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Constant on a Uniform Berry-Esseen Bound on a Closed Sphere via Stein's Method[1]

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Abstract : For each $n, k \in \mathbb{N}$, let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik})$, $i = 1, 2, \ldots, n$, be independent random vectors in R^k such that Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Without assuming the existence of the third moments, a uniform Berry-Esseen bound for multidimensional central limit theorem on a closed sphere is presented in this paper.

Keywords : Berry-Esseen inequality, central limit theorem, Stein's Method.
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1 Introduction

Central limit theorem is one of the well-known theorem is probability theory that can be applied to the area of statistics. This theorem guarantee that, under some conditions, the distribution of sum or mean of a large number of independent random variables tend to be close to the normal distribution. The rate of this convergence was independently quantified by Berry [1] and Esseen [2]. Their results have been known as the Berry-Esseen inequality and been studied by many researchers such as Chaidee [3], Chen and Shao [4, 5], Nagaev [6], Neammanee and Thongtha [7], Paditz [8] and Shevtsova [9]. The extension of the theorem to multidimension, multidimension central limit theorem, was first investigated by Bergström [10]. Bergström proved that for a fixed $k \in \mathbb{N}$, the

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distribution of sum of independent and identically distributed random vectors $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$, where the random vectors Y_i satisfy:

$$EY_i = \bar{0}, \quad \sum_{i=1}^{n} EY_{ij}^2 = 1 \text{ for } j = 1, 2, \ldots, k \quad \text{and}$$
$$EY_{ij}Y_{il} = 0 \text{ for } j \neq l,$$

converges weakly to the Gaussian distribution in R^k. This means that the distribution of sum of the random vectors Y_i can be approximated by the Gaussian distribution. A uniform bound of the approximation was first investigated by Esseen [2]. He gave a bound over the set of closed sphere,

$$B_k(r) = \{w \in \mathbb{R}^k \mid w_1^2 + w_2^2 + \cdots + w_k^2 \leq r^2\}$$

for $r > 0$, under the assumption that $\sum_{j=1}^{k} E|Y_{ij}|^4 < \infty$. His result is

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k}{n^{\frac{k}{k+1}}}$$

where F_n is the distribution of sum of $Y_i, i = 1, 2, ..., n$, and Φ_k is the Gaussion distribution in $\mathbb{R}^k$. A few decades later, many researchers put their effort to find the uniform bound. The bound was improved in many directions such as: extending the result to more general sets, see [11], ralaxing the assumption about $Y_i, i = 1, 2, \ldots, n$, see [12], [13] and [14], improving the rate of convergence, see [15] and computing the constant C_k on the bounds of the approximation, see [13], [16] and [17]. In the last direction, Götze[13] calculated the constant in the case that the random vectors Y_i may not be identically distributed. He assumed the finiteness of the third moments and used the Stein's method to find a uniform bound on any measurable convex set C in $\mathbb{R}^k$. His estimation is

$$|F_n(C) - \Phi_k(C)| \leq C_k\gamma_3$$

where $\gamma_3 = \sum_{i=1}^{n} E||Y_i||^3$, $||\cdot||$ is the Euclidean norm in $\mathbb{R}^k$ and $C_k = 124.4a_k\sqrt{k} + 10.7$, where $a_k = 2.04, 2.4, 2.69, 2.94$ for $k = 2, 3, 4, 5$, respectively and $a_k \leq 1.27\sqrt{k}$ for $k \geq 6$. Thongtha and Neammanee [17] assumed an independence of all component of Y_i and used the Stein's method to investigate a constant on a uniform bound over the set of closed sphere $B_k(r)$ as shown in the following theorem.

Theorem 1.1. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$ be independent random vectors in $\mathbb{R}^k$ with zero means and Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. Assume that*

$\sum_{i=1}^{n} EY_{ij}^2 = 1$ *for* $j = 1, 2, \ldots, k$ *and* $\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\sup_{r\in\mathbb{R}} |F_n(B_k(r)) - \Phi_k(B_k(r))| \leq c\beta_3$$

where $c = \frac{4.55}{k} + \frac{3}{k\sqrt{k}}$ *and* $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

All of the above mentioned research, at least the finiteness of the third moments are assumed. In this work, we will give a constant on a uniform bound of the approximation over the set of closed sphere without assuming the existence of the third moments.

The contents of this paper are organized as follows. The Stein's method which is a main tool to prove the main result is described in section 2 while the statement and the proof of the result are given in section 3.

2 Stein's Method

The Stien's method, introduced by Stein [18] in 1972, is now widely used technique for estimating Berry-Esseen bounds. This technique was initially introduced in order to investigate the bounds in one dimension. Over several decades, the method have been developed by many researchers in order to find bounds of multivariate normal approximation such as Barbour [19], Götze [13], Chatterjee and Meckes [20], Reinert and Röllin [14] and Thongtha and Neammanee [17]. The Stein's technique is based on the a partial differential equation. The keys of this method consist of two parts, the partial differrential equation and its corresponding solution. Thongtha and Neammanee [17] introduced the Stein's equation and the corresponding solution and used it to estimate the bounds over the set of closed shpere $B_k(r)$. The equation is

$$\sum_{i=1}^{k} f_{w_i}(w) - \sum_{i=1}^{k} w_i f_B(w) = \sqrt{k}[h_B(w) - \Phi_k(B)], \tag{2.1}$$

where $w = (w_1, w_2, \ldots, w_k) \in \mathbb{R}^k$, B is a Borel set in $\mathbb{R}^k$,f_{w_i} are the partial derivatives of f_B with respect to w_i for $i = 1, 2, \ldots, k$, and h_B is the indicator function on B, that is

$$h_B(w) = \begin{cases} 1 & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases}$$

The corresponding solution of the equation (2.1) is given as follows.

$$f_B(w) = \begin{cases} -\sqrt{2\pi}e^{\frac{1}{2}\bar{w}^2}(1-\Phi_k(B))(1-\Phi_1(\bar{w})) & \text{if } w \in B, \bar{w} \geq 0, \\ \sqrt{2\pi}e^{\frac{1}{2}\bar{w}^2}(1-\Phi_k(B))\Phi_1(\bar{w}) & \text{if } w \in B, \bar{w} < 0, \\ \sqrt{2\pi}e^{\frac{1}{2}\bar{w}^2}\Phi_k(B)[1-\Phi_1(\bar{w}))] & \text{if } w \notin B, \bar{w} \geq 0, \\ -\sqrt{2\pi}e^{\frac{1}{2}\bar{w}^2}\Phi_k(B)\Phi_1(\bar{w})) & \text{if } w \notin B, \bar{w} < 0 \end{cases} \tag{2.2}$$

where $\bar{w} = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} w_i$. If $W_n = \sum_{i=1}^{n} Y_i$ and $B = B_k(r)$, we can easily compute from the equation (2.1) that

$$P(W_n \in B_k(r)) - \Phi_k(B_k(r)) = \frac{1}{\sqrt{k}}E[\sum_{i=1}^{k} f_{r_{w_i}}(W_n) - \sum_{i=1}^{k} W_{ni}f_r(W_n)], \tag{2.3}$$

where f_r denote the solution in equation (2.2) in the case that $B = B_k(r)$. From the equation (2.3), we can approximate the left hand side by estimating the right hand side which depends on the solution f_r. Thongtha and Neammanee [17] gave some properties of f_r which is used to approximate the right hand side of the equation (2.3). Here are the properties.

Proposition 2.1 For $k \in \mathbb{N}$, $w \in \mathbb{R}^k$ and $r > 0$, we have

1. $|f_r(w)| \leq \frac{1}{|\bar{w}|}$ for $\bar{w} \neq 0$,
2. $|f_r(w)| \leq 2$ and
3. $|f_{r_{w_i}}(w)| \leq \frac{2}{\sqrt{k}}$ for $i = 1, 2, \ldots, k$.

In Propostion 2.2, Thongtha and Neammanee [21] gave a bound of a function concerning f_B in the equation (2.2).

Proposition 2.2 For $k \in \mathbb{N}$, $i = 1, 2, \ldots, k$ and any Borel set B in $\mathbb{R}^k$, the function $g_i : \mathbb{R}^k \to \mathbb{R}$ is defined by

$$g_i(w) = \frac{\partial}{\partial w_i}(\sum_{j=1}^{k} w_j)f_B(w)$$

is bounded by 2.

3 The Main Result

In this section, we give a constant on a uniform Berry- Esseen bound for multidimensional central limit theorem over the set of $B_k(r)$ for $r > 0$. The used technique is the Stein's method together with the concentration inequality.

For $n \in \mathbb{N}$, let $X_i, i = 1, 2, \ldots, n$, be independent random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Define $S_n = \sum_{i=1}^{n} X_i$ and $S_n^{(i)} = S_n - X_i$, and let

$$\alpha = \sum_{i=1}^{n} EX_i^2 I(|X_i| > 1) \text{ and } \beta = \sum_{i=1}^{n} E|X_i|^3 I(|X_i| \leq 1).$$

The following proposition is a concentration inequality which is presented in Chen and Shao [4].

Proposition 3.1. *If $\alpha + \beta \leq 0.14$, then for $a < b$,*

$$P(a \leq S_n^{(i)} \leq b) \leq 1.5(b - a) + 3.3\delta_1$$

where $\delta_1 = \frac{1}{2}(0.28\alpha + \beta)$.

The concentration inequality has an important role in proving the main result. To charpen the result, we decide to create a new concentration inequality which is proved by using the same idea as in Proposition 3.1. Here is our concentration inequality.

Proposition 3.2. *If $\alpha + \beta \leq 0.25$, then for $a < b$,*

$$P(a \leq S_n^{(i)} \leq b) \leq 1.67(b - a) + 4.77\delta_2$$

where $\delta_2 = \frac{1}{2}(0.5\alpha + \beta)$.

Proof. Follows an idea of Proposition 3.1. □

The Proposition 3.2 is used to prove the main result which is stated as follows.

Theorem 3.3. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$, be independent random vectors in $\mathbb{R}^k$ with zero means and Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. Assume that $\sum_{i=1}^{n} EY_{ij}^2 = 1$ for $j = 1, 2, \ldots, k$. Then,*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \left(\frac{7.2}{\sqrt{k}} + 4\right)\delta$$

for all polsitive real numbers r where

$$\delta = \sum_{j=1}^{k} \sum_{i=1}^{n} \left\{EY_{ij}^2 I(|Y_{ij}| > 1) + E|Y_{ij}|^3 I(|Y_{ij}| \leq 1)\right\}.$$

Theorem 3.3 is proved by combining the ideas in [4] and [17]. To prove the theorem, we introduce the following notations.
For $k, n \in \mathbb{N}$, $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$, let

$$\overline{Y}_{ij} = Y_{ij} I(|Y_{ij}| \leq 1), \quad \overline{W}_{nj} = \sum_{i=1}^{n} \overline{Y}_{ij}, \quad \overline{W}_{nj}^{(i)} = \overline{W}_{nj} - \overline{Y}_{ij},$$

$$\alpha_j = \sum_{i=1}^{n} EY_i^2 I(|Y_{ij}| > 1), \quad \beta_j = \sum_{i=1}^{n} E|Y_i|^3 I(|Y_{ij}| \leq 1) \quad \text{and}$$

$$\bar{K}_{ij}(t) = E\overline{Y}_{ij}[I(0 \leq t \leq \overline{Y}_{ij}) - I(\overline{Y}_{ij} \leq t < 0)]$$

for $t \in \mathbb{R}$ where I is an indicator function on Ω. We can easily compute that

$$\bar{K}_{ij}(t) \geq 0 \quad \text{for all } t \in \mathbb{R}, \tag{3.1}$$

$$\sum_{i=1}^{n} \int_{-\infty}^{\infty} \bar{K}_{ij}(t)dt = 1 - \alpha_j \quad \text{and} \tag{3.2}$$

$$\sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|Y_{ij}| + |t|)\bar{K}_{ij}(t)dt = \frac{3}{2}\beta_j \tag{3.3}$$

for all $j = 1, 2, \ldots, k$, see an idea in [5].
We are now ready to prove Theorem 3.3.

Proof. If there exists an element $i \in \{1, 2, \ldots, k\}$ such that

$$\alpha_i + \beta_i \geq 0.25,$$

then the theorem is done by the fact that $|P(W_n \in B_k(r)) - \Phi_k(B_k(r))| \leq 1$ and

$$\left(\frac{7.2}{\sqrt{k}} + 4\right)\delta \geq \left(\frac{7.2}{\sqrt{k}} + 4\right)(\alpha_i + \beta_i) \geq 1. \tag{3.4}$$

Next, we assume that $\alpha_i + \beta_i \leq 0.25$ for all $i \in \{1, 2, \ldots, k\}$. Let f_r be the solution defined in (2.2) in the case that $B = B_k(r)$ and $f_{r_{w_i}}$ the derivative of f_r with respect to w_i for $i = 1, 2, \ldots, k$. By equation (2.3),

$$P(W_n \in B_k(r)) - \Phi_k(B_k(r)) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} [S_i - T_i] \tag{3.5}$$

where

$$S_i = Ef_{r_{w_i}}(W_{n1}, W_{n2}, \ldots, W_{nk}), \quad \text{and} \quad T_i = EW_{ni} f_r(W_{n1}, W_{n2}, \ldots, W_{nk}).$$

To prove the theorem, we claim that for all $i \in \{1, 2, \ldots, n\}$

$$|S_i - T_i| \leq \left(\frac{7.2}{\sqrt{k}} + 4\right)(\alpha_i + \beta_i) \tag{3.6}$$

This equation will be proved in case of $k=2$. For multidimension, we can prove it by using the same argument. Follows an idea in [5], we can show that

$$\begin{aligned} S_1 &= \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(W_{n1}^{(i)}+Y_{i1},W_{n2})\bar{K}_{i1}(t)dt \\ &\quad +\alpha_1 E f_{r_{w_1}}(W_{n1},W_{n2}), \quad \text{and} \\ T_1 &= \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(W_{n1}^{(i)}+t,W_{n2})\bar{K}_{i1}(t)dt \\ &\quad +\sum_{i=1}^{n} EY_{i1}I(|Y_{i1}|>1)[f_r(W_{n1},W_{n2})-f_r(W_{n1}^{(i)},W_{n2})]. \end{aligned}$$

Thus,

$$S_1 - T_1 = R_1 + R_2 + R_3 + R_4 \tag{3.7}$$

where

$$\begin{aligned} R_1 &= \sum_{i=1}^{n} EI(|Y_{i1}|\le 1)\int_{-\infty}^{\infty}[f_{r_{w_1}}(W_{n1}^{(i)}+Y_{i1},W_{n2})-f_{r_{w_1}}(W_{n1}^{(i)}+t,W_{n2})]\bar{K}_{i1}(t)dt \\ R_2 &= \sum_{i=1}^{n} EI(|Y_{i1}|> 1)\int_{-\infty}^{\infty}[f_{r_{w_1}}(W_{n1}^{(i)}+Y_{i1},W_{n2})-f_{r_{w_1}}(W_{n1}^{(i)}+t,W_{n2})]\bar{K}_{i1}(t)dt \\ R_3 &= \alpha_1 E f_{r_{w_1}}(W_{n1},W_{n2}) \\ R_4 &= -\sum_{i=1}^{n} EY_{i1}I(|Y_{i1}|>1)[f_r(W_{n1},W_{n2})-f_r(W_{n1}^{(i)},W_{n2})]. \end{aligned}$$

By Proposition 2.1(3) and (3.2),

$$\begin{aligned} |R_2+R_3| &\le \frac{4}{\sqrt{2}}\sum_{i=1}^{n} EI(|Y_{i1}|>1)\int_{-\infty}^{\infty}\bar{K}_{i1}(t)dt + \frac{2}{\sqrt{2}}\alpha_1 \\ &\le \frac{4}{\sqrt{2}}\sum_{i=1}^{n} EI(|Y_{i1}|>1) + \frac{2}{\sqrt{2}}\alpha_1 \\ &\le \frac{6}{\sqrt{2}}\alpha_1. \end{aligned} \tag{3.8}$$

By Proposition 2.1(2),

$$|R_4| \le 4\sum_{i=1}^{n} EY_{i1}I(|Y_{i1}|>1) \le 4\alpha_1. \tag{3.9}$$

By the same argument as of (3.7) in [17], we write

$$R_1 = \frac{1}{\sqrt{2}}R_{11} + \frac{1}{2}R_{12} \tag{3.10}$$

where

$$R_{11} = \sum_{i=1}^{n} E[I(|Y_{i1}| \leq 1) \int_{-\infty}^{\infty} [h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2}) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})]\bar{K}_{i1}(t)dt]$$

$$R_{12} = \sum_{i=1}^{n} E[I(|Y_{i1}| \leq 1) \int_{-\infty}^{\infty} [(W_{n1}^{(i)} + Y_{i1} + W_{n2}) f_r(W_{n1}^{(i)} + Y_{i1}, W_{n2}) - (W_{n1}^{(i)} + t + W_{n2}) f_r(W_{n1}^{(i)} + t, W_{n2})]\bar{K}_{i1}(t)dt].$$

For $i = 1, 2, \ldots, n$, let

$$A_{i1} = \{w \in \Omega \mid -t + \eta(w) < W_{n1}^{(i)}(w) \leq -Y_{i1}(w) + \eta(w)\},$$
$$B_{i1} = \{w \in \Omega \mid -Y_{i1}(w) - \eta(w) \leq W_{n1}^{(i)}(w) < -t - \eta(w)\}$$

where

$$\eta(w) = \sqrt{r^2 - W_{n2}^2(w) I(w \in \Lambda)} \text{ and } \Lambda = \{w \in \Omega \mid W_{n2}^2(w) \leq r^2\}.$$

Using the argument of (3.9)–(3.11) in [17] together with Proposition 3.2 and (3.2), we obtain that

$$\begin{aligned} R_{11} &\leq \sum_{i=1}^{n} E[I(|Y_{i1}| \leq 1) \int_{-\infty}^{\infty} (I(A_{i1}) + I(B_{i1})\bar{K}_{i1}(t)dt] \\ &= \sum_{i=1}^{n} EI(|Y_{i1}| \leq 1) \int_{-\infty}^{0} P(A_{i1} \mid Y_{i1}, W_{n2})\bar{K}_{i1}(t)dt \\ &+ \sum_{i=1}^{n} EI(|Y_{i1}| \leq 1) \int_{0}^{\infty} P(B_{i1} \mid Y_{i1}, W_{n2})\bar{K}_{i1}(t)dt \\ &\leq \sum_{i=1}^{n} EI(|Y_{i1}| \leq 1) \int_{-\infty}^{\infty} [1.67|Y_{i1} - t| + 4.77(\frac{0.5\alpha_1 + \beta_1}{2})]\bar{K}_{i1}(t)dt \\ &\leq 1.67 \sum_{i=1}^{n} EI(|Y_{i1}| \leq 1) \int_{-\infty}^{\infty} (|Y_{i1}| + |t|)\bar{K}_{i1}(t)dt + 1.2\alpha_1 + 2.39\beta_1 \\ &\leq 4.9\beta_1 + 1.2\alpha_1 \end{aligned} \tag{3.11}$$

where we used the equation (3.3) in the last inequality. Similarly, we can show that

$$\begin{aligned} R_{11} &\geq -\sum_{i=1}^{n} E[I(|Y_{i1}| < 1) \int_{-\infty}^{\infty} (I(C_{i1}) + I(D_{i1})\bar{K}_{i1}(t)dt] \\ &\geq -(4.9\beta_1 + 1.2\alpha_1) \end{aligned} \tag{3.12}$$

where

$$C_{i1} = \{w \in \Omega \mid -Y_{i1}(w) + \eta(w) < W_{n1}^{(i)}(w) \leq -t + \eta(w)\} \text{ and}$$
$$D_{i1} = \{w \in \Omega \mid -t - \eta(w) \leq W_{n1}^{(i)}(w) < -Y_{i1}(w) - \eta(w)\}.$$

By (3.11) and (3.12), we have

$$|R_{11}| \le 4.9\beta_1 + 1.2\alpha_1. \tag{3.13}$$

To prove the equation (3.6), it remains to estimate $|R_{12}|$. Since g_1 in Proposition 2.2 is piecewise continuous, by Proposition 2.2, (3.1),(3.3) and the fundamental theorem of calculus,

$$\begin{aligned}|R_{12}| &= |\sum_{i=1}^{n} EI(|Y_{i1}| \le 1)\int_{-\infty}^{\infty}\int_{t}^{Y_{i1}} g_1(W_{n1}^{(i)} + u, W_{n2})du\bar{K}_{i1}(t)dt| \\ &\le 2\sum_{i=1}^{n} EI(|Y_{i1}| \le 1)\int_{-\infty}^{\infty}(|Y_{i1}| + |t|)\bar{K}_{i1}(t)dt \\ &\le 3\beta_1. \end{aligned} \tag{3.14}$$

Combining (3.7)–(3.10), (3.13) and (3.14) yields

$$|S_1 - T_1| \le \left(\frac{7.2}{\sqrt{2}} + 4\right)\alpha_1 + \left(\frac{4.9}{\sqrt{2}} + \frac{3}{2}\right)\beta_1 \le \left(\frac{7.2}{\sqrt{2}} + 4\right)(\alpha_1 + \beta_1) \tag{3.15}$$

Similarly, we have the same conclusion as in (3.15) for the other elements i in $\{2, 3, \ldots, k\}$. Hence, the equation (3.6) is proved. By equations (3.4)–(3.6), we have the theorem. □

Corollary 3.4. *Assume the assumption as in Theorem 3.3. If* $\alpha_i + \beta_i \le 0.25$ *for all* $i = 1, 2, \ldots, k$ *, then*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \le \left(\frac{7.2}{k} + \frac{4}{\sqrt{k}}\right)\delta$$

for all polsitive real numbers r.

Proof. If $\alpha_i + \beta_i \le 0.25$ for all $i = 1, 2, \ldots, k$, the result is done by the equation (3.5) and (3.6). □

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