Thai Journal of Mathematics
Volume 13 (2015) Number 1 : 213–225
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Factorisable Monoid of Generalized Hypersubstitutions of Type $\tau = (2)$

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Abstract : A generalized hypersubstitution of type τ maps any operation symbol to the set of all terms of the same type which does not necessarily preserve the arity. Every generalized hypersubstitution can be extended to a mapping on the set of all terms. We define a binary operation on the set of all generalized hypersubstitutions by using this extension. It turns out that this set together with the binary operation forms a monoid. In this paper, we characterize all unit elements and determine the set of all unit-regular elements of this monoid of type $\tau = (2)$. We conclude a submonoid of the moniod of all generalized hypersubstitutions of type $\tau = (2)$ which is factorisable.

Keywords : Generalized hypersubstitution; Unit element; Unit-regular element.
2010 Mathematics Subject Classification : 20B30; 20M05; 20M17.

1 Introduction

The notions of hyperidentities and hypervarieties of a given type τ without nullary operations originated by J.Aczèl [1], V.D. Belousov [2], W.D. Neumann [3] and W. Taylor [4]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [5]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [6]. We

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defined a binary operation on the set of all generalized hypersubstitutions and then proved that this set together with the binary operation forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids. The present paper gives the characterization of all unit elements and determine the set of all unit-regular elements of this monoid of type $\tau = (2)$ that leads to a submonoid of this monoid which is factorisable.

2 Preliminaries

Let $\tau = (n_i)_{i \in I}$ be a type indexed by a set I, f_i be an operation symbol of arity n_i for $n_i \in \mathbb{N}$. Let $X_n := \{x_1, x_2, ..., x_n\}$ be an n-element alphabet and $X := \{x_1, x_2, ...\}$ be a countably infinite set of variables. An *n-ary term of type τ*, for simply an n-ary term, is defined inductively as follows:

(i) The variables $x_1, x_2, ..., x_n$ are n-ary terms.

(ii) If $t_1, t_2, ..., t_{n_i}$ are n-ary terms then $f_i(t_1, t_2, ..., t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the smallest set which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii). Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ and called the set of all terms of type τ.

A generalized hypersubstitution of type τ, for simply a generalized hypersubstitution, is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which does not necessarily preserve an arity. The set of all generalized hypersubstitutions of type τ is denoted by $Hyp_G(\tau)$. To define a binary operation on this set, we need the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ which is defined by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, ..., t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, ..., t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, ..., s_{n_i})$, then
$S^m(t, t_1, ..., t_m) := f_i(S^m(s_1, t_1, ..., t_m), ..., S^m(s_{n_i}, t_1, ..., t_m))$.

For any generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, t_2, ..., t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then a binary operation $\circ_G$ on $Hyp_G(\tau)$ is defined by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$. In [6], S. Leeratanavalee and K. Denecke proved that:

For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], ..., \widehat{\sigma}[t_n]) = \widehat{\sigma}[S^n(t, t_1, ..., t_n)]$,

(ii) $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Then $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.

3 Main Results

To characterize all unit elements and determine the set of all unit-regular elements of $\underline{Hyp_G(2)}$, we first introduce some notations which will be used throughout this paper.

For a type $\tau = (n)$ with an n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote

σ_t := the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t)$:= the set of all variables occurring in t,

$leftmost(t)$:= the first variable (from the left) occurring in t,

$rightmost(t)$:= the last variable occurring in t.

Definition 3.1. For any monoid S, an element $u \in S$ is called *unit* if there exists $u^{-1} \in S$ such that $uu^{-1} = e = u^{-1}u$ where e is the identity element of S. The set of all unit elements of S is denoted by $U(S)$.

Lemma 3.2. *Let $\sigma_t \in Hyp_G(n)$ where $t = f(t_1, t_2, ..., t_n) \in W_{(n)}(X)$. If $t_i \in W_{(n)}(X) \backslash X$ for some $i \in \{1, 2, ..., n\}$, then σ_t is not unit.*

Proof. Let $t = f(t_1, ..., t_i, ..., t_n) \in W_{(n)}(X)$ where $t_i \in W_{(n)}(X) \backslash X$ for some $i \in \{1, 2, ..., n\}$. Let $\sigma_s \in Hyp_G(n)$ and $s = f(s_1, s_2, ..., s_n) \in W_{(n)}(X)$ where $s_i \in W_{(n)}(X)$ for all $i \in \{1, 2, \ldots, n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2, ..., s_n)] \\ &= S^n(f(t_1, ..., t_i, ..., t_n), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]) \\ &= f(S^n(t_1, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), ..., S^n(t_i, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), \\ &\quad ..., S^n(t_n, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n])).\end{aligned}$$

Since $t_i \in W_{(n)}(X) \backslash X, \widehat{\sigma}_t[s_i] \in W_{(n)}(X) \backslash X$. So $(\sigma_t \circ_G \sigma_s)(f) \neq f(x_1, x_2, \ldots, x_n) = \sigma_{id}$. Then $\sigma_t \circ_G \sigma_s \neq \sigma_{id}$ for all $\sigma_s \in Hyp_G(n)$. Hence σ_t is not unit in $Hyp_G(n)$. □

Lemma 3.3. *Let $\sigma_t \in Hyp_G(n)$ where $t = f(x_{m_1}, x_{m_2}, ..., x_{m_n}) \in W_{(n)}(X)$. If $m_i > n$ for some $i \in \{1, 2, ..., n\}$, then σ_t is not unit in $Hyp_G(n)$.*

Proof. Let $t = f(x_{m_1}, ..., x_{m_i}, ..., x_{m_n})$ where $m_i > n$ for some $i \in \{1, 2, ..., n\}$. Let $\sigma_s \in Hyp_G(n)$ where $s = f(s_1, s_2, ..., s_n)$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2, ..., s_n)] \\
&= S^n(f(x_{m_1}, ..., x_{m_i}, ..., x_{m_n}), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]) \\
&= f(S^n(x_{m_1}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), ..., S^n(x_{m_i}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], \\
&\quad ..., \widehat{\sigma}_t[s_n]), ..., S^n(x_{m_n}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n])) \\
&= f(S^n(x_{m_1}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n]), ..., x_{m_i}, ..., \\
&\quad S^n(x_{m_n}, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2], ..., \widehat{\sigma}_t[s_n])) \\
&\neq f(x_1, x_2, ..., x_n) \quad \text{since } m_i > n \\
&= \sigma_{id}(f).
\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \neq \sigma_{id}$ for all $\sigma_s \in Hyp_G(n)$. Hence σ_t is not unit in $Hyp_G(n)$. □

Theorem 3.4. *An element $\sigma_t \in U(Hyp_G(n))$ if and only if $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ where $\pi \in S_n$ and S_n is a set of all permutation of $\{1, 2, ..., n\}$.*

Proof. Assume that $\sigma_t \in U(Hyp_G(n))$, then there exists $\sigma_s \in U(Hyp_G(n))$ such that $\sigma_t \circ_G \sigma_s = \sigma_{id} = \sigma_s \circ_G \sigma_t$. By Lemma 3.2 and Lemma 3.3, if $t = f(t_1, t_2, ..., t_n)$ and $s = f(s_1, s_2, ..., s_n)$ then $t_1, ..., t_n, s_1, ..., s_n \in \{x_1, x_2, ..., x_n\}$. Let $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ and $s = f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})$ where $\pi, \pi' : \{1, 2, ..., n\} \to \{1, 2, .., n\}$. Consider

$$\begin{aligned}
\sigma_{id}(f) &= (\sigma_t \circ_G \sigma_s)(f) \\
f(x_1, x_2, ..., x_n) &= \widehat{\sigma}_t[f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})] \\
&= S^n(f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}), x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)}) \\
&= f(x_{\pi'(\pi(1))}, x_{\pi'(\pi(2))}, ..., x_{\pi'(\pi(n))}) \\
&= f(x_{(\pi' \circ \pi)(1)}, x_{(\pi' \circ \pi)(2)}, ..., x_{(\pi' \circ \pi)(n)})
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{id}(f) &= (\sigma_s \circ_G \sigma_t)(f) \\
f(x_1, x_2, ..., x_n) &= \widehat{\sigma}_s[f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})] \\
&= S^n(f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)}), x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)}) \\
&= f(x_{\pi(\pi'(1))}, x_{\pi(\pi'(2))}, ..., x_{\pi(\pi'(n))}) \\
&= f(x_{(\pi \circ \pi')(1)}, x_{(\pi \circ \pi')(2)}, ..., x_{(\pi \circ \pi')(n)}).
\end{aligned}$$

Then $\pi \circ \pi' = (1) = \pi' \circ \pi$ and $\pi \circ \pi', \pi' \circ \pi$ are bijective. Next, we show that π is bijective. Let $\pi(i) = \pi(j)$ for some $i, j \in \{1, 2, .., n\}$. Then $(\pi' \circ \pi)(i) = (\pi'(\pi(i)) = \pi'(\pi(j)) = (\pi' \circ \pi)(j)$. Since $\pi' \circ \pi$ is one-to-one, $i = j$. Thus π is one-to-one. Let $i \in \{1, 2, ..., n\}$. Since $\pi \circ \pi'$ is onto, there exists $j \in \{1, 2, ..., n\}$ such that $(\pi \circ \pi')(j) = i$. Then $\pi(\pi'(j)) = i$ for some $\pi'(j) \in \{1, 2, ..., n\}$. Hence π is onto, so $\pi \in S_n$.

Conversely, let $\sigma_t \in Hyp_G(n)$ where $t = f(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ such that $\pi \in S_n$. Since $(S_n, \circ)$ is a group, there exists $\pi' \in S_n$ such that $\pi \circ \pi' = (1) = \pi' \circ \pi$. Let $\sigma_s \in Hyp_G(n)$ where $s = f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})$. Then

$$\begin{aligned}(\sigma_t \circ \sigma_s)(f) &= \widehat{\sigma}_t[f(x_{\pi'(1)}, x_{\pi'(2)}, ..., x_{\pi'(n)})] \\ &= f(x_{(\pi'\circ\pi)(1)}, x_{(\pi'\circ\pi)(2)}, ..., x_{(\pi'\circ\pi)(n)}) \\ &= f(x_1, x_2, ..., x_n) \\ &= \sigma_{id}(f).\end{aligned}$$

Similarly, we have $\sigma_s \circ \sigma_t = \sigma_{id}$. Hence $\sigma_t \in U(Hyp_G(n))$. □

Corollary 3.5. $|U(Hyp_G(n))| = n!$.

Corollary 3.6. $U(Hyp_G(2)) = \{\sigma_{f(x_1,x_2)} = \sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Definition 3.7. An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup S is called *regular* if all its elements are regular.

Definition 3.8. An element e of a semigroup S is called *idempotent* if $e^2 = ee = e$, and we denote the set of all idempotent elements in S by $E(S)$.

Next, we fix a type $\tau = (2)$ with the binary operation f. For $\sigma_t \in Hyp_G(2)$, we denote

$R_1 := \{\sigma_t | t = f(x_2, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin var(t')\}$,
$R_2 := \{\sigma_t | t = f(t', x_1) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin var(t')\}$,
$R_3 := \{\sigma_t | t = f(x_1, t') \text{ where } t' \in W_{(2)}(X) \text{ such that } x_2 \notin var(t')\}$,
$R_4 := \{\sigma_t | t = f(t', x_2) \text{ where } t' \in W_{(2)}(X) \text{ such that } x_1 \notin var(t')\}$,
$R_5 := \{\sigma_t | t \in \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}\}$ and
$R_6 := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

In 2010, W. Puninagool and S. Leeratanavalee [7] showed that : $\bigcup_{i=1}^{6} R_i$ is a set of all regular elements in $Hyp_G(2)$ and $(\bigcup_{i=3}^{6} R_i) \setminus \{\sigma_{f(x_2,x_1)}\} = E(Hyp_G(2))$.

Definition 3.9. An element a of a monoid S is called *unit-regular* if there exists $u \in U(S)$ such that $aua = a$. A monoid S is called *unit-regular* if all its elements are unit-regular.

Proposition 3.10. *$\bigcup_{i=1}^{6} R_i$ is a set of all unit-regular elements in $Hyp_G(2)$.*

Proof. Let $\sigma_t \in \bigcup_{i=1}^{6} R_i$, then $\sigma_t \in R_1$ or $\sigma_t \in R_2$ or $\sigma_t \in (\bigcup_{i=3}^{6} R_i) \setminus \{\sigma_{f(x_2,x_1)}\}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$.

Case 1: $\sigma_t \in R_1$. Then $t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[f(x_2,t')]] \\ &= \widehat{\sigma}_t[S^2(f(x_2,x_1), x_2, \widehat{\sigma}_{f(x_2,x_1)}[t'])] \\ &= \widehat{\sigma}_t[f(\widehat{\sigma}_{f(x_2,x_1)}[t'], x_2)] \\ &= S^2(f(x_2,t'), \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[t']], x_2) \\ &= f(x_2,t') \quad \text{since } x_1 \notin var(t') \\ &= \sigma_t(f).\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_t$.
Case 2: $\sigma_t \in R_2$. Then $t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[f(t',x_1)]] \\ &= \widehat{\sigma}_t[S^2(f(x_2,x_1), \widehat{\sigma}_{f(x_2,x_1)}[t'], x_1)] \\ &= \widehat{\sigma}_t[f(x_1, \widehat{\sigma}_{f(x_2,x_1)}[t'])] \\ &= S^2(f(t',x_1), x_1, \widehat{\sigma}_t[\widehat{\sigma}_{f(x_2,x_1)}[t']]) \\ &= f(t',x_1) \quad \text{since } x_2 \notin var(t') \\ &= \sigma_t(f).\end{aligned}$$

Hence $\sigma_t \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_t$.
Case 3: $\sigma_t \in (\bigcup\limits_{i=3}^{6} R_i) \setminus \{\sigma_{f(x_2,x_1)}\} = E(Hyp_G(2))$. Then $\sigma_t \circ_G \sigma_{id} \circ_G \sigma_t = \sigma_t \circ_G \sigma_t = \sigma_t$.
Case 4: $\sigma_t = \sigma_{f(x_2,x_1)}$. Then

$$\sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{id} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{f(x_2,x_1)}.$$

Therefore, for any $\sigma_t \in \bigcup\limits_{i=1}^{6} R_i$, there exists $\sigma_u \in U(Hyp_G(2))$ such that $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$. Hence $\bigcup\limits_{i=1}^{6} R_i$ is a set of all unit-regular elements in $Hyp_G(2)$. □

Remark 3.11. $\bigcup\limits_{i=1}^{6} R_i$ is not closed under $\circ_G$, i.e. $\bigcup\limits_{i=1}^{6} R_i$ is not a subsemigroup of $Hyp_G(2)$.

Example 3.12. (1) Let $\sigma_t \in R_1$ such that $t = f(x_2, t')$ where $t' = f(x_3, x_2)$.

Then

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_2, f(x_3, x_2))]\\ &= S^2(f(x_2, f(x_3, x_2)), \widehat{\sigma}_t[x_2], \widehat{\sigma}_t[f(x_3, x_2)])\\ &= S^2(f(x_2, f(x_3, x_2)), x_2, f(x_2, f(x_3, x_2)))\\ &= f(f(x_2, f(x_3, x_2)), f(x_3, f(x_2, f(x_3, x_2)))).\end{aligned}$$

We see that, if $rightmost(t') = x_2$ then $x_2 \in var((\sigma_t \circ_G \sigma_t)(f))$. So $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_i$.

(2) Let $\sigma_t \in R_2$ such that $t = f(t', x_1)$ where $t' = f(x_1, x_5)$. Then

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(f(x_1, x_5), x_1)]\\ &= S^2(f(f(x_1, x_5), x_1), \widehat{\sigma}_t[f(x_1, x_5)], \widehat{\sigma}_t[x_1])\\ &= S^2(f(f(x_1, x_5), x_1), f(f(x_1, x_5), x_1), x_1)\\ &= f(f(f(f(x_1, x_5), x_1), x_5), f(f(x_1, x_5), x_1)).\end{aligned}$$

We see that, if $leftmost(t') = x_1$ then $x_1 \in var((\sigma_t \circ_G \sigma_t)(f))$. So $\sigma_t \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_i$.

(3) Let $\sigma_t \in R_3$ and $\sigma_s \in R_4$ such that $t = f(x_1, t')$ and $s = f(s', x_2)$ where $t' = f(x_5, x_1)$ and $s' = f(x_2, x_3)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(f(x_2, x_3), x_2)]\\ &= S^2(f(x_1, f(x_5, x_1)), \widehat{\sigma}_t[f(x_2, x_3)], \widehat{\sigma}_t[x_2])\\ &= S^2(f(x_1, f(x_5, x_1), f(x_2, f(x_5, x_2)), x_2)\\ &= f(f(x_2, f(x_5, x_2)), f(x_5, f(x_2, f(x_5, x_2)))).\end{aligned}$$

Since $x_1 \in var(t)$ and $leftmost(s') = x_2$, $x_2 \in var((\sigma_t \circ_G \sigma_s)(f))$. So $\sigma_t \circ_G \sigma_s \notin \bigcup_{i=1}^{6} R_i$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, f(x_5, x_1))]\\ &= S^2(f(f(x_2, x_3), x_2), \widehat{\sigma}_s[x_1], \widehat{\sigma}_s[f(x_5, x_1)])\\ &= S^2(f(f(x_2, x_3), x_2), x_1, f(f(x_1, x_3), x_1))\\ &= f(f(f(f(x_1, x_3), x_1), x_3), f(f(x_1, x_3), x_1)).\end{aligned}$$

Since $x_2 \in var(s)$ and $rightmost(s') = x_1$, $x_1 \in var((\sigma_t \circ_G \sigma_s)(f))$. So $\sigma_s \circ_G \sigma_t \notin \bigcup_{i=1}^{6} R_i$.

By (1), (2) and (3), we have $\bigcup_{i=1}^{6} R_i$ is not a subsemigroup of $Hyp_G(2)$.

Let $\sigma_t \in Hyp_G(2)$, we denote

$R'_1 := \{\sigma_t | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') \neq x_2\}$,

$R'_2 := \{\sigma_t | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $leftmost(t') \neq x_1\}$,

$R'_3 := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1\}$,

$R'_4 := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $leftmost(t') \neq x_2\}$,

$R_5 := \{\sigma_t | t = x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$ and

$R_6 := \{\sigma_t | var(t) \cap \{x_1, x_2\} = \emptyset\}$.

Denote $(UR)_{Hyp_G(2)} = (\bigcup_{i=1}^{4} R'_i) \cup R_5 \cup R_6$.

Proposition 3.13. *$(UR)_{Hyp_G(2)}$ is unit-regular submonoid of $Hyp_G(2)$.*

Proof. Since $(UR)_{Hyp_G(2)} \subseteq Hyp_G(2)$ and every element in $(UR)_{Hyp_G(2)}$ is unit-regular. It suffices to show that $(UR)_{Hyp_G(2)}$ is a submonoid of $Hyp_G(2)$.

Case 1: $\sigma_t \in R'_1$. Then $t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$. Let $\sigma_s \in (UR)_{Hyp_G(2)}$.

Case 1.1: $\sigma_s \in R'_1$. Then $s = f(x_2, s')$ where $s' \in W_{(2)}(X)$ such that $x_1 \notin var(s')$ and $rightmost(s') \neq x_2$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_2, s')]\\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[x_2], \widehat{\sigma}_t[s'])\\ &= S^2(f(x_2, t'), x_2, \widehat{\sigma}_t[s'])\\ &= f(S^2(x_2, x_2, \widehat{\sigma}_t[s']), S^2(t', x_2, \widehat{\sigma}_t[s']))\\ &= f(\widehat{\sigma}_t[s'], S^2(t', x_2, \widehat{\sigma}_t[s'])).\end{aligned}$$

Since $x_1 \notin var(s')$ and $rightmost(s') \neq x_2$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_1 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', x_2, \widehat{\sigma}_t[s'])$. So that $\sigma_t \circ_G \sigma_s \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

Case 1.2: $\sigma_s \in R'_2$. Then $s = f(s', x_1)$ where $s' \in W_{(2)}(X)$ such that $x_2 \notin var(s')$ and $leftmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s', x_1)]\\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[s'], \widehat{\sigma}_t[x_1])\\ &= f(S^2(x_2, \widehat{\sigma}_t[s'], x_1), S^2(t', \widehat{\sigma}_t[s'], x_1))\\ &= f(x_1, S^2(t', \widehat{\sigma}_t[s'], x_1)).\end{aligned}$$

We have $x_2 \notin var(S^2(t', \widehat{\sigma}_t[s'], x_1))$ and since $rightmost(t') \neq x_2$, so

$rightmost(S^2(t', \widehat{\sigma}_t[s'], x_1)) \neq x_1$. Then $\sigma_t \circ_G \sigma_s \in R_3' \subseteq (UR)_{HypG(2)}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')]\\ &= S^2(f(s', x_1), \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[t'])\\ &= f(S^2(s', x_2, \widehat{\sigma}_s[t']), S^2(x_1, x_2, \widehat{\sigma}_s[t'])\\ &= f(S^2(s', x_2, \widehat{\sigma}_s[t']), x_2).\end{aligned}$$

We have $x_1 \notin var(S^2(s', x_2, \widehat{\sigma}_s[t']))$ and since $leftmost(t') \neq x_1$, so $leftmost(S^2(s', x_2, \widehat{\sigma}_s[t'])) \neq x_2$. Then $\sigma_s \circ_G \sigma_t \in R_4' \subseteq (UR)_{HypG(2)}$.

Case 1.3: $\sigma_s \in R_3'$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ such that $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')]\\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[s'])\\ &= S^2(f(x_2, t'), x_1, \widehat{\sigma}_t[s'])\\ &= f(S^2(x_2, x_1, \widehat{\sigma}_t[s']), S^2(t', x_1, \widehat{\sigma}_t[s']))\\ &= f(\widehat{\sigma}_t[s'], S^2(t', x_1, \widehat{\sigma}_t[s']))\end{aligned}$$

Since $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_1 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', x_1, \widehat{\sigma}_t[s']))$. So that $\sigma_t \circ_G \sigma_s \in R_6 \subseteq (UR)_{HypG(2)}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')]\\ &= S^2(f(x_1, s'), \widehat{\sigma}_s[x_2], \widehat{\sigma}_s[t'])\\ &= f(S^2(x_1, x_2, \widehat{\sigma}_s[t']), S^2(s', x_2, \widehat{\sigma}_s[t']))\\ &= f(x_2, S^2(s', x_2, \widehat{\sigma}_s[t'])).\end{aligned}$$

We have $x_1 \notin var(S^2(s', x_2, \widehat{\sigma}_s[t']))$ and since $rightmost(s') \neq x_1$, so $rightmost(S^2(s', x_2, \widehat{\sigma}_s[t'])) \neq x_2$. So that $\sigma_s \circ_G \sigma_t \in R_1' \subseteq (UR)_{HypG(2)}$.

In case of $\sigma_s \in R_4'$, we can prove in the same manner as in Case 1.3 that $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{HypG(2)}$.

Case 1.4: $\sigma_s \in R_5$, $s = x_1$ or $s = x_2$ or $s = f(x_1, x_2)$ or $s = f(x_2, x_1)$.

If $s = x_1$, then

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_1] = x_1$$
$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_1}[f(x_2, t')] = S^2(x_1, x_2, \widehat{\sigma}_{x_1}[t']) = x_2.$$

If $s = x_2$, then

$$(\sigma_t \circ_G \sigma_s)(f) = \widehat{\sigma}_t[x_2] = x_2$$
$$(\sigma_s \circ_G \sigma_t)(f) = \widehat{\sigma}_{x_2}[f(x_2, t')] = S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']).$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $S^2(x_2, x_2, \widehat{\sigma}_{x_2}[t']) = x_i \notin \{x_1, x_2\}$.

If $s = f(x_1, x_2)$, then $\sigma_s = \sigma_{id}$ such that $\sigma_t \circ_G \sigma_{id} = \sigma_t = \sigma_{id} \circ_G \sigma_t$.

If $s = f(x_2, x_1)$, then

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_s[f(x_2, x_1)] \\ &= S^2(f(x_2, t'), x_2, x_1) \\ &= f(S^2(x_2, x_2, x_1), S^2(t', x_2, x_1)) \\ &= f(x_1, S^2(t', x_2, x_1)).\end{aligned}$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $x_2 \notin var(S^2(t', x_2, x_1))$ and $rightmost(S^2(t', x_2, x_1)) \neq x_1$. So that $\sigma_t \circ_G \sigma_s \in R'_1$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(x_2, x_1), x_2, \widehat{\sigma}_s[t']) \\ &= f(\widehat{\sigma}_s[t'], x_2).\end{aligned}$$

Since $x_1 \notin var(t')$ and $rightmost(t') \neq x_2$, so $x_1 \notin var(\widehat{\sigma}_s[t'])$ and $leftmost(\widehat{\sigma}_s[t']) \neq x_2$. So that $\sigma_s \circ_G \sigma_t \in R'_4$.

Therefore $\sigma_s \circ_G \sigma_t, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

Case 1.5: $\sigma_s \in R_6$. Then $s = f(s_1, s_2)$ where $x_1, x_2 \notin var(s)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s_1, s_2)] \\ &= S^2(f(x_2, t'), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \\ &= f(S^2(x_2, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]), S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2])) \\ &= f(\widehat{\sigma}_t[s_2], S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2])).\end{aligned}$$

Since $x_1, x_2 \notin var(s)$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s_1]) \cup var(\widehat{\sigma}_t[s_2])$ and then $x_1, x_2 \notin var(S^2(t', \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]))$. So that $\sigma_t \circ_G \sigma_s \in R_6 \subseteq (UR)_{Hyp_G(2)}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_2, t')] \\ &= S^2(f(s_1, s_2), x_2, \widehat{\sigma}_s[t']) \\ &= f(s_1, s_2) \qquad \text{since } x_1, x_2 \notin var(s).\end{aligned}$$

So that $\sigma_s \circ_G \sigma_t \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in R'_2$ and $\sigma_s \in (\bigcup_{i=2}^{4} R'_i) \cup R_5 \cup R_6$. We can prove similar to Case 1. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

Case 3: $\sigma_t \in R'_3$. Then $t = f(x_1, t')$ where $t' \in W_{(2)(X)}$ such that $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$. Let $\sigma_s \in R'_3 \cup R'_4 \cup R_5 \cup R_6$.

Case 3.1: $\sigma_s \in R'_3$. Then $s = f(x_1, s')$ where $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(x_1, s')] \\ &= S^2(f(x_1, t'), x_1, \widehat{\sigma}_t[s']) \\ &= f(S^2(x_1, x_1, \widehat{\sigma}_t[s']), S^2(t', x_1, \widehat{\sigma}_t[s'])) \\ &= f(x_1, t') \qquad \text{since } x_2 \notin var(t').\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_3' \subseteq (UR)_{Hyp_G(2)}$.

Case 3.2: $\sigma_s \in R_4'$. Then $s = f(s', x_2)$ where $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, t')] \\ &= S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t']) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), S^2(x_2, x_1, \widehat{\sigma}_s[t'])) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), \widehat{\sigma}_s[t']).\end{aligned}$$

Since $x_2 \notin var(t')$ and $rightmost(t') \neq x_1$, so $x_1, x_2 \notin var(\widehat{\sigma}_s[t'])$. Since $x_1 \notin var(s')$ and $x_1, x_2 \notin var(\widehat{\sigma}_s[t'])$, so $x_1, x_2 \notin var(S^2(s', x_1, \widehat{\sigma}_s[t']))$. Then $\sigma_s \circ_G \sigma_t \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \widehat{\sigma}_t[f(s', x_2)] \\ &= S^2(f(x_1, t'), \widehat{\sigma}_t[s'], x_2) \\ &= f(S^2(x_1, \widehat{\sigma}_t[s'], x_2), S^2(t', \widehat{\sigma}_t[s'], x_2,)) \\ &= f(\widehat{\sigma}_t[s'], S^2(t', \widehat{\sigma}_t[s'], x_2,)).\end{aligned}$$

Since $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$, so $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$. Since $x_2 \notin var(t')$ and $x_1, x_2 \notin var(\widehat{\sigma}_t[s'])$, so $x_1, x_2 \notin var(S^2(t', \widehat{\sigma}_t[s'], x_2))$. Then $\sigma_t \circ_G \sigma_s \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

If $\sigma_s \in R_5$ and $\sigma_s \in R_6$, we can prove similar to Case 1.4 and Case 1.5. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

Case 4: $\sigma_t \in R_4'$ and $\sigma_s \in R_4' \cup R_5 \cup R_6$. We can prove similar to Case 3. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

Case 5: $\sigma_t \in R_5$ and $\sigma_s \in R_5 \cup R_6$. We can prove similar to Case 1.4. Then we have $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (UR)_{Hyp_G(2)}$.

Case 6: $\sigma_t \in R_6$ and $\sigma_s \in R_6$. Then $\sigma_t \circ_G \sigma_s = \sigma_t \in R_6 \subseteq (UR)_{Hyp_G(2)}$.

Therefore $((UR)_{Hyp_G(2)}; \circ_G)$ is a submonoid of $Hyp_G(2)$. □

Proposition 3.14. $(UR)_{Hyp_G(2)}$ *is a maximal unit-regular semigroup of* $Hyp_G(2)$.

Proof. Let H be a proper unit-regular semigroup of $Hyp_G(2)$ such that $(UR)_{Hyp_G(2)} \subseteq H \subset Hyp_G(2)$. Let $\sigma_t \in H$, then σ_t is unit-regular element.

Case 1: $\sigma_t \in R_1 \setminus R_1'$. Then $t = f(x_2, t')$ where $x_1 \notin var(t')$ and $rightmost(t') = x_2$. Since $x_2 \in var(t)$ and $rightmost(t') = x_2$, $\widehat{\sigma}_t[t'] = t$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(x_2, t')] \\ &= S^2(f(x_2, t'), x_2, \widehat{\sigma}_t[t']) \\ &= S^2(f(x_2, t'), x_2, t) \\ &= f(S^2(x_2, x_2, t), S^2(t', x_2, t)) \\ &= f(t, S^2(t', x_2, t)).\end{aligned}$$

Since $x_2 \in var(t')$, t occurs in $S^2(t', x_2, t)$, so that $x_2 \in var(S^2(t', x_2, t))$. Since $x_2 \in var(S^2(t', x_2, t)) \cup var(t)$, $\sigma_t \circ_G \sigma_t$ is not unit-regular, so $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(x_2, t')$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $rightmost(t') = x_2\} \nsubseteq H$.
Case 2: $\sigma_t \in R_2 \backslash R_2'$. Then $t = f(t', x_1)$ where $x_2 \notin var(t')$ and $leftmost(t') = x_1$. Since $x_1 \in var(t')$ and $leftmost(t') = x_1$, $\widehat{\sigma}_t[t'] = t$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_t)(f) &= \widehat{\sigma}_t[f(t', x_1)] \\ &= S^2(f(t', x_1), \widehat{\sigma}_t[t'], x_1) \\ &= S^2(f(t', x_1), t, x_1) \\ &= f(S^2(t', t, x_1), S^2(x_1, t, x_1)) \\ &= f(S^2(t', t, x_1), t).\end{aligned}$$

Since $x_1 \in var(t')$, t occurs in $S^2(t', t, x_1)$, so that $x_1 \in var(S^2(t', t, x_1))$. Since $x_1 \in var(S^2(t', t, x_1)) \cup var(t)$, $\sigma_t \circ_G \sigma_t$ is not unit-regular, so $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(t', x_1)$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $leftmost(t') = x_1\} \nsubseteq H$.
Case 3: $\sigma_t \in R_3 \setminus R_3'$. Then $t = f(x_1, t')$ where $x_2 \notin var(t')$ and $rightmost(t') = x_1$. Choose $\sigma_s \in R_4' \subseteq H$, then $s = f(s', x_2)$ such that $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(x_1, t')] \\ &= S^2(f(s', x_2), x_1, \widehat{\sigma}_s[t']) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), S^2(x_2, x_1, \widehat{\sigma}_s[t'])) \\ &= f(S^2(s', x_1, \widehat{\sigma}_s[t']), \widehat{\sigma}_s[t'])\end{aligned}$$

Since $x_2 \in var(s)$ and $rightmost(t') = x_1$,so $x_1 \in var(\widehat{\sigma}_s[t'])$. Since $x_1 \in var(\widehat{\sigma}_s[t'])$, $\sigma_s \circ_G \sigma_t$ is not unit-regular, so $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ such that $x_2 \notin var(t')$ and $rightmost(t') = x_1\} \nsubseteq H$.
Case 4: $\sigma_t \in R_4 \setminus R_4'$. Then $t = f(t', x_2)$ where $x_1 \notin var(t')$ and $leftmost(t') = x_2$. Choose $\sigma_s \in R_3' \subseteq H$, then $s = f(x_1, s')$ such that $x_2 \notin var(s')$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \widehat{\sigma}_s[f(t', x_2)] \\ &= S^2(f(x_1, s'), \widehat{\sigma}_s[t'], x_2) \\ &= f(S^2(x_1, \widehat{\sigma}_s[t'], x_2), S^2(s', \widehat{\sigma}_s[t'], x_2)) \\ &= f(\widehat{\sigma}_s[t'], S^2(s', \widehat{\sigma}_s[t'], x_2)).\end{aligned}$$

Since $x_1 \in var(s)$ and $leftmost(t') = x_2$, so $x_2 \in var(\widehat{\sigma}_s[t'])$. Since $x_2 \in var(\widehat{\sigma}_s[t'])$, $\sigma_s \circ_G \sigma_t$ is not unit-regular, so $\sigma_t \notin H$. Hence $\{\sigma_t \in Hyp_G(2) | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ such that $x_1 \notin var(t')$ and $leftmost(t') = x_2\} \nsubseteq H$. Therefore $H = (UR)_{Hyp_G(2)}$. □

Definition 3.15. Let S be a semigroup and $E(S)$ be the set of all idempotents in S. We say S is *left* [*right*] *factorisable* if $S = GE(S)$ [$S = E(S)H$] for some subgroup $G[H]$ of S. S is *factorisable* if S is both left and right factorisable.

Theorem 3.16. [8] *A monoid S is factorisable if and only if it is unit-regular.*

Corollary 3.17. $(UR)_{Hyp_G(2)}$ *is factorisable.*

Acknowledgement : The corresponding author was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

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(Received 16 November 2013)
(Accepted 21 June 2014)