad P_3^2 = \begin{pmatrix} 0, & -\frac{2}{3} \\ -\frac{4}{3}, & \frac{2}{3} \\ -\frac{4}{3}, & -\frac{2}{3} \end{pmatrix}.$$

The main result of this paper is as follows:

Lemma 2.1. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8) *and suppose that there is an* $N \geq 0$ *such that* $y_N = -x_N - 2 \geq 0$. *Then* $(x_{N+1}, y_{N+1}) = (0, -2) \in P_3^1$.

Proof. We have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 2 &&= 0, \text{ and} \\ y_{N+1} &= x_N + |y_N| &&= -2. \end{aligned}$$

Hence $(x_{N+1}, y_{N+1}) = (0, -2) \in P_3^1$. □

Lemma 2.2. *Let* $\mathcal{L}_1 = \{(0, y) | y \geq 0\}$ *and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8). *Then every solution with initial condition in* $\mathcal{L}_1$ *is eventually periodic with prime period* 3.

Proof. Let $(x_0, y_0) \in \mathcal{L}_1$, we have

$$\begin{aligned} x_1 &= |x_0| - y_0 - 2 &&= -y_0 - 2 < 0, \text{ and} \\ y_1 &= x_0 + |y_0| &&= y_0 > 0. \end{aligned}$$

We see that $y_1 = -x_1 - 2$, and applying Lemma 2.1, $(x_2, y_2) = (0, -2) \in P_3^1$. □

Lemma 2.3. *Let* $\mathcal{L}_2 = \{(0, y) | y < 0\}$*and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8). *Then every solution with initial condition in* $\mathcal{L}_2$ *is eventually periodic with prime period* 3.

Proof. Let $(x_0, y_0) \in \mathcal{L}_2$, we have

$$\begin{aligned} x_1 &= |x_0| - y_0 - 2 &&= 0 - y_0 - 2 &&= -y_0 - 2, \text{ and} \\ y_1 &= x_0 + |y_0| &&= 0 - y_0 &&= -y_0 > 0. \end{aligned}$$

Case1: $x_1 = -y_0 - 2 > 0, y_0 < -2$. Thus,

$$\begin{aligned} x_2 &= |x_1| - y_1 - 2 &&= -4 < 0 \\ y_2 &= x_1 + |y_1| &&= -2y_0 - 2 > 0, \\ x_3 &= |x_2| - y_2 - 2 &&= 2y_0 + 4 < 0 \\ y_3 &= x_2 + |y_2| &&= -2y_0 - 6. \end{aligned}$$

If $y_3 = -2y_0 - 6 \geq 0, y_0 \leq -3$, then we see that $y_3 = -x_3 - 2 \geq 0$, and applying Lemma 2.1, $(x_4, y_4) = (0, -2) \in P_3^1$.

Suppose that $y_3 = -2y_0 - 6 < 0, -3 < y_0 < -2$, we will prove that for $y_0 \in (-3, -2)$ the solution is eventually prime period $3(P_3^1$ or $P_3^2)$ by Mathematical induction.

For each integer n with $n \geq 1$, let P(n) be the following statement: "for $y_0 \in (l_{n-1}, u_{n-1})$,

$$x_{3n+1} = 0, \quad y_{3n+1} = 2^{2n}y_0 + \delta_n,$$

such that y_{3n+1} is non-negative when $y_0 \in [u_n, u_{n-1})$, and so $(x_{3n+3}, y_{3n+3}) = (0, -2) \in P_3^1$. On the other hand, y_{3n+1} is negative when $y_0 \in [l_{n-1}, u_n)$, and so

$$\begin{aligned} x_{3n+2} &= -2^{2n}y_0 - (\delta_n + 2) < 0, & y_{3n+2} &= -2^{2n}y_0 - \delta_n > 0, \\ x_{3n+3} &= 2^{2n+1}y_0 + 2\delta_n < 0, & y_{3n+3} &= -2^{2n+1}y_0 - (2\delta_n + 2), \end{aligned}$$

such that y_{3n+3} is non-negative when $y_0 \in (l_{n-1}, l_n]$, and so $(x_{3n+4}, y_{3n+4}) = (0, -2) \in P_3^1$ where as y_{3n+3} is negative when $y_0 \in (l_n, u_n)$ where

$$l_n = \frac{-2^{2n+4} - 2}{3 \times 2^{2n+1}}, u_n = \frac{-2^{2n+3} + 2}{3 \times 2^{2n}}, \delta_n = \frac{2^{2n+3} - 2}{3}."$$

We shall show that P(1) is true. Since $x_3 = 2y_0 + 4 < 0$ and $y_3 = -2y_0 - 6 < 0$ when $y_0 \in (l_0, u_0) = (-3, -2)$, we have that

$$\begin{aligned} x_{3(1)+1} = x_4 &= |x_3| - y_3 - 2 &&= 0, \text{ and} \\ y_{3(1)+1} = y_4 &= x_3 + |y_3| &&= 4y_0 + 10 \quad = 2^{2(1)}y_0 + \delta_1. \end{aligned}$$

If $y_0 \in [u_1, u_{1-1}) = [u_1, u_0) = [-\frac{10}{4}, -2)$, then $y_{3(1)+1} = 4y_0 + 10 \geq 0$. We apply Lemma 2.2 to obtain $(x_{3(1)+3}, y_{3(1)+3}) = (0, -2) \in P_3^1$.

If $y_0 \in (l_{1-1}, u_1) = (l_0, u_1) = (-3, -\frac{10}{4})$, then $y_{3(1)+1} = 4y_0 + 10 < 0$, and so

$$\begin{aligned} x_{3(1)+2} = x_5 &= |x_4| - y_4 - 2 &&= -4y_0 - 12 &&= -2^{2(1)}y_0 - (\delta_1 + 2) < 0 \\ y_{3(1)+2} = y_5 &= x_4 + |y_4| &&= -4y_0 - 10 &&= -2^{2(1)}y_0 - \delta_1 > 0, \\ x_{3(1)+3} = x_6 &= |x_5| - y_5 - 2 &&= 8y_0 + 20 &&= 2^{2(1)+1}y_0 + 2\delta_1 < 0 \\ y_{3(1)+3} = y_6 &= x_5 + |y_5| &&= -8y_0 - 22 &&= -2^{2(1)+1}y_0 - (2\delta_1 + 2). \end{aligned}$$

If $y_0 \in (l_{1-1}, l_1] = (l_0, l_1] = (-3, -\frac{22}{8}]$, then $y_{3(1)+3} = -8y_0 - 22 \geq 0$, we apply Lemma 2.1 to obtain $(x_{3(1)+4}, y_{3(1)+4}) = (0, -2) \in P_3^1$.

If $y_0 \in (l_1, u_1) = (-\frac{22}{8}, -\frac{10}{4})$, then $y_{3(1)+3} = -8y_0 - 22 < 0$. Hence, P(1) is true.

Next, assume that P(N) is true. We shall show that P$(N+1)$ is true. Since P(N) is true, we have

$$x_{3N+3} = 2^{2N+1}y_0 + 2\delta_N < 0 \text{ and } y_{3N+3} = -2^{2N+1}y_0 - (2\delta_N + 2) < 0$$

when

$$y_0 \in (l_N, u_N) = \left(\tfrac{-2^{2N+4}-2}{3\times 2^{2N+1}}, \tfrac{-2^{2N+3}+2}{3\times 2^{2N}}\right).$$

Then,

$$\begin{aligned} x_{3(N+1)+1} = x_{3N+4} &= |x_{3N+3}| - y_{3N+3} - 2 &&= 0, \text{ and} \\ y_{3(N+1)+1} = y_{3N+4} &= x_{3N+3} + |y_{3N+3}| &&= 2^{2(N+1)}y_0 + 4\delta_N + 2 \\ & &&= 2^{2(N+1)}y_0 + \delta_{N+1}. \end{aligned}$$

Note that

$$\delta_{N+1} = \frac{2^{2N+5}-2}{3} = 4\left(\frac{2^{2N+3}-2}{3}\right) + \frac{6}{3} = 4\delta_N + 2.$$

If

$$y_0 \in [u_{N+1}, u_{(N+1)-1}) = [u_{N+1}, u_N) = \left[\frac{-2^{2N+5}+2}{3\times 2^{2N+2}}, \frac{-2^{2N+3}+2}{3\times 2^{2N}}\right),$$

then $y_{3(N+1)+1} = 2^{2(N+1)}y_0 + \delta_{N+1} = 2^{2(N+1)}y_0 + \left(\frac{2^{2N+5}-2}{3}\right) \geq 0$. We apply Lemma 2.2 to obtain $(y_{3(N+1)+3}, y_{3(N+1)+3}) = (0, -2) \in P_3^1$.

If

$$y_0 \in (l_{(N+1)-1}, u_{(N+1)}) = (l_N, u_{N+1}) = \left(\frac{-2^{2N+4}-2}{3\times 2^{2N+1}}, \frac{-2^{2N+5}+2}{3\times 2^{2N+2}}\right),$$

then

$$\begin{aligned} y_{3(N+1)+1} &= 2^{2(N+1)}y_0 + \delta_{N+1} &&= 2^{2(N+1)}y_0 + \left(\frac{2^{2N+5}-2}{3}\right) < 0, \text{ so} \\ x_{3(N+1)+2} &= |x_{3N+4}| - y_{3N+4} - 2 &&= -2^{2(N+1)}y_0 - (\delta_{N+1} + 2) \\ & &&= -2^{2N+2}y_0 + \left(\frac{-2^{2N+5}-4}{3}\right) < 0 \\ y_{3(N+1)+2} &= x_{3N+4} + |y_{3N+4}| &&= -2^{2(N+1)}y_0 - \delta_{N+1} > 0, \\ x_{3(N+1)+3} &= |x_{3N+5}| - y_{3N+5} - 2 &&= 2^{2(N+1)+1}y_0 + 2\delta_{N+1} < 0 \\ y_{3(N+1)+3} &= x_{3N+5} + |y_{3N+5}| &&= -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 2). \end{aligned}$$

If

$$y_0 \in (l_{(N+1)-1}, l_{N+1}] = (l_N, l_{N+1}] = \left(\frac{-2^{2N+4}-2}{3\times 2^{2N+1}}, \frac{-2^{2N+6}-2}{3\times 2^{2N+3}}\right],$$

then

$$y_{3(N+1)+3} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 2) = -2^{2N+3}y_0 + \left(\frac{-2^{2N+5}\cdot 2-2}{3}\right) \geq 0.$$

We apply Lemma 2.1 to obtain $(x_{3(N+1)+4}, y_{3(N+1)+4}) = (0, -2) \in P_3^1$.

If

$$y_0 \in (l_{N+1}, u_{N+1}) = \left(\frac{-2^{2N+6}-2}{3\times 2^{2N+3}}, \frac{-2^{2N+5}+2}{3\times 2^{2N+2}}\right),$$

then

$$y_{3(N+1)+3} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + 2) = -2^{2N+3}y_0 + \left(\frac{-2^{2N+5}\cdot 2-2}{3}\right) < 0.$$

Hence, P$(N+1)$ is true. By Mathematical induction, P(n) is true for all $n \geq 1$.

Note that

$$\lim_{n\to\infty} l_n = -\frac{8}{3} = \lim_{n\to\infty} u_n.$$

We also note that if $(x_0, y_0) = (0, -\frac{8}{3})$, then $(x_3, y_3) = (-\frac{4}{3}, -\frac{2}{3}) \in P_3^2$. We can conclude that in Case1, every solution of the system is eventually prime period 3 (P_3^1 or P_3^2).

Case2: $x_1 = -y_0 - 2 \leq 0, y_0 \geq -2$, then

$$\begin{aligned} x_2 &= |x_1| - y_1 - 2 &&= y_0 + 2 + y_0 - 2 &&= 2y_0 < 0 \\ y_2 &= x_1 + |y_1| &&= -y_0 - 2 - y_0 &&= -2y_0 - 2. \end{aligned}$$

If $y_2 = -2y_0 - 2 \geq 0, -2 \leq y_0 \leq -1$, then we apply Lemma 2.1:

$$(x_3, y_3) = (0, -2) \in P_3^1.$$

Suppose that $y_2 = -2y_0 - 2 < 0, y_0 > -1$, then

$$\begin{aligned} x_3 &= |x_2| - y_3 - 2 &&= 0 \\ y_3 &= x_2 + |y_3| &&= 4y_0 + 2. \end{aligned}$$

If $y_3 = 4y_0 + 2 \geq 0$, then we apply Lemma 2.2, and so

$$(x_5, y_5) = (0, -2) \in P_3^1.$$

Suppose that $y_3 = 4y_0 + 2 < 0, -1 < y_0 < -\frac{2}{4}$. We will prove that for $y_0 \in (-1, -\frac{2}{4})$ the solution is eventually prime period 3 by Mathematical induction.

For each integer n with $n \geq 1$, let Q(n) be the following statement; "for $y_0 \in (\mathfrak{l}_n, \mathfrak{u}_n)$, we have

$$\begin{aligned} x_{3n+1} &= -2^{2n}y_0 - (\gamma_n + 2) < 0, & y_{3n+1} &= -2^{2n}y_0 - \gamma_n > 0 \\ x_{3n+2} &= 2^{2n+1}y_0 + 2\gamma_n < 0, & y_{3n+2} &= -2^{2n+1}y_0 - (2\gamma_n + 2) \end{aligned}$$

such that y_{3n+2} is non-negative when $y_0 \in (\mathfrak{l}_n, \mathfrak{l}_{n+1}]$, and so $(x_{3n+3}, y_{3n+3}) = (0, -2) \in P_3^1$ where as y_{3n+2} is negative when $y_0 \in (\mathfrak{l}_{n+1}, \mathfrak{u}_n)$, and so

$$x_{3n+3} = 0, \quad y_{3n+3} = 2^{2n+2}y_0 + (4\gamma_n + 2).$$

Thus y_{3n+3} is non-negative when $y_0 \in [\mathfrak{u}_{n+1}, \mathfrak{u}_n)$, and so $(x_{3n+5}, y_{3n+5}) = (0, -2) \in P_3^1$ where as y_{3n+3} is negative when $y_0 \in (\mathfrak{l}_{n+1}, \mathfrak{u}_{n+1})$ where

$$\mathfrak{l}_n = \frac{-2^{2n}-2}{3\times 2^{2n-1}}, \mathfrak{u}_n = \frac{-2^{2n+1}+2}{3\times 2^{2n}}, \gamma_n = \frac{2^{2n+1}-2}{3}."$$

The proof is similar the above case. So we will omit the proof. Then we have Q(n) is true for all $n \geq 1$.

Note that

$$\lim_{n\to\infty} \mathfrak{l}_n = -\frac{2}{3} = \lim_{n\to\infty} \mathfrak{u}_n$$

and we also note that $(x_0, y_0) = (0, -\frac{2}{3}) \in P_3^2$.

The proof is complete. □

Lemma 2.4. *Let* $Q_1 = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x \geq 0 \text{ and } y \geq 0\}$ *and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8). *Then every solution with initial condition in* Q_1 *is eventually in* $\mathcal{L}_1$ *or* $\mathcal{L}_2$.

Proof. Let $(x_0, y_0) \in Q_1$. Then

$$\begin{array}{lll} x_1 & = |x_0| - y_0 - 2 & = x_0 - y_0 - 2 \\ y_1 & = x_0 + |y_0| & = x_0 + y_0 \geq 0. \end{array}$$

Case 1: When $x_1 = x_0 - y_0 - 2 \geq 0$,

$$\begin{array}{lll} x_2 & = |x_1| - y_1 - 2 & = -2y_0 - 4 < 0 \\ y_2 & = x_1 + |y_1| & = 2x_0 - 2 > 0, \\ x_3 & = |x_2| - y_2 - 2 & = -2x_0 + 2y_0 + 4 \leq 0 \\ y_3 & = x_2 + |y_2| & = 2x_0 - 2y_0 - 6, \\ x_4 & = |x_3| - y_3 - 2 & = 0. \end{array}.$$

Then, (x_4, y_4) is in $\mathcal{L}_1$ or $\mathcal{L}_2$.
Case 2: When $x_1 = x_0 - y_0 - 2 < 0$,

$$\begin{array}{lll} x_2 & = |x_1| - y_1 - 2 & = -2x_0 \leq 0 \\ y_2 & = x_1 + |y_1| & = 2x_0 - 2, \\ x_3 & = |x_2| - y_2 - 2 & = 0. \end{array}$$

Then (x_4, y_4) is in $\mathcal{L}_1$ or $\mathcal{L}_2$.

□

Lemma 2.5. *Let* $Q_2 = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x < 0 \text{ and } y \geq 0\}$ *and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8). *Then every solution with initial condition in* Q_2 *is eventually periodic with prime period* 3.

Proof. Let $(x_0, y_0) \in Q_2$. Then,

$$\begin{array}{lll} x_1 & = |x_0| - y_0 - 2 & = -x_0 - y_0 - 2 \\ y_1 & = x_0 + |y_0| & = x_0 + y_0. \end{array}$$

If $x_1 = -x_0 - y_0 - 2 < 0$ and $y_1 = x_0 + y_0 \geq 0$, then we apply Lemma 2.1, that $(x_2, y_2) = (0, -2) \in P_3^1$.

If $x_1 = -x_0 - y_0 - 2 < 0$ and $y_1 = x_0 + y_0 < 0$, then $x_2 = |x_1| - y_1 - 2 = 0$, (x_2, y_2) is in $\mathcal{L}_1$ or $\mathcal{L}_2$.

If $x_1 = -x_0 - y_0 - 2 \geq 0$ and $y_1 = x_0 + y_0 < 0$, then

$$\begin{array}{lll} x_2 & = |x_1| - y_1 - 2 & = -2x_0 - 2y_0 - 4 \geq 0 \\ y_2 & = x_1 + |y_1| & = -2x_0 - 2y_0 - 2 > 0, \\ x_3 & = |x_2| - y_2 - 2 & = -4 \\ y_3 & = x_2 + |y_2| & = -4x_0 - 4y_0 - 6 \geq 0, \\ x_4 & = |x_3| - y_3 - 2 & = 4x_0 + 4y_0 + 8 \leq 0 \\ y_4 & = x_3 + |y_3| & = -4x_0 - 4y_0 - 10, \\ x_5 & = |x_3| - y_3 - 2 & = 0. \end{array}$$

Thus (x_5, y_5) is in $\mathcal{L}_1$ or $\mathcal{L}_2$.

In above cases, the solutions are eventually in $\mathcal{L}_1$ or $\mathcal{L}_2$. We apply Lemma (2.2) and Lemma (2.3).

If $x_1 = -x_0 - y_0 - 2 \geq 0$ and $y_1 = x_0 + y_0 \geq 0$, we apply Lemma 2.4, then the proof is complete. □

Lemma 2.6. *Let* $Q_4 = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x \geq 0 \text{ and } y < 0\}$ *and* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8)*. Then every solution with initial condition in* Q_4 *is eventually periodic with prime period* 3*.*

Proof. Let $(x_0, y_0) \in Q_4$. Then

$$\begin{array}{lll} x_1 & = |x_0| - y_0 - 2 & = x_0 - y_0 - 2 \\ y_1 & = x_0 + |y_0| & = x_0 - y_0 > 0. \end{array}$$

If $x_1 = x_0 - y_0 - 2 \geq 0$, then $(x_1, y_1) \in Q_1$, and we apply Lemma 2.4.

If $x_1 = x_0 - y_0 - 2 < 0$, then $(x_1, y_1) \in Q_2$, and we apply Lemma 2.5. The proof is complete. □

Theorem 2.7. *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (1.8) *with initial condition* $(x_0, y_0) \in \mathbf{R}^2 - \{(x, y) : x < 0 \text{ and } y < 0\}$*. Then the solution is eventually prime period 3 solution (*P_3^1 *or* P_3^2*.)*

3 Discussion and Conclusion

We begin with the first result in Lemma2.1 as a tool to prove the next lemma which is about the solution of the system satisfying some conditions such that the next iteration will be a point in P_3^1. The second lemma states that if we begin with initial condition $(x_0, y_0) \in \mathcal{L}_1$(nonnegative y axis) then the solution of the system is eventually P_1^3 in two iterations by using Lemma 2.1. The third lemma states that if we begin with initial condition $(x_0, y_0) \in \mathcal{L}_2$(negative y axis) then the solution of the system is eventually P_1^3 or P_2^3. We separate the solutions into two cases and use Lemma 2.1, Lemma 2.2 and mathematical induction to prove a couple of induction statements in the third lemma. Now we know that if we begin with initial condition on the y axis then the solution of the system is eventually periodic with prime period 3. The forth lemma states that if we begin with initial condition in the first quadrant the solution will be in $\mathcal{L}_1$ or $\mathcal{L}_2$ which means the solution of the system is eventually periodic with prime period 3. The fifth lemma state that if we begin with initial condition in the second quadrant then the solution is eventually periodic with prime period 3 by using Lemma 2.1 Lemma 2.2 and Lemma 2.3. The last lemma states that if we begin with initial condition in the forth quadrant, then the solution will be in the first quadrant or the second quadrant where we can apply Lemma 2.4 and Lemma 2.5. This allows us to conclude that the solution of the system is eventually periodic with prime period 3 when the initial condition is in $\mathbf{R}^2$ except for initial conditions in the third quadrant. We conjecture that the solution of the system is eventually periodic with prime period 3 for every initial condition in $\mathbf{R}^2$.

Acknowledgements : We would like to thank the referees for comments and suggestions on the manuscript. This work was supported by the Thailand Research Fund(MRG5580088) National Research Council of Thailand and Pibulsongkram Rajabht University. The first and third authors are supported by the Centre of Excellence in Mathematics, CHE, Thailand.

References

[1] E.A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman Hall/CRC Press, New York, 2005.

[2] R. M. May, Host-parasitoid systems in patchy environments A PhenomenologicalModel, J ANIM ECOL. 47 (1987) 833 - 843.

[3] T. Awerbuch-Friedlander, R. Levins, E. Camouzis, E. A. Grove G. Ladas, A Non-Linear System of Difference Equations Linking Mosquitoes Habitats and Community Interventions, Communications on Applied Nonlinear Analysis. 15 (2008) 77 - 88.

[4] V.L.Kocic G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher order with Applications, Kluwer academic publishers, Boston, 1993.

[5] M.F. Barnsley, R.L. Devaney, B.B. Mandelbrot, H. O. Peitgen, D. Saupe R.F. Voss, The Science of Fractal Images, Springer-Verlag, New York,1991.

[6] R.L. Devanney, A piecewise linear model of the the zones of instability of an area-preservingmap, Physica. 10D (1984) 387 - 393.

[7] E.A. Grove, E. Lapierre, W. Tikjha, On the Global Behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$, Cubo Mathematical Journal. 14 (2012) 125 - 166.

(Received 25 September 2014)
(Accepted 19 March 2015)