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Matrix Transformations of Orlicz Sequence Spaces

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Abstract: In this paper, we give the matrix characterizations from the Nakano vector-valued sequence space $\ell(X,p)$ into the Orlicz sequence space ℓ_M and by applying this result we also obtain necessary and sufficient conditions for infinite matrices mapping the sequence spaces $F_r(X,p)$ and $M_0(X,p)$ into ℓ_M, where $p = (p_k)$ is a bounded sequence of positive real numbers such that $p_k \leq 1$ for all $k \in N$ and $r \geq 0$.

Keywords: Matrix transformations, Orlicz sequence space, Nakano sequence space.
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1 Introduction

Let $(X, \|.\|)$ be a real Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. We write $x = (x_k)$ with x_k in X for all $k \in N$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $F_r(X,P)$, $E_r(X,p)$, and $M_0(X,p)$ are defined by

$$c_0(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k\|^{p_k} = 0\},$$
$$c(X,p) = \{x = (x_k) : \lim_{k\to\infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X\},$$
$$\ell_\infty(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty\},$$
$$\ell(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty\},$$
$$F_r(X,p) = \{x = (x_k) : \sum_{k=1}^{\infty} k^r \|x_k\|^{p_k} < \infty \},$$
$$E_r(X,p) = \{x = (x_k) : \sup_k \|x_k\|^{p_k}/k^r < \infty.\},$$

$$M_0(X,p) = \{x = (x_k) \ : \ \sum_{k=1}^{\infty} n^{-1/p_k} \|x_k\| < \infty \ \text{ for some } n \in N \ \}.$$

When $X = R$ the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $F_r(p)$, $E_r(p)$, and $M_0(p)$, respectively and the first three spaces are known as *the sequence spaces of Maddox*. These spaces were first introduced and studied by Simons [12], Maddox [6, 7], and Nakano [10]. When $p_k = 1$ for all $k \in N$, the space $F_r(p)$ is written as F_r. This space was first definded by Cooke [2]. The space $M_0(p)$ was definded by Grosse-Erdmann [3].

The structure of sequence spaces $c_0(p)$, $c(p)$, $\ell(p)$, and $\ell_\infty(p)$ have been investigated in [3]. Grosse-Erdmann [4] has given characterizations of matrix transformations between scalar-valued sequence spaces of Maddox. Wu and Liu [16] deal with the problem of characterizations of infinite matrices mapping $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$.

Suantai [13], and Suantai and Sudsukh [15] gave the matrix characterizations from $\ell(X,p)$ into ℓ_∞ and E_r. The problem of matrix transformations concerning the Orlicz sequence space was done by Suantai [14]. In that work, he gave necessary and sufficient conditions for infinite matrices mapping $\ell_\infty(X,p)$ and $c_0(X)$ into the Orlicz sequence space. In this paper we consider the problem of characterizing those infinite matrices mapping $\ell(X,p)$, $F_r(X,p)$ and $M_0(X,p)$ into the Orlicz sequence space. Because the Orlicz sequence space is a generalization of the ℓ_r space, so we also obtain characterizations of infinite matrices mapping those vector-valued sequence spaces into the ℓ_r space as a special case.

2 Notation and Definitions

Let $(X, \|.\|)$ be a real Banach space, the space of all sequences in X is denoted by $W(X)$ and $\Phi(X)$ is denoted for the space of all finite sequences in X. When $X = R$, the corresponding spaces are written as W and Φ.

A sequence spaces in X is a linear subspace of $W(X)$. Let E be any X-valued sequence space. For $x \in E$ and $k \in N$, we write x_k stands for the k^{th} term of X. For $k \in N$ denote by e_k the sequence $(0, 0, ..., 0, 1, 0, ...)$ with 1 in the k^{th} position and by e the sequence $(1, 1, 1, ...)$. For $x \in X$ and $k \in N$, let $e^k(x)$ be the sequence $(0, 0, ..., 0, x, 0, ...)$ with x in the k^{th} position and let $e(x)$ be the sequence $(x, x, x, ...)$. For a fixed scalar sequence $\mu = (\mu_k)$ the sequence space E_μ is defined by

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\} \ .$$

The sequence space E is called normal if $x \in E$ and $y \in W(X)$ with $\|y_k\| \leq \|x_k\|$ for all $k \in N$ implies that $y \in E$.

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \rightarrow F$ if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$, and the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the set of all infinite matrices mapping E into F. If $u = (u_k)$ and $v = (v_k)$ are scalar sequences, let

$$_u(E, F)_v = \{A = (f_k^n) : (u_n v_k f_k^n)_{n,k} \in (E, F)\ \}$$

If $u_k \neq 0$ for all $k \in N$, we write $u^{-1} = (\frac{1}{u^k})$.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if for each $k \in N$ the k^{th} coordinate mapping $p_k : E \rightarrow X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E, τ) is an Fréchet (Banach, LF-, LB-) space, then E is called an *FK- (BK-, LFK-, LBK-) space.* Now, suppose that E contains $\Phi(X)$. Then E is said to have *property AB* if the set $\{\sum_{k=1}^{n} e^k(x_k) : n \in N\ \}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property AK* if $\sum_{k=1}^{n} e^k(x_k) \rightarrow x$ in E as $n \rightarrow \infty$ for every $x = (x_k) \in E$. It has *property AD* if $\Phi(X)$ is dense in E.

The space $\ell(p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} |x_k|^{p_k}\right)^{1/K}$, where $K = \max\{1, \sup_k p_k\}$ (see [8]). The space $c_0(p)$ is an FK-space with AK, $c(p)$ is an FK-space and $\ell_\infty(p)$ is a complete LBK-space with AB (see [3, 8]). It is known that the space $\ell(X, p)$ is an FK-space with AK under the paranorm $g(x) = \left(\sum_{k=1}^{\infty} \|x_k\|^{p_k}\right)^{1/K}$, where $K = \max\{1, \sup_k p_k\}$.

A function $M : R \rightarrow [0, \infty)$ is said to be an *Orlicz function* if it is even, convex, continuous and vanishing only at 0. We define the Orlicz sequence space by the formula

$$\ell_M = \{x = (x_k) \in \ell^0\ :\ \rho_M(cx) = \sum_{k=1}^{\infty} M(cx_k) < \infty\ for some\ c > 0\ \}$$

where ℓ^0 stands for the space of all real sequences. We consider ℓ_M equipped with the Luxemburg norm

$$\|x\| = \inf\{\varepsilon > 0 : \rho_M(\frac{x}{\varepsilon}) \leq 1\ \}.$$

Let h_M denote the subspace order continuous elements, i.e

$$h_M = \{x = (x_k) \; : \; \rho_M(cx) < \infty \text{ for any } \; c > 0 \;\}.$$

It is known that ℓ_M is a BK-space and h_M is a closed subspace of ℓ_M.

We say that an Orlicz function M *satisfies the* δ_2 *condition* ($M \in \delta_2$ for short) if there exist constants $k \geq 2$ and $u_0 > 0$ such that

$$M(2u) \leq KM(u)$$

whenever $|u| \leq u_0$.

It is known that if $M \in \delta_2$, then $h_M = \ell_M$ (see [1]). For more details on Orlicz sequence space we refer to [1], [5], [9], and [11].

Now let us quote some known results that will be used to reduce our problems into simple forms.

Proposition 2.1 *Let E and $E_n (n \in N)$ be X-valued sequence spaces, and F and $F_n (n \in N)$ scalar sequence spaces, and let u and v be sequences of real numbers with $u_k \neq 0$, $v_k \neq 0$ for all $k \in N$. Then we have*

(i) $(\cup_{n=1}^{\infty} E_n, F) = \cap_{n=1}^{\infty}(E_n, F)$

(ii) $(E, \cap_{n=1}^{\infty} F_n) = \cap_{n=1}^{\infty}(E, F_n)$

(iii) $(E_u, F_v) = \; _v(E, F)_{u^{-1}}$,

(iv) $(E_1 + E_2, F) = (E_1, F) \cap (E_2, F)$,

(v) $(E, F_1) = (E, F_2) \cap (\Phi(X), F_1)$ *if E is an FK-space with AD, F_2 is an FK-space and F_1 is a closed subspace of F_2.*

Proof. See [14, Proposition 2.1]. □

Proposition 2.2 *Let M be an Orlicz function and $x \in \ell_M$.*

(i) If $\|x\| \leq 1$, *then* $\rho_M(x) \leq \|x\|$.

(ii) If $\|x\| > 1$, *then* $\rho_M(x) > \|x\|$.

(iii) If $M \in \delta_2$, *then* $\|x\| = 1 \implies \rho_M(x) = 1$.

Proof. *See [1, Theorem 1.38 and Theorem 1.39].* □

3 Main Results

We now turn to our main objective. We begin with giving characterizations of infinite matrices mapping the Nakano vector-valued sequence space $\ell(X,p)$ into the Orlicz sequence space.

Theorem 3.1 *Let $p=(p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$ and $A=(f_k^n)$ an infinite matrix. Then $A \in (\ell(X,p), \ell_M)$ if and only if*

(1) for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$ for all $x \in X$ and

(2) there exists $m_0 \in N$ such that

$$\sup_k \sup_{\|x\|\le 1} \sum_{n=1}^{\infty} \big(M \circ (m_0^{-1/p_k} f_k^n)\big)(x) \le 1 \ .$$

Proof. Suppose that $A \in (\ell(X,p), \ell_M)$. Since $e^k(x) \in \ell(X,p)$ for all $x \in X$ and all $k \in N$, we have $Ae^k(x) \in \ell_M$, so (1) is obtained. Now, we shall show that the condition (2) is satisfied. By Zeller's theorem, we have that $A : \ell(X,p) \to \ell_M$ is continuous. Then there exists $m_0 \in N$ such that

$$x=(x_k) \in \ell(X,p),\ \ \|x\| \le \frac{1}{m_0} \ \Rightarrow\ \|Ax\| \le 1 \ . \tag{3.1}$$

Let $x \in X$ with $\|x\| \le 1$ and $k \in N$. We have $m_0^{-1/p_k} e^k(x) \in \ell(X,p)$ and $\|m_0^{-1/p_k} e^k(x)\| \le \frac{1}{m_0}$. By (3.1) we have $\|(m_0^{-1/p_k} f_k^n(x))_{n=1}^{\infty}\| = \|A(m_0^{-1/p_k} e^k(x))\| \le 1$. By Proposition 2.2 (1) we obtain that $\sum_{n=1}^{\infty} M(m_0^{-1/p_k} f_k^n(x)) \le 1$. This implies that

$$\sup_k \sup_{\|x\|\le 1} \sum_{n=1}^{\infty} \big(M \circ (m_0^{-1/p_k} f_k^n)\big)(x) \le 1 \ ,$$

so that (2) holds.
Conversely, assume that the conditions (1) and (2) hold. By (2), there is $m_0 \in N$ such that

$$\sum_{n=1}^{\infty} M\big((m_0^{-1/p_k} f_k^n(x)\big) \le 1 \ \text{ for all } k \in N \text{ and all } x \in X \text{ with } \|x| \ \le 1.$$

It follows from Proposition 2.2 (1) that

$$\|A(m_0^{-1/p_k} e^k(x))\| = \|\big(m_0^{-1/p_k} f_k^n(x)\big)_{n=1}^{\infty}\| \le 1$$

for all $k \in N$ and all $x \in X$ with $\|x\| \leq 1$. Hence, for $x \in X$ with $x \neq 0$, we have

$$\|A(m_0^{-1/p_k} e^k(x))\| = \|\big(m_0^{-1/p_k} f_k^n(x)\big)_{n=1}^{\infty}\| \leq \|x\| \,. \tag{3.2}$$

Let $x = (x_k) \in \ell(X,p)$ and $k \in N$. By (3.2), we have

$$\begin{aligned}\|Ae^k(x_k)\| &= \|A(m_0^{1/p_k}(m_0^{-1/p_k} e^k(x_k)))\| \\ &= m_0^{1/p_k}\|A(m_0^{-1/p_k} e^k(x_k))\| \\ &\leq m_0^{1/p_k}\|x_k\|. \end{aligned} \tag{3.3}$$

Since $(m_0^{1/p_k} x_k) \in \ell(X,p)$, we have $(m_0^{1/p_k} x_k) \in c_0(X,p) \subseteq c_0(X)$, hence there is a $k_0 \in N$ such that $m_0^{1/p_k}\|x_k\| < 1$ for all $k > k_0$. Since $0 < p_k \leq 1$, we obtain

$$m_0^{1/p_k}\|x_k\| \leq \big(m_0^{1/p_k}\|x_k\|\big)^{p_k} = m_0\|x_k\|^{p_k} \tag{3.4}$$

for all $k > k_0$.

It follows from (3.3) and (3.4) that

$$\begin{aligned}\sum_{k=1}^{\infty}\|Ae^k(x_k)\| &\leq \sum_{k=1}^{\infty} m_0^{1/p_k}\|x_k\| \\ &= \sum_{k=1}^{k_0} m_0^{1/p_k}\|x_k\| + \sum_{k=k_0+1}^{\infty} m_0^{1/p_k}\|x_k\| \\ &\leq \sum_{k=1}^{k_0} m_0^{1/p_k}\|x_k\| + m_0 \sum_{k=k_0+1}^{\infty} \|x_k\|^{p_k} \\ &< \infty \,. \end{aligned}$$

Hence $\sum_{k=1}^{\infty} Ae^k(x_k)$ converges absolutely in ℓ_M. Since ℓ_M is Banach, $\sum_{k=1}^{\infty} Ae^k(x_k)$ converges in ℓ_M. Let $y = (y_k) \in \ell_M$ be the sum of the series $\sum_{k=1}^{\infty} Ae^k(x_k)$. By the continuity of p_m, we have for each $m \in N$,

$$y_m = p_m(y) = \lim_{n\to\infty} \sum_{k=1}^{n} p_m(Ae^k(x_k)) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k^m(x_k)$$

This implies that Ax exists and $(Ax)_m = \sum_{k=1}^{\infty} f_k^m(x_k) = y_m$, so that $Ax \in \ell_M$. □

Theorem 3.2 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A \in (\ell(X,p), h_M)$ if and only if*

(1) for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in h_M$ for all $x \in X$ and

(2) there exists $m_0 \in N$ such that

$$\sup_k \sup_{\|x\| \le 1} \sum_{n=1}^{\infty} \big(M \circ (m_0^{-1/p_k} f_k^n)\big)(x) \le 1 \ .$$

Proof. Since h_M is a closed subspace of ℓ_M, the theorem is obtained directly by applications of Theorem 3.1 and Proposition 2.1(v). □

When $p_k = 1$ for all $k \in N$, the following result is obtained directly by Theorem 3.1.

Theorem 3.3 *For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X),\ \ell_M)$ if and only if*

(1) for each $k \in N$, $(f_k^n(x))_{n=1}^{\infty} \in \ell_M$ for all $x \in X$ and

(2) there exists $m_0 \in N$ such that

$$\sup_k \sup_{\|x\| \le 1} \sum_{n=1}^{\infty} \big(M \circ (m_0^{-1} f_k^n)\big)(x) \le 1$$

□

When $M(t) = |t|^r$, $r \ge 1$, we have $\ell_M = \ell_r$. By an application of Theorem 3.1 the following result is obtained.

Corollary 3.4 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$ and $r \ge 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), \ell_r)$ if and only if*

(1) for each $k \in N$, $\sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$ for all $x \in X$ and

(2) there exists $m_0 \in N$ such that

$$\sup_k \sup_{\|x\| \le 1} \sum_{n=1}^{\infty} |m_0^{-1/p_k} f_k^n(x)|^r \le 1 \ .$$

When $p_k = 1$ for all $k \in N$, the following result is obtained directly from Corollary 3.4.

Corollary 3.5 *For $r \geq 1$ and for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X), \ell_r)$ if and only if*

(1) for each $k \in N$, $\sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$ for all $x \in X$ and

(2) $\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |f_k^n(x)|^r < \infty$.

Theorem 3.6 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and $A = (f_k^n)$ an infinite matrix. Then $A \in (M_0(X,p), \ell_M)$ if and only if*

(1) for each $m \in N$ and $k \in N$, $(m^{1/p_k} f_k^n(x))_{n=1}^{\infty} \in \ell_M$ for all $x \in X$ and

(2) for each $m \in N$, there exists $r_m \in N$ such that

$$\sup_k \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} \big(M \circ (r_m^{-1} m^{1/p_k} f_k^n)\big)(x) \leq 1 \ .$$

Proof. It is easy to see that $M_0(X,p) = \cup_{n=1}^{\infty} \ell(X,p)_{(n^{-1/p_k})}$, so it implies by Proposition 3.1 (i) that

$$A \in (M_0(X,p), \ell_M) \iff A \in (\ell(X)_{(m^{-1/p_k})},\ \ell_M) \text{ for all } m \in N \ .$$

By Proposition 2.1(iii), we have, for each $m \in N$,

$$A \in (\ell(X)_{(m^{-1/p_k})},\ \ell_M) \iff (m^{1/p_k} f_k^n)_{n,k} \in (\ell(X),\ \ell_M) \ .$$

This implies by Theorem 3.3 that

$$A \in (M_0(X,p),\ \ell_M) \iff \text{(1) and (2) are satisfied.}$$

□

By putting $M(t) = |t|^r$, where $r \geq 1$, Theroem 3.6 yields the following result.

Corollary 3.7 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and $r \geq 1$. Then for an infinite matrix $A = (f_k^n)$, $A \in (M_0(X,p), \ell_r)$ if and only if*

(1) for each $m \in N$ and $k \in N$, $\sum_{n=1}^{\infty} |m^{1/p_k} f_k^n(x)|^r < \infty$ and

(2) for each $m \in N$, there exists $r_m \in N$ such that

$$\sup_k \sup_{||x|| \le 1} \sum_{n=1}^{\infty} |r_m^{-1} m^{1/p_k} f_k^n(x)|^r \le 1 .$$

Theorem 3.8 *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$, $r \ge 0$ and $A = (f_k^n)$ an infinite matrix. Then $A \in (F_r(X,p), \ell_M)$ if and only if*

(1) for each $k \in N$, $(k^{-r/p_k} f_k^n(x))_{n=1}^{\infty} \in \ell_M$ for all $x \in X$ and

(2) there exists $m_0 \in N$ such that

$$\sup_k \sup_{||x|| \le 1} \sum_{n=1}^{\infty} \big(M \circ (m_0^{-1/p_k} k^{-r/p_k} f_k^n)\big)(x) \le 1 .$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it follows from Proposition 2.1 (iii) that

$$A \in (F_r(X,p),\ \ell_M) \iff \big(k^{-r/p_k} f_k^n\big)_{n,k} \in (\ell(X,p),\ \ell_M) .$$

It implies by Theorem 4.1 that

$$\big(k^{-r/p_k} f_k^n\big)_{n,k} \in (\ell(X,p),\ \ell_M) \iff \text{the conditions (1) and (2) hold.}$$

□

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