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Modules which are Reduced over their Endomorphism Rings

N. Agayev[†], S. Halicioglu[‡], A. Harmanci[‡] and B. Ungor[‡,1]

[†]Fatih Sultan Mehmet Vakif University, Faculty Of Engineering,
Computer Engineering Department, Istanbul, Turkey
e-mail : naghayev@fsm.edu.tr

[‡] Department of Mathematics, Hacettepe University, 06800 Ankara, Turkey
e-mail : halici@ankara.edu.tr (S. Halicioglu)
harmanci@hacettepe.edu.tr (A. Harmanci)
bungor@science.ankara.edu.tr (B. Ungor)

Abstract : Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. The module M is called *reduced* if for any $m \in M$ and $f \in S$, $fm = 0$ implies $fM \cap Sm = 0$. In this paper, we investigate properties of reduced modules and rigid modules.

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1 Introduction

Throughout this paper R denotes an associative ring with identity. For a module M, $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ denote the left annihilator of a subset

[1]Corresponding author.

of M in S and the right annihilator of a subset of S in M, respectively. A ring R is *reduced* if it has no nonzero nilpotent elements. Recently, the reduced ring concept was extended to R-modules by Lee and Zhou in [1], that is, an R-module M is called *reduced* if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module M is called *semicommutative* [2], if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$. *Baer rings* [3] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. The module M is called *Baer* [4] if for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A submodule N of M is said to be *fully invariant* if for any $f \in S$, $f(N) \leq N$. A ring R is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. The module M is said to be *quasi-Baer* [4] if for every fully invariant submodule N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A ring R is called *right principally quasi-Baer* if the right annihilator of a principal right ideal of R is generated by an idempotent. The module M is called *principally quasi-Baer* [5] if for any $m \in M$, $l_S(Sm) = Sf$ for some $f^2 = f \in S$. A ring R is called *right (left) principally projective* if every right (left) ideal is projective [6]. The module M is called *Rickart* [7] if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. The ring R is called *right Rickart* if R_R is a Rickart module, that is, the right annihilator of any element is generated by an idempotent. It is obvious that the module R_R is Rickart if and only if the ring R is right principally projective. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_n$ and $\mathbb{Z}/\mathbb{Z}n$ we denote, respectively, integers, rational numbers, real numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n.

2 Reduced Modules

Let M be an R-module with $S = \mathrm{End}_R(M)$. Some properties of R-modules do not characterize the ring R, namely there are reduced R-modules but R need not be reduced and there are abelian R-modules but R is not an abelian ring. Because of that reduced, rigid, symmetric, semicommutative and Armendariz modules in terms of endomorphism rings S are introduced by the present authors (see [8]). In this section we study properties of modules which are reduced over their endomorphism rings.

We start with the following proposition.

Proposition 2.1. *Let M be an R-module with $S = End_R(M)$. Consider the following conditions for $f \in S$.*
(1) $S(Kerf) \cap Imf = 0$.
(2) *Whenever $m \in M$, $fm = 0$ if and only if $Imf \cap Sm = 0$.*
Then (1) $\Rightarrow$ (2). *If M is a semicommutative module, then* (2) $\Rightarrow$ (1).

Proof. Clear. ☐

Following the definition of Lee and Zhou [1], M is a reduced module if and only if condition (2) of Proposition 2.1 holds for each $f \in S$. If M is a reduced module, then it is semicommutative and so condition (1) of Proposition 2.1 also holds for each $f \in S$.

As an illustration we state the following examples.

Example 2.2. Let p be any prime integer and M denote the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p)\oplus\mathbb{Q}$. Then S is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$. It is evident that M is a reduced module.

Note that every module need not be reduced.

Example 2.3. Let p be any prime integer and $M = \mathbb{Z}(p^\infty)$ the Prüfer p-group as a $\mathbb{Z}$-module. Let $\{v_i\}$ $(i = 1, 2, 3, \cdots)$ be elements in M which they satisfy the equalities $pv_1 = 0$, $pv_i = v_{i-1}$ $(i = 2, 3, \cdots)$. By [9, page 54], S is isomorphic to the ring of p-adic integers $\mathbb{A}(p)$. Define f as $f(v_1) = 0$ and $f(v_i) = v_{i+1}$ for $(i = 2, 3, 4, \cdots)$. Let $m = v_2$. Then $f(v_2) = v_1$ and $f^2(v_2) = 0$. Hence M is not reduced.

Lemma 2.4. *Let M be an R-module with $S = End_R(M)$. If M is a reduced module, then S is a reduced ring.*

Proof. It is clear from [8, Lemma 2.11] and [8, Proposition 2.14]. □

Definition 2.5. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *principally projective* if for any $m \in M$, $l_S(m) = Se$ for some $e^2 = e \in S$.

It is obvious that the module ${}_RR$ is principally projective if and only if the ring R is left principally projective. It is straightforward that all Baer and quasi-Baer modules are principally projective. And every quasi-Baer module is principally quasi-Baer. There are principally projective modules which are not quasi-Baer or Baer (see [10, Example 8.2]).

Example 2.6. Let R be a Prüfer domain (a commutative ring with an identity, no zero divisors and all finitely generated ideals are projective) and M the right R-module $R \oplus R$. By ([3], page 17), S is a 2×2 matrix ring over R and it is a Baer ring. Hence M is Baer and so principally projective module.

Note that the endomorphism ring of a principally projective module may not be a right principally projective ring in general. For if M is a principally projective module and $\varphi \in S$, then we have two cases. $Ker\varphi = 0$ or $Ker\varphi \neq 0$. If $Ker\varphi = 0$, then for any $f \in r_S(\varphi)$, $\varphi f = 0$ implies $f = 0$. Hence $r_S(\varphi) = 0$. Assume that $Ker\varphi \neq 0$. There exists a nonzero $m \in M$ such that $\varphi m = 0$. By hypothesis, $\varphi \in l_S(m) = Se$ for some $e^2 = e \in S$. In this case $\varphi = \varphi e$ and so $r_S(\varphi) \leq (1-e)S$. The following example shows that this inclusion is strict.

Example 2.7. Let Q be the ring and N the Q-module constructed by Osofsky in [11]. Since Q is commutative, we can just as well think of N as a right Q-module. Let $S = \mathrm{End}_Q(N)$. It is easy to see that N is a principally projective module. Identify S with the ring $\begin{bmatrix} Q & 0 \\ Q/I & Q/I \end{bmatrix}$ in the obvious way, and consider $\varphi = \begin{bmatrix} 0 & 0 \\ 1+I & 0 \end{bmatrix} \in S$. Then $r_S(\varphi) = \begin{bmatrix} I & 0 \\ Q/I & Q/I \end{bmatrix}$. This is not a direct summand of S because I is not a direct summand of Q. Therefore S is not a right principally projective ring.

Proposition 2.8. *Let M be an R-module with $S = End_R(M)$. If M is semicommutative, then we have the followings.*
(1) *M is a Baer module if and only if M is a quasi-Baer module.*
(2) *M is a principally projective module if and only if M is a principally quasi-Baer module.*

Proof. Let M be an R-module with M semicommutative.
(1) The necessity is clear. By Theorem 2.14 of [12] and [2, Lemma 2.15], the sufficiency follows.
(2) The necessity follows from the proof of Lemma 2.15 of [12]. The sufficiency is clear from the semicommutativity. □

Recall that a ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced by Roos in [13] and studied by Goodearl and Boyle [14], Roman and Rizvi [15]. Following Roos [13], a module M is called *abelian* if all idempotents of S are central.

Remark 2.9. *It is easy to show that if M is a semicommutative module, then S is an abelian ring. It follows from Theorem 2.14 of [12], every reduced module M is semicommutative, and every semicommutative module M is abelian. The converses hold if M is a principally projective module. Note that for a prime integer p, the cyclic group M of p^2 elements is a $\mathbb{Z}$-module for which $S = \mathbb{Z}_{p^2}$. The module M is neither reduced nor principally projective although it is semicommutative.*

Proposition 2.10. *Let M be a uniform R-module with $S = End_R(M)$. If M is a reduced module, then S is a domain.*

Proof. For $f,\ g \in S$, suppose $fg = 0$ with $f \neq 0$. We show that $g = 0$. For any $m \in M$, $fgmR = 0$ and so $fM \cap SgmR = 0$. By hypothesis $fM = 0$ or $SgmR = 0$. Then $Sgm = 0$ and so $gm = 0$ for all $m \in M$. Hence $g = 0$. □

Lemma 2.11. [16, Lemma 1. 9] *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then M_1 is a fully invariant submodule of M if and only if $Hom(M_1, M_2) = 0$.*

We observe in Example 3.7 that the direct sum of reduced modules need not be reduced. Note the following fact.

Proposition 2.12. *Let M be an R-module with $S = End_R(M)$. Let $M = M_1 \oplus M_2$ be a decomposition of M where M_1 and M_2 are fully invariant submodules of M with $S_1 = End_R(M_1)$ and $S_2 = End_R(M_2)$.*
(1) *If M_1 and M_2 are reduced over S, then M is reduced.*
(2) *If M_1 and M_2 are reduced over S_1 and S_2 respectively, then M is reduced.*

Proof. (1) Let $f \in S$, $m \in M$ and $fm = 0$. There exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2$. Hence $fm_1 + fm_2 = 0$. Since M_1 and M_2 are fully invariant submodules of M, $fm_1 = 0$ and $fm_2 = 0$ by Lemma 2.11. So $fM_1 \cap Sm_1 = 0$ and $fM_2 \cap Sm_2 = 0$. Let $x \in fM \cap Sm$. Then $x = fm^{'} = gm$ for some $m^{'} \in M$ and $g \in S$. For $m^{'} \in M$ there exist $m_1^{'} \in M_1$ and $m_2^{'} \in M_2$ such that $m^{'} = m_1^{'} + m_2^{'}$. So $fm_1^{'} - gm_1 = gm_2 - fm_2^{'} \in M_1 \cap M_2 = 0$. It follows that $fm_1^{'} = gm_1 = 0$ and $fm_2^{'} = gm_2 = 0$. Therefore $x = 0$.
(2) Let $f \in S$, $m \in M$ and $fm = 0$. There exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m = m_1 + m_2$. Hence $fm_1 + fm_2 = 0$. Since M_1 and M_2 are fully invariant submodules of M, $fm_1 = 0$ and $fm_2 = 0$. Let the restrictions of f to M_1 and M_2 be denoted by the same f. Then $fM_1 \cap S_1m_1 = 0$ and $fM_2 \cap S_2m_2 = 0$. Let $x \in fM \cap Sm$. Then $x = fm^{'} = gm$ for some $m^{'} \in M$ and $g \in S$. For $m^{'} \in M$, there exist $m_1^{'} \in M_1$ and $m_2^{'} \in M_2$ such that $m^{'} = m_1^{'} + m_2^{'}$. So $fm_1^{'} + fm_2^{'} = gm_1 + gm_2$. It follows that $fm_1^{'} = gm_1 = 0$ and $fm_2^{'} = gm_2 = 0$. Therefore $x = 0$. □

Corollary 2.13. *Let M be an R-module with $S = End_R(M)$. Let $M = M_1 \oplus M_2$ where M_1 and M_2 are submodules of M with $S_1 = End_R(M_1)$ and $S_2 = End_R(M_2)$. If M is semicommutative, then we have the following.*
(1) *If M_1 and M_2 are reduced over S, then M is reduced.*
(2) *If M_1 and M_2 are reduced over S_1 and S_2 respectively, then M is reduced.*

Proof. Let M be a semicommutative module. It is enough to show that every direct summand N of M is fully invariant. We write $M = N \oplus L$. Let π denote the natural projection of M onto N. From $\pi(1-\pi) = 0$ and $(1-\pi)\pi = 0$ we have $\pi g(1-\pi) = 0$ and $(1-\pi)g\pi = 0$ for each $g \in S$. Then π is a central idempotent in S. Hence $g(N) = g(\pi(M)) = \pi(g(M)) \leq N$. This completes the proof. □

We end this section with some observations relating to being M an reduced module and S an reduced ring. Recall that a module M is called *n-epiretractable* [17] if every n-generated submodule of M is a homomorphic image of M.

Theorem 2.1. *Let M be an R-module with $S = End_R(M)$. Then the following hold.*
(1) *If M is a 1-epiretractable module and S is a reduced ring, then M is reduced.*
(2) *If M is a principally projective module and S is a reduced ring, then M is reduced.*

Proof. (1) Let $fm = 0$ for $f \in S$ and $m \in M$. Since M is 1-epiretractable, there exists $g \in S$ such that $gM = mR$. We have $fgM = 0$ and $fg = gf = 0$ since S is reduced. Let $fm' = hm \in fM \cap Sm$ where $m' \in M, h \in S$. Then $gfm' = ghm = 0$ and so $ghmR = 0$. This implies $ghgM = 0$, i.e., $ghg = 0$. Therefore $gh = hg = 0$. Now by assumption, there exists $m_1 \in M$ such that $m = gm_1$. Then $fm' = hm = hgm_1 = 0$. Hence M is reduced.
(2) Let $fm = 0$ for $f \in S$ and $m \in M$, and $fm' = gm \in fM \cap Sm$. Since $fm = 0 \in mR$, we may find an idempotent e in S such that $f \in l_S(mR) = Se$.

By hypothesis, e is central in S. So $f = fe = ef$, $em = 0$. Then $fm' = egm = gem = 0$. Hence $fM \cap Sm = 0$. Thus M is reduced. □

Theorem 2.2. *Let M be an R-module with $S = End_R(M)$. If M is a reduced module, then the following hold.*
(1) *Assume that for every submodule N of M there exist $e^2 = e \in S$ and $f \in S$ such that $N \subseteq eM$ and $f(N) = eM$. Then M is a Baer module.*
(2) *If every fully invariant submodule is a direct summand of M, then M is a Baer module.*
(3) *If M is a uniform module, then each nonzero element of S is a monomorphism.*

Proof. (1) Let N be a submodule of M. Then there exist an idempotent homomorphism $e \in S$ and $f \in S$ such that $N \subseteq eM$ and $fN = eM$. We prove that $l_S(N) = S(1-e)$. It is trivial that $S(1-e) \leq l_S(N)$ since $N \subseteq eM$. Let $g \in l_S(N)$. By hypothesis $gN = 0$ implies $gfN = 0$. Then $gfN = geM = 0$, and so $ge = 0$. Hence $g = g(1-e) \in S(1-e)$. So $l_S(N) \leq S(1-e)$. This completes the proof.
(2) Since M is a reduced module, if $fm = 0$ where $f \in S$, then for all $g \in S$, $fgm \in fM \cap Sm = 0$. This implies that for all $f \in S$, $Kerf$ is a fully invariant submodule of M. Let I be an ideal of S. Since $r_M(I) = \cap_{f\in I} Kerf$ and all the $Kerf$ are fully invariant submodules of M, $r_M(I)$ is a fully invariant submodule of M. So it is a direct summand of M and therefore M is a Baer module.
(3) Let $fm = 0$ where $f \in S$, $m \in M$. Then $fmR = 0$. By hypothesis, $fM \cap SmR = 0$ and so $fM = 0$ or $SmR = 0$. Hence $f = 0$ or $m = 0$. □

3 Rigid Modules

Let M be an R-module with $S = \mathrm{End}_R(M)$. Rigid R-modules are introduced and studied in [18] and [19] by the present authors. Recently, rigid modules over their endomorphism rings are studied in [8]. In this section we continue to investigate further properties of a rigid module over its endomorphism ring as a generalization of a reduced module over its endomorphism ring and relations between reduced, semicommutative and $\mathcal{K}$-co(non)singular modules.

We mention the following obvious proposition.

Proposition 3.1. *Let M be an R-module with $S = End_R(M)$. For any $f \in S$, the following are equivalent.*
(1) *$Kerf \cap Imf = 0$.*
(2) *For $m \in M$, $f^2m = 0$ if and only if $fm = 0$.*

A module M is called *rigid* if it satisfies Proposition 3.1 for every $f \in S$. By [8, Lemma 2.20], if M is a rigid module, then S is a reduced ring and therefore abelian.

Rickart modules provide a generalization of a right principally projective ring to the general module theoretic setting. It is clear that every Baer module is a Rickart module while the converse is not true. For example, $\mathbb{Z}^{(\mathbb{R})}$ is Rickart but not Baer as a $\mathbb{Z}$-module.

Proposition 3.2. *Let M be an R-module with $S = End_R(M)$. If M is a reduced module, then M is a rigid module. The converse holds if M satisfies one of the following conditions.*
(1) M is a semicommutative module.
(2) M is a principally projective module.
(3) M is a Rickart module.

Proof. For any $f \in S$, $S(Kerf) \cap Imf = 0$ by hypothesis. Since $Kerf \cap Imf \subset S(Kerf) \cap Imf$, $Kerf \cap Imf = 0$. By Proposition 3.1, M is a rigid module. Conversely,
(1) Assume that M is a rigid and semicommutative module. Let $f \in S$ and $m \in M$ with $fm = 0$. Let $fm' = gm \in fM \cap Sm$. We multiply it by f from the left and we have $f^2m' = fgm$. Since M is semicommutative and $fm = 0$, $f^2m' = fgm = 0$. By hypothesis $fm' = 0$.
(2) Let M be a rigid and principally projective module. Assume that $fm = 0$ for $f \in S$ and $m \in M$. Then there exists $e^2 = e \in S$ such that $l_S(mR) = Se$. Since e is central in S, $fe = ef = f$ and $eg = gf$ for each $g \in S$ and $em = 0$. Let $fm' = gm \in fM \cap Sm$. Multiply $fm' = gm$ by e from the left to obtain $efm' = fm' = gem = 0$. Therefore M is a reduced module.
(3) Let M be a Rickart and rigid module. Assume that $fm = 0$ for $f \in S$ and $m \in M$. Then there exists $e^2 = e \in S$ such that $r_M(f) = eM$. Since e is central in S, $fe = ef = 0$ and $m = em$. Let $fm' = gm \in fM \cap Sm$. We multiply $fm' = gm$ from the left by e to obtain $efm' = fem' = egm = gem = gm = 0$. Therefore M is a reduced module. □

There are semicommutative modules which are neither rigid nor principally projective.

Example 3.3. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and the right R-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$.

For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,

$$fg\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f\begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + adc' + bc'c \end{bmatrix}$$

and

$$gf\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g\begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & acc' \\ acc' & acd' + ac'd + bcc' \end{bmatrix}.$$

Hence $fg = gf$ for all $f, g \in S$. Therefore S is commutative and so M is semicommutative. Define $f \in S$ by $f\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ where $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. Then $f\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $f^2\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Hence M is not rigid. Let $m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $l_S(m) \neq 0$ since the endomorphism f defined preceding belongs to $l_S(m)$. M is indecomposable as a right R-module, therefore S does not have any idempotents other than zero and identity. Hence $l_S(m)$ can not be generated by an idempotent as a left ideal of S.

An R-module M is called *Hopfian* provided every surjective endomorphism of M is an isomorphism. For example, every Noetherian module is Hopfian (see [9, Lemma 11.6]).

Theorem 3.1. *Let T be a ring and M a left T-module. If $t \in T$ satisfies $M = tM$ and M is rigid over T, then $tm = 0$ implies $m = 0$ for any $m \in M$.*

Proof. Let $m \in M$ with $tm = 0$. Since $M = tM$, there exists $u \in M$ such that $m = tu$. Then $0 = tm = t^2u$. It implies $tu = 0$ by hypothesis. Hence $m = 0$. □

Corollary 3.4. *Let M be an R-module with $S = End_R(M)$. If M is rigid, then M_R is Hopfian.*

Proof. It is clear from Theorem 3.1. □

A right R-module M is said to be *nonsingular* if for any $m \in M$, $mE = 0$ for an essential right ideal E of R implies $m = 0$, and M is called *cononsingular* if each submodule N of M with $r_R(N) = \{r \in R \mid Nr = 0\} \neq 0$ is essential in M. In [4], a module M is said to be $\mathcal{K}$-*nonsingular* if for every $\varphi \in S$, $Ker\varphi$ is essential in M implies $\varphi = 0$. Also the module M is said to be $\mathcal{K}$-*cononsingular* if for every submodule N of M, $\varphi N \neq 0$ for all $0 \neq \varphi \in S$ implies N is essential in M.

Proposition 3.5. *Let M be an R-module with $S = End_R(M)$. If M is a rigid module, then M is a $\mathcal{K}$-nonsingular module.*

Proof. Let $f \in S$. Assume that $Kerf$ is an essential submodule of M. Since M is rigid, $Kerf \cap Imf = 0$. Then $Imf = 0$ and so $f = 0$. Hence M is $\mathcal{K}$-nonsingular. □

Corollary 3.6. *Let M be an R-module with $S = End_R(M)$. If M is a reduced module, then M is $\mathcal{K}$-nonsingular.*

Example 3.7 shows that the converse statement of Corollary 3.6 need not be true in general. There exists a $\mathcal{K}$-nonsingular module which is neither reduced nor $\mathcal{K}$-cononsingular.

Example 3.7. Let M denote the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$. We show that for any $f \in S$ with $Kerf$ essential in M we have $f = 0$. Since S is isomorphic to the ring $\left\{\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a \in \mathbb{Z}, b, c \in \mathbb{Q}\right\}$, we may assume S as this ring. We write the elements of S as matrices and the elements of $\mathbb{Z}\oplus\mathbb{Q}$ as 2×1 columns. Let $f = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in S$ and $m = \begin{bmatrix} n \\ q \end{bmatrix}$, $a, n \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ with $fm = 0$. Then we have $an = 0$, $bn + cq = 0$. Assume that $Kerf$ is essential in M. Then $Kerf \cap (\mathbb{Z} \oplus (0)) \neq 0$. There exists $m \in Kerf$ such that n is nonzero and $an = 0$ and $bn = 0$. Hence $a = b = 0$. Similarly, $Kerf \cap ((0) \oplus \mathbb{Q}) \neq 0$. We may find $m' = \begin{bmatrix} 0 \\ q' \end{bmatrix} \in Kerf$ such that q' is nonzero. So $cq' = 0$ and then $c = 0$. It follows $f = 0$ and M is $\mathcal{K}$-nonsingular. Let $f = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \in S$ and $m = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $fm = 0$. Let $g = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \in S$ and $m' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $fm' = gm \in fM \cap Sm \neq 0$. Therefore M is not reduced. Let $N = (1, 1/2)\mathbb{Z} + (1, 1/3)\mathbb{Z}$. Then N is not essential in M. If $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in l_S(N)$, then $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = 0$ and $\begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 1 \\ 1/3 \end{bmatrix} = 0$ implies $a = 0$ and $b + c/2 = 0$, $b + c/3 = 0$. It follows that $a = 0$, $b = 0$ and $c = 0$. Hence M is not $\mathcal{K}$-cononsingular.

The proof of Theorem 3.2 is clear from Rizvi and Roman [4, Theorem 2.12]. We give a proof for the sake of completeness.

Theorem 3.2. *Let M be an R-module with $S = End_R(M)$. If M is a rigid and extending module, then it is Baer and $\mathcal{K}$-cononsingular.*

Proof. If M is a rigid module, from Proposition 3.5, M is a $\mathcal{K}$-nonsingular module. Since a $\mathcal{K}$-nonsingular and extending module is a Baer module by [4, Theorem 2.12], M is Baer. Let N be a submodule of M with $l_S(N) = 0$. We claim N is essential in M. We may find a direct summand K of M so that N is an essential submodule of K. Let $M = K \oplus L$ and π_L denote the canonical projection from M onto L. Then $\pi_L(N) = 0$. Hence $\pi_L \in l_S(N)$. Thus $\pi_L = 0$ and so $L = 0$, $M = K$ and N is essential in M. □

Corollary 3.8. *Let M be an R-module with $S = End_R(M)$. If M is a reduced and extending module, then M is Baer and $\mathcal{K}$-cononsingular.*

Corollary 3.9. *Let M be an R-module with $S = End_R(M)$. If M is a rigid and extending module, then M is a Rickart module.*

Proof. It is clear from Theorem 3.2 since Baer modules are Rickart modules. □

Corollary 3.10. *Let M be an R-module with $S = End_R(M)$. If M is a reduced and extending module, then M is a Baer module.*

In the following result we give the relations between principally projective modules, reduced modules, semicommutative modules, abelian modules and rigid modules.

Theorem 3.3. *Let M be an R-module with $S = End_R(M)$. If M is a principally projective module, then the following conditions are equivalent.*
(1) *M is a reduced module.*
(2) *M is a semicommutative module.*
(3) *M is an abelian module.*
(4) *M is a rigid module.*
(5) *S is a reduced ring.*

Proof. (1) $\Leftrightarrow$ (2) Clear from Lemma 3.2.
(2) $\Rightarrow$ (3) Clear from Remark 2.9.
(3) $\Rightarrow$ (2) Let $f \in S$, $m \in M$ with $fm = 0$. There exists $e^2 = e \in S$ such that $l_S(m) = Se$. Then $f = ef = fe$, $em = 0$ and e is central in S. So $0 = em = Sem = fSem = feSm = fSm$. Hence M is semicommutative.
(3) $\Rightarrow$ (4) Let $f^2m = 0$ for $f \in S$, $m \in M$. For some $e^2 = e \in S$ we have $f \in l_S(fm) = Se$. Then $fe = f$ and $efm = 0$. By hypothesis, $efm = fem$. Hence $0 = efm = fem = fm$. So M is rigid.
(4) $\Rightarrow$ (3) Let $e^2 = e \in S$. For any $f \in S$, $(ef - efe)^2m = 0$ for all $m \in M$ since $(ef - efe)^2 = 0$. We have $(ef - efe)m = 0$ for all $m \in M$ by hypothesis. Hence $ef - efe = 0$. Similarly, $(fe - efe)^2m = 0$ for all $m \in M$ implies $fe - efe = 0$. It follows that $ef = fe = efe$ and so S is abelian, therefore M is abelian.
(1) $\Rightarrow$ (5) It follows from Lemma 2.4.
(5) $\Rightarrow$ (1) Let $f \in S$ and $m \in M$ with $fm = 0$. Assume that $fm = 0$. There exists $e^2 = e \in S$ such that $f \in l_S(m) = Se$. Then $em = 0$, $f = fe$. By hypothesis, e is a central idempotent in S. Hence $f = fe = ef$. Let $fm' = gm \in fM \cap Sm$. Then $fm' = efm' = egm = gem = 0$. It follows that $fM \cap Sm = 0$ and (1) holds. □

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