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On the Generalized Nakano Sequence Space

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Abstract : The main purpose of this note is to define and to investigate the generalized Nakano sequence space $\mathcal{A}(p)$ and show that the sequence space $\mathcal{A}(p)$ equipped with the Luxemburg norm is rotund and posses property-H when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

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1 Introduction

By w, we shall denote the space of all real or complex valued sequences. Each linear subspace of w is called a *sequence space*. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, ...\}$.

A K- space λ is called an FK- space provided λ is a complete linear metric space. An FK- space whose topology is normable is called a BK- space [2, pp. 272-273].

A lower triangular matrix is called *factorable* if one can write each $A = (a_{nk}) = a_n b_k$ where a_n depends only n and b_k depends only $0 \leq k \leq n$. A triangle is a lower triangular matrix with no zeros on the principal diagonal. A matrix A is called *regular* if A is limit preserving over c, where c denote the space of convergent sequences. For a Banach space λ, we denote by $S(\lambda)$ and $B(\lambda)$ the unit sphere and unit ball of λ, respectively. A point $x_0 \in S(\lambda)$ is called:

(a) an *extreme point* if for every $x, y \in S(\lambda)$ the equality $2x_0 = x + y$ implies $x = y$;

(b) an *H point* if for any sequence (x_n) in λ such that $\| x \| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x implies that $\| x_n - x \| \to 0$ as $n \to \infty$;

A Banach space λ is said to be *rotund*, if every point of $S(\lambda)$ an extreme point. A Banach space λ is said *posses property H* provided every point of $S(\lambda)$ is H point.

Let λ be an arbitrary vector space over $\mathbb{C}$.

(a) A functional $m : \lambda \to [0, \infty]$ is called *modular* if

M1 : $m(x) = 0 \Leftrightarrow x = 0$,

M2 : $m(\alpha x) = m(x)$ for $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) with $|\alpha| = 1$, for all $x \in \lambda$,

M3 : $m(\alpha x + \beta y) \leq m(x) + m(y)$ if $\alpha, \ \ \beta \geq 0, \ \ \alpha + \beta = 1$, for all $x, y \in \lambda$.

(b) If M3 is replaced by

M4 : $m(\alpha x + \beta y) = \alpha^s m(x) + \beta^s m(y)$ if $\alpha, \ \ \beta \geq 0, \ \ \alpha^s + \beta^s = 1$, with an $s \in [0, 1]$ then the modular m is called an *s-convex modular*; and if $s = 1$, m is called *convex modular*.

(c) A modular m defines a corresponding modular space, i.e, the space λ_m given by

$$\lambda_m = \Big\{ x \in w : m(tx) \to 0 \ \ as \ \ t \to \infty \Big\}.$$

Recall that a sequence (x_n) is said to be an ϵ-separated sequence if, for some $\epsilon > 0$

$$\text{sep}(x_n) = \inf \Big\{ \parallel x_n - x_k \parallel : n \neq k \Big\} > \epsilon.$$

A Banach space λ has property β if and only if, for every $\epsilon > 0$ such that, for each element $x \in B(\lambda)$ and each sequence $(x_n) \in B(\lambda)$ with $\text{sep}(x_n) \geq \epsilon$, there an index k such that

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

The Nakano sequence space $\ell(p)$ is defined by

$$\ell(p) = \Big\{ x = (x_k) \in w : m(tx) < \infty \quad \text{for some } \ t > 0 \Big\},$$

where $m(x) = \sum_k |x_k|^{p_k}$ and $p = (p_k)$ is a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The space $\ell(p)$ is a Banach space with the norm

$$\parallel x \parallel = \inf \left\{ t > 0 : m\left(\frac{x}{t}\right) \leq 1 \right\}.$$

If $p = (p_k)$ is bounded, we have

$$\ell(p) = \left\{ x \in w : \sum_k |x_k|^{p_k} < \infty \right\}.$$

Also, some geometric properties of $\ell(p)$ were studied in [1] and [3].

For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$ces_p = \left\{ x = (x_k) \in w : \left(\sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p \right)^{\frac{1}{p}} < \infty \right\} \tag{1.1}$$

equipped with the norm

$$\| x \| = \left(\sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}}.$$

This space was introduced by Shue [9]. Some geometric properties of the Cesàro sequence space ces_p were studied in [10]. It is known that ces_p is locally uniform rotund and posses property H [5]. Cui and Hudzik [3] proved that ces_p has the Banach- Saks of type p if $p > 1$, and it was shown in [4] that ces_p has property β.

2 The sequence space $\mathcal{A}(p)$

The space $ces(p)$ [8] is defined by

$$ces(p) = \{x \in w : \rho(tx) < \infty \quad \text{for some} \quad t > 0\}, \tag{2.1}$$

where

$$\rho(x) = \sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{p_n}.$$

The space $ces(p)$ is a Banach space with the norm

$$\| x \| = \inf \left\{ t > 0 : \rho\left(\frac{x}{t}\right) \leq 1 \right\}$$

and if $p = (p_k)$ is bounded then we have

$$ces(p) = \left\{ x \in w : \sum_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^{p_n} < \infty \right\}.$$

Several geometric properties of $ces(p)$ were studied in [8]. Define the sequence $y = (y_n)$, which will be frequently used, as the A-transform of a sequence $x = (x_k)$, i.e.,

$$(Ax)_n = y_n = a_n \sum_{k=0}^{n} x_k \tag{2.2}$$

where, $A = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} a_n, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases} ; \ (n, \ k \in \mathbb{N}), \tag{2.3}$$

$a_n > 0$ for all $n \in \mathbb{N}$, $a = (a_n)$ is monotone decreasing and A is regular.

Now, we wish to introduce the generalized Nakano sequence space $\mathcal{A}(p)$, as the set of all sequences such that A-transforms of them are in the space $\ell(p)$, that is

$$\mathcal{A}(p) = \{x = (x_k) \in w : (Ax) \in \ell(p)\} \tag{2.4}$$

or, the other word

$$\mathcal{A}(p) = \Big\{x \in w : m(tx) < \infty \ \text{ for some } \ t > 0 \ \Big\},$$

where

$$m(x) = \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} < \infty.$$

We consider the space $\mathcal{A}(p)$ equipped with the so - called *Luxemburg norm*

$$\| x \| = \inf \Big\{ t > 0 : m(\frac{x}{t}) \leq 1 \Big\}.$$

If $p = (p_n)$ is bounded, then we have

$$\mathcal{A}(p) = \left\{ x \in w : \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} < \infty \right\}.$$

The main purpose of this note is to define and to investigate the generalized Nakano sequence space $\mathcal{A}(p)$ and show that the sequence space $\mathcal{A}(p)$ equipped with the Luxemburg norm is rotund and posses property H when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Clearly, in the special cases $a_n = (n+1)^{-1}$ and $a_n = 1$, we have $\mathcal{A}(p) = ces(p)$ and $\mathcal{A}(p) = \ell(p)$, respectively. Also, throughout this paper we assume that $p = (p_i)$ is bounded with $p_i > 1$ for all $i \in \mathbb{N}$ and $K = \sup_i p_i$.

Now, we may begin with the following theorem which is essential in the text:

Theorem 2.1 *The set $\mathcal{A}(p)$ is the BK- spaces with the norm $\| x \|_{\mathcal{A}(p)} = \| Ax \|_{\ell(p)}$.*

Proof. Since (2.2) holds and $\ell(p)$ is the $BK-$space [7] with the respect to it norm and the matrix A is normal, Theorem 4.3.2 of Wilansky [11, pp. 61] gives the fact that the space $\mathcal{A}(p)$ is $BK-$ space. □

Proposition 2.2 *The functional m on the space $\mathcal{A}(p)$ is a convex modular.*

Proof. $m(x) = 0 \Leftrightarrow x = 0$ and $m(\alpha x) = m(x)$ for all scalar α with $|\alpha| = 1$ is clear so, we omit it. Let $x, y \in \mathcal{A}(p)$ and $\alpha \geq 0, \quad \beta \geq 0$ with $\alpha + \beta = 1$ by the convexity of the function $u \to |u|^{p_n}$; $n \in \mathbb{N}$, we have:

$$\begin{aligned} m(\alpha x+\beta y) &= \sum_n (a_n \sum_{i=0}^{n} |(\alpha x_i+\beta y_i)|)^{p_n} \\ &\le \sum_n \left((a_n \sum_{i=0}^{n} |\alpha x_i|) + (a_n \sum_{i=0}^{n} |\beta y_i|) \right)^{p_n} \\ &\le \alpha \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} + \beta \sum_n \left(a_n \sum_{i=0}^{n} |y_i| \right)^{p_n} \\ &= \alpha m(x) + \beta m(y). \end{aligned}$$

□

Proposition 2.3 *For $x \in \mathcal{A}(p)$ the modular m on $\mathcal{A}(p)$ satisfies the following properties :*

P1. if $0 < r < 1$ then $r^K m(xr^{-1}) \le m(x)$ and $m(rx) \le rm(x)$,

P2. if $r > 1$, then $m(x) \le r^K m(xr^{-1})$,

P3. if $r \ge 1$, then $m(x) \le rm(x) \le m(rx)$.

Proof. It is obvious that *P3* is satisfied by the convexity of m. It remains to prove *P1* and *P2*. For $0 < r < 1$, we have

$$\begin{aligned} m(x) = \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} &= \sum_n \left(ra_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} \\ = \sum_n r^{p_n} \left(a_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} &\ge \sum_n r^K \left(a_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} \\ = r^K \sum_n \left(a_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} &= r^K m(xr^{-1}), \end{aligned}$$

and it implies by the convexity of m that $m(rx) \le rm(x)$, hence *P1* is satisfied. Now, assume that $r \ge 1$. Then we have

$$\begin{aligned} m(x) = \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} &= \sum_n r^{p_n} \left(a_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} \\ \le r^K \sum_n \left(a_n \sum_{i=0}^{n} |x_i r^{-1}| \right)^{p_n} &= r^K m(xr^{-1}) \end{aligned}$$

hence *P2* is obtained. □

Now, we give relations between the Lxemburg norm and the modular m on the space $\mathcal{A}(p)$.

Proposition 2.4 *For any $x \in \mathcal{A}(p)$ we have*

P4. *if* $\| x \| < 1$ *then* $m(x) \leq \| x \|$

P5. *if* $\| x \| > 1$ *then* $m(x) \geq \| x \|$

P6. $\| x \| = 1$ *if and only if* $m(x) = 1$

P7. $\| x \| < 1$ *if and only if* $m(x) < 1$

P8. $\| x \| > 1$ *if and only if* $m(x) > 1$

P9. *if* $0 < r < 1$ *and* $\| x \| > r$ *then* $m(x) > r^K$

P10. *if* $r \geq 1$ *and* $\| x \| < r$ *then* $m(x) < r^K$

Proof. (*P4*) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \| x \|$. Then we have $\| x \| + \epsilon < 1$. By definition of $\| . \|$ there exists $\mu > 0$ such that $\| x \| + \epsilon > \mu$ and $m(x\mu^{-1})$ From Proposition 2.3 *P1.* and *P3*, we have

$$m(x) \leq m\left((\| x \| + \epsilon) x \mu^{-1}\right) \leq (\| x \| + \epsilon) m\left(x\mu^{-1}\right) \leq \| x \| + \epsilon$$

which implies that $m(x) \leq \| x \|$ so *P4* is satisfied.

(*P5*) Let $\epsilon > 0$ be such that $0 < \epsilon < (\| x \| - 1) \| x \|^{-1}$ then $1 < (1 - \epsilon) \| x \| < \| x \|$. By definition of $\| . \|$ and by Proposition 2.3 *P1* we have

$$1 < m(x[(1 - \epsilon) \| x \|]^{-1}) \leq [(1 - \epsilon) \| x \|]^{-1} m(x)$$

so $(1 - \epsilon) \| x \| < m(x)$ for all $\epsilon \in (0, (\| x \| - 1) \| x \|^{-1})$. This implies that $\| x \| \leq m(x)$, hence *P5* is obtained.

(*P6*) Assume that $\| x \| = 1$. By definition of $\| x \|$ we have that for $\epsilon > 0$ there exists $\mu > 0$ such that $1 + \epsilon > \mu > \| x \|$ and $m(x\mu^{-1}) \leq 1$. From Proposition 2.3 *P2*, we have

$$m(x) \leq \mu^K m(x\mu^{-1}) \leq \mu^K < (1 + \epsilon)^K$$

so $(m(x))^{K^{-1}} < 1 + \epsilon$ for all $\epsilon > 0$, which implies $m(x) \leq 1$. If $m(x) < 1$, then we can choose $r \in (0, 1)$ such that $m(x) < r^K < 1$. From Proposition 2.3 *P1* we have $m(xr^{-1}) \leq (r^K)^{-1} m(x) < 1$ hence $\| x \| \leq r < 1$ which is a contradiction. Therefore $m(x) = 1$. On the other hand; assume that $m(x) = 1$. Then $\| x \| \leq 1$. If $\| x \| < 1$, we have by *P4* that $m(x) \leq \| x \| \leq 1$ which contradicts our assumption. Therefore $\| x \| = 1$.

(*P7*) follows directly from *P4* and *P6*.

(*P8*) follows from *P6* and *P7*.

(*P9*) Suppose $0 < r < 1$ and $\| x \| > r$. Then $\| xr^{-1} \| > 1$. By *P5* we have $m(xr^{-1}) > 1$. Hence by Proposition 2.3 *P1* we obtain that $m(x) \geq r^K m(xr^{-1}) > r^K$.

(*P10*) Suppose that $r \geq 1$ and $\| x \| < r$. Then $\| xr^{-1} \| < 1$. By *P7* we have $\| xr^{-1} \| < 1$. If $r = 1$, it is obvious that $m(x) < 1 = r^K$. If $r > 1$, then by Proposition 2.3 *P2*; we obtain that $m(x) \leq r^K m(xr^{-1}) < r^K$. □

Proposition 2.5 *Let (x_n) be a sequence in $\mathcal{A}(p)$.*

P11. If $\| x \| \to 1$ as $n \to \infty$, thenm$(x) \to 1$ as $n \to \infty$.

P12. If $m(x) \to 0$ as $n \to \infty$, then $\| x \| \to 0$ as $n \to \infty$.

Proof. (*P11*) Suppose that $\| x \| \to 1$ as $n \to \infty$. Let $\epsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \| x_n \| < 1 + \epsilon$ for all $n \in \mathbb{N}$. By Proposition 2.4 *P9* and *P10* we have $(1 - \epsilon)^K < m(x_n) < (1 + \epsilon)^K$ for all $n \geq N$ which implies $m(x_n) \to 1$ as $n \to \infty$.

(*P12*) Suppose that $\| x_n \| \nrightarrow 0$ as $n \to \infty$. Then there is an $\epsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $\| x_{n_k} \| > \epsilon$ for all $k \in \mathbb{N}$. By Proposition 2.4 *P9* we have $m(x_{n_k}) > \epsilon^K$ for all $k \in \mathbb{N}$. This implies $m(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. □

Now we shall show that $\mathcal{A}(p)$ has the property H but we firstly give a lemma:

Lemma 2.6 *Let $x \in \mathcal{A}(p)$ and $(x^n) \subseteq \mathcal{A}(p)$. If $\lim_n m(x^n) = m(x)$ and $\lim_n x_i^n = x_i$ for all $i \in \mathbb{N}$ then $\lim_n x^n = x$.*

Proof. Let $\epsilon > 0$ be given. Since $m(x) = \sum_n \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} < \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} < \epsilon (2^{K+1} 3)^{-1}. \tag{2.5}$$

Since

$$m(x^n) - \sum_{n=0}^{n_0} \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} \to m(x) - \sum_{n=0}^{n_0} \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n}$$

as $(n \to \infty)$ and $x_i^n \to x_i$ as $n \to \infty$ as for all $i \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that

$$m(x^n) - \sum_{n=0}^{n_0} \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} < m(x) - \sum_{n=0}^{n_0} \left(a_n \sum_{i=0}^{n} |x_i| \right)^{p_n} + (2^{K+1} 3)^{-1} \tag{2.6}$$

for all $n \geq n_0$, and

$$\sum_{n=0}^{n_0} \left(a_n \sum_{i=0}^{n} |x_i^n - x_i| \right)^{p_n} < 3^{-1} \epsilon. \tag{2.7}$$

for all $n \geq n_0$. It follows from (2.5), (2.6) and (2.7) that for $n \geq n_0$

$$m(x^n-x)=\sum_n\left(a_n\sum_{i=0}^{n}|x_i^n-x_i|\right)^{p_n}$$
$$=\sum_{n=0}^{n_0}\left(a_n\sum_{i=0}^{n}|x_i^n-x_i|\right)^{p_n}+\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i^n-x_i|\right)^{p_n}$$
$$<3^{-1}\epsilon+2^M\left[\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i^n|\right)^{p_n}+\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}\right]$$
$$=3^{-1}\epsilon+2^M\left[m(x^n)-\sum_{n=0}^{n_0}\left(a_n\sum_{i=0}^{n}|x_i^n|\right)^{p_n}+\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}\right]$$
$$<3^{-1}\epsilon+2^M\left[m(x^n)-\sum_{n=0}^{n_0}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}+(2^K3)^{-1}\epsilon+\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}\right]$$
$$=3^{-1}\epsilon+2^M\left[\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}+(2^K3)^{-1}\epsilon+\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}\right]$$
$$=3^{-1}\epsilon+2^M\left[(2^K3)^{-1}\epsilon+2\sum_{n=n_0+1}^{\infty}\left(a_n\sum_{i=0}^{n}|x_i|\right)^{p_n}\right]<3^{-1}\epsilon+3^{-1}\epsilon+3^{-1}\epsilon=\epsilon$$

This show that $m(x^n-x)\to 0$ as $n\to\infty$. Hence by *P8* of Proposition 2.5 , we have $\| x^n-x \|\to 0$ as $n\to\infty$. □

Theorem 2.7 *The $\mathcal{A}(p)$ has the property H.*

Proof. Let $x\in S(\mathcal{A}(p))$ and $(x^n)\subseteq\mathcal{A}(p)$ such that $\| x \|\to 1$ and $x^n \underset{\rightarrow}{w} x$ as $n\to\infty$. From Proposition 2.2 we have $m(x)=1$ so it follows from Proposition 2.3 that $m(x^n)\to m(x)$ as $n\to\infty$. Since the mapping $p_i:\mathcal{A}(p)\longrightarrow\mathbb{R}$, defined by $p_i(y)=y_i$ is a continuous linear functional on $\mathcal{A}(p)$ it follows that $x_i^n\to x_i$ as $n\to\infty$ for all $i\in\mathbb{N}$. Thus, we have obtain by Lemma 2.6 that $x^n\to x$ as $n\to\infty$. □

Theorem 2.8 *The space $\mathcal{A}(p)$ is rotund.*

Proof. Let $x\in S(\mathcal{A}(p))$ and $y,z\in B(\mathcal{A}(p))$ with $x=2^{-1}(y+z)$. By Proposition 2.2 and convexity of m we have

$$1=m(x)\le 2^{-1}(m(y)+m(z))\le 2^{-1}(1+1),$$

so that $m(x)=2^{-1}(m(y)+m(z))=1$. This implies that

$$\left(a_n\sum_{i=0}^{n}|2^{-1}(y_i+z_i)|\right)^{p_n}=2^{-1}\left(a_n\sum_{i=0}^{n}|y_i|\right)^{p_n}+2^{-1}\left(a_n\sum_{i=0}^{n}|z_i|\right)^{p_n}\qquad(2.8)$$

for all $k \in \mathbb{N}$. We shall show that $y_i = z_i$ for all $i \in \mathbb{N}$. From (2.8), we have

$$|x_1|^{p_1} = 2^{-1}[|y_1| + |z_1|]^{p_1}. \tag{2.9}$$

Since the mapping $u \to |u|^{p_1}$ is strictly convex, it implies by (2.8) that $y_1 = z_1$. Now assume that $y_i = z_i$ for all $i = 1, 2, ..., k-1$. Then $y_i = z_i = x_i$ for all $i = 1, 2, ..., k-1$. From (2.8) we have

$$\left(a_n \sum_{i=0}^{n} |2^{-1}(y_i + z_i)|\right)^{p_n} = \left(2^{-1}[a_n \sum_{i=0}^{n} |y_i| + a_n \sum_{i=0}^{n} |z_i|]\right)^{p_n} \tag{2.10}$$

$$= 2^{-1}\left(a_n \sum_{i=0}^{n} |y_i|\right)^{p_n} + 2^{-1}\left(a_n \sum_{i=0}^{n} |z_i|\right)^{p_n} \tag{2.11}$$

By the convexity of the mapping $u \to |u|^{p_1}$ it implies that $a_n \sum_{i=0}^{n} |y_i| = a_n \sum_{i=0}^{n} |z_i|$. Since $y_i = z_i$ for all $i = 1, 2, ..., k-1$ we get that

$$|y_k| = |z_k|. \tag{2.12}$$

If $y_k = 0$, then we have $y_k = z_k = 0$. Suppose that $y_k \neq 0$. Then $z_k \neq 0$. If $y_k z_k < 0$ it follows from (2.12) that $y_k + z_k = 0$. This implies by (2.10) and (2.12)

$$\left(a_n \sum_{i=0}^{n-1} |x_i|\right)^{p_n} = \left(a_n \left(\sum_{i=0}^{n-1} |x_i| + |y_i|\right)\right)^{p_n},$$

which is contradiction. Thus, we have $y_k z_k > 0$. This implies that by (2.9) that $y_k = z_k$. Thus we have by induction that $y_i = z_i$ for all $i \in \mathbb{N}$, so $y = z$. □

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