A Unified Local Convergence for Jarratt-type Methods in Banach Space Under Weak Conditions

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Abstract : We present a unified local convergence analysis for Jarratt-type methods in order to approximate a solution of a nonlinear equation in a Banach space setting. Our methods include the Jarratt;Inverse free Jarratt; super-Halley and other high order methods. The convergence ball and error estimates are given for these methods under the same conditions. Numerical examples are also provided in this study.

Keywords : Jarratt-type methods; Inexact Newton method; Banach space; Convergence ball; local convergence.
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1 Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

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where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [1, 2, 3, 4]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of equation (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev-Halley-type methods [5, 6, 1, 2], [7]-[15], [4]–[17] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive or for quadratic equations the second Fréchet-derivative is constant. Moreover, in some applications involving stiff systems, high order methods are useful. That is why in a unified way we study the local convergence of Jarratt-type methods(JTM) defined for each $n = 0, 1, 2, \cdots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1} F(x_n), \\
H_n &= F'(x_n)^{-1} [F(x_n) + \frac{2}{3}\gamma(y_n - x_n) - F'(x_n)], \\
x_{n+1} &= y_n - \frac{3\alpha}{4}\left(I + (\delta + \beta)H_n\right)^{-1}(I + \beta H_n)H_n(y_n - x_n),
\end{aligned}
\tag{1.2}
$$

where x_0 is an initial point, I is the identity operator and $\alpha, \beta, \gamma, \delta$ are real parameters. Many popular iterative methods are special cases of (JTM) method. For example, if $\alpha = 0$, we obtain Newton's method [1, 2, 3, 4, 26], if $\alpha = \gamma = 1, \delta = \frac{3}{2}$ and $\beta = 0$ we obtain the Jarratt method [18, 19] and for if $\alpha = \gamma = 0, \delta = \frac{3}{2}$ and $\beta = -\frac{3}{2}$, we obtain the inverse free Jarratt method [20]. Other choices of parameters α, β, γ and δ are also possible. The usual conditions for the semi-local convergence of these methods are $(\mathcal{C})$:

$(\mathcal{C}_1)$ There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \leq \beta$;

$(\mathcal{C}_2)$
$$\|\Gamma_0 F(x_0)\| \leq \eta;$$

$(\mathcal{C}_3)$
$$\|F''(x)\| \leq \beta_1 \quad \text{for each} \ x \in D;$$

$(\mathcal{C}_4)$
$$\|F'''(x)\| \leq \beta_2 \quad \text{for each} \ x \in D;$$

$(\mathcal{C}_5)$
$$\|F'''(x) - F'''(y)\| \le \beta_3 \|x - y\| \quad \text{for each } x, y \in D.$$

The local convergence conditions are similar but x_0 is x^* in $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. There is a plethora of local and semi-local convergence results under the $(\mathcal{C})$ conditions [1]–[31]. The conditions $(\mathcal{C}_4)$ and $(\mathcal{C}_5)$ restrict the applicability of these methods. That is why, in our study we assume the conditions $(\mathcal{A})$:

$(\mathcal{A}_1)$ $F : D \to Y$ is Fréchet-differentiable and there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$;

$(\mathcal{A}_2)$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0 \|x - x^*\|^p \quad \text{for each } x \in D \text{ and some } p \in (0, 1];$$

$(\mathcal{A}_3)$
$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L \|x - y\|^p \quad \text{for each } x, y \in D \text{ and some } p \in (0, 1];$$

$(\mathcal{A}_4)$
$$\|F'(x^*)^{-1} F'(x)\| \le K \quad \text{for each } x \in D.$$

and

$(\mathcal{A}_5)$
$$\left|1 - \frac{2}{3}\gamma\right| + \frac{2}{3}|\gamma| \le 1.$$

Notice that the $(\mathcal{A})$ conditions are weaker than the $(\mathcal{C})$ conditions. Hence, the applicability of (JTM) is expanded under the $(\mathcal{A})$ conditions. Moreover, in this study we extend the local convergence results for (JTM) method by considering the inexact Newton method (INM) defined for each $n = 0, 1, 2, \cdots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1} F(x_n), \\
x_{n+1} &= y_n - z_n,
\end{aligned}
\tag{1.3}
$$

where $\{z_n\} \in X$ is a null sequence, chosen to force convergence of sequence $\{x_n\}$ to x^*. Notice that if for each $n = 0, 1, 2, \cdots$

$$z_n = \frac{3\alpha}{4} \left(I + (\delta + \beta)H_n)\right)^{-1} \left(I + \beta H_n\right) H_n (y_n - x_n), \tag{1.4}$$

then (JTM) reduces to (INM). Several other choices of sequence $\{z_n\}$ are also possible [1, 2, 3, 4].

The paper is organized as follows: In Section 2 we present the local convergence of these methods. The numerical examples are given in the concluding Section 3.

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

2 Local convergence

In this section we present the local convergence of method (JTM) under the ($\mathcal{A}$) conditions. It is convenient for the local convergence of (JTM) to introduce some functions and parameters. Define parameters R and r_A by

$$R = (\frac{1}{L_0})^p \text{ and } r_A = \left(\frac{1+p}{(1+p)L_0 + L}\right)^{\frac{1}{p}}. \tag{2.1}$$

Notice that

$$r_A < R$$

and

$$\frac{Lt^{1+p}}{(1+p)(1-L_0t^p)} \le t \text{ for each } t \in [0, r_A] \tag{2.2}$$

Define function g on $[0, R]$ by

$$g(t) = (1 - L_0t^p) - (\frac{2|\gamma|}{3})^p|\delta + \beta|LK^pt^p. \tag{2.3}$$

We have that $g(0) = 1 > 0$ and $g(R) = -(\frac{2|\gamma|}{3L_0})|\delta + \beta|LK^p < 0$. Hence, it follows from the intermediate value theorem that function g has zeros in $(0, R)$. Denote by r_0 the smallest such root. Then, we also have that

$$g(t) > 0 \text{ for each } t \in [0, r_0). \tag{2.4}$$

Define functions f and f_1 on $[0, r_0)$ by

$$f(t) = \frac{Lt^p}{(1+p)(1-L_0t^p)} + \frac{3}{4}(\frac{2|\gamma|}{3})^p|\alpha|\frac{[(1-L_0t^p) + (\frac{2|\gamma|}{3})^pLK^p|\beta|t^p]}{(1-L_0t^p)^{1+p}}\frac{LK^{1+p}t^p}{g(t)} \tag{2.5}$$

and

$$f_1(t) = f(t) - 1. \tag{2.6}$$

We have that

$$f_1(0) = f(0) - 1 = -1 < 0$$

and

$$f_1(t) \to \infty \text{ as } t \to r_0^-.$$

Hence, function f_1 has zeros in $(0, r_0)$. Denote by r_1 the smallest such zero. Set

$$r^* = \min\{r_A, r_1\}. \tag{2.7}$$

Choose

$$r \in [0, r^*). \tag{2.8}$$

Then, we have that

$$f(t) < 1 \text{ for each } t \in [0, r]. \tag{2.9}$$

Next, we show the main local convergence result for (JTM) under ($\mathcal{A}$) conditions.

Theorem 2.1. *Suppose that the (A) conditions and $\overline{U}(x^*, r) \subseteq D$, hold, where r is given by (2.8). Then, sequence $\{x_n\}$ generated by (JTM) method (1.2) for any $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$.*

$$\|y_n - x^*\| \le \frac{L\|x_n - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_n - x^*\|^p)} \le \|x_n - x^*\| \qquad (2.10)$$

and

$$\|x_{n+1} - x^*\| \le f(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \qquad (2.11)$$

Proof. We shall use induction to show that estimates (2.10), (2.11) hold and $y_n, x_{n+1} \in U(x^*, r)$ for each $n = 0, 1, 2, \cdots$. Using $(\mathcal{A}_2)$ and the hypothesis $x_0 \in U(x^*, r)$ we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\|^p < L_0 r^p < 1. \qquad (2.12)$$

It follows from (2.12) and the Banach Lemma on invertible operators [1, 2, 3, 4] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|^p} < \frac{1}{1 - L_0 r^p}. \qquad (2.13)$$

Using the first substep of method (JTM) for $n = 0$, (2.13), $(\mathcal{A}_3)$, (2.8) and $F(x^*) = 0$ we get that

$$
\begin{aligned}
y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F'(x_0) \\
&= -[F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1}\int_0^1 (F'(x^* + \tau(x_0 - x^*)) \\
&\quad - F'(x_0))d\tau(x_0 - x^*)] \qquad (2.14)
\end{aligned}
$$

so

$$
\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \|F'(x^*)^{-1}\int_0^1 (F'(x^* + \tau(x_0 - x^*)) \\
&\quad - F'(x_0))d\tau\|\|x_0 - x^*\| \\
&\le \frac{L\|x_0 - x^*\|^{1+p}}{1 - L_0\|x_0 - x^*\|^p} \\
&= \frac{L\|x_0 - x^*\|^p}{1 - L_0\|x_0 - x^*\|^p} \\
&\le \|x_0 - x^*\| < r, \qquad (2.15)
\end{aligned}
$$

which shows (2.10) for $n = 0$ and that $y_0 \in U(x^*, r)$. Note that $u_0 = x_0 + \frac{2}{3}\gamma(y_0 - x_0) \in U(x^*, r)$. Indeed, we have by $(\mathcal{A}_5)$ that

$$
\begin{aligned}
\|u - x^*\| &= \|x_0 - x^* + \frac{2}{3}\gamma(y_0 - x^* + x^* - x_0)\| \\
&\leq |1 - \frac{2}{3}\gamma|\|x_0 - x^*\| + \frac{2}{3}|\gamma|\|y_0 - x^*\| \\
&\leq (|1 - \frac{2}{3}\gamma| + \frac{2}{3}|\gamma|)r \leq r.
\end{aligned}
$$

We also have by (2.13), (1.2), $(\mathcal{A}_3)$ that

$$
\begin{aligned}
\|H_0\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}(F'(x_0 + \frac{2}{3}\gamma(y_0 - x_0)) - F'(x_0)\| \\
&\leq \frac{1}{1 - L_0\|x_0 - x^*\|^p}L\|\frac{2}{3}\gamma(y_0 - x_0)\|^p \\
&\leq \frac{L(\frac{2|\gamma|}{3})^p}{1 - L_0\|x_0 - x^*\|^p}\|F'(x_0)^{-1}F'(x^*)\|^p \\
&\quad \|F'(x^*)^{-1}\int_0^1 F'(x^* + \tau(x_0 - x^*))d\tau(x_0 - x^*)\|^p \\
&\leq \frac{L(\frac{2|\gamma|}{3})^p}{1 - L_0\|x_0 - x^*\|^p}\frac{K^p\|x_0 - x^*\|^p}{(1 - L_0\|x_0 - x^*\|^p)^p} \\
&\leq \frac{L(\frac{2|\gamma|}{3})^pK^p\|x_0 - x^*\|^p}{(1 - L_0\|x_0 - x^*\|^p)^{1+p}} \\
&\leq \frac{L(\frac{2|\gamma|}{3})^pK^pr^p}{(1 - L_0r^p)^{1+p}}. \quad\quad (2.16)
\end{aligned}
$$

Moreover, we have by (2.4), (2.7), (2.8), (2.13), (2.16) that

$$
\begin{aligned}
|\delta + \beta|\|H_0\| &\leq |\delta + \beta|\frac{(\frac{2|\gamma|}{3})^pK^pL\|x_0 - x^*\|^p}{(1 - L_0\|x_0 - x^*\|^p)^{1+p}} \\
&\leq |\delta + \beta|\frac{(\frac{2|\gamma|}{3})^pK^pLr^p}{(1 - L_0r^p)^{1+p}} < 1. \quad\quad (2.17)
\end{aligned}
$$

It follows from (2.17) and the Banach Lemma that $(I + (\delta + \beta)H_0)^{-1}$ exists and

$$
\|(I + (\delta + \beta)H_0)^{-1}\| \leq \frac{1}{1 - |\delta + \beta|\frac{(\frac{2|\gamma|}{3})^pK^pLr^p}{(1 - L_0r^p)^{1+p}}}. \quad\quad (2.18)
$$

Furthermore, using (2.16), we get that

$$
\begin{aligned}
\|I + \beta H_0\| &\leq 1 + |\beta| \|H_0\| \\
&\leq 1 + \frac{(\frac{2|\gamma|}{3})^p |\beta| K^p L \|x_0 - x^*\|^p}{(1 - L_0 \|x_0 - x^*\|^p)^{1+p}} \\
&\leq 1 + \frac{(\frac{2|\gamma|}{3})^p |\beta| K^p L r^p}{(1 - L_0 r^p)^{1+p}}.
\end{aligned}
\tag{2.19}
$$

Then, using the second substep in (JTM) for $n = 0$, (2.15), (2.5), (2.8), (2.9), (2.16), (1.2), (2.18) and (2.19) we get in turn that

$$
\begin{aligned}
\|x_1 - x_0\| &\leq \|y_0 - x^*\| + \frac{3}{4}|\alpha| \|(I + (\delta + \beta)H_0)^{-1}\| \|I + \beta H_0\| \|H_0\| \|y_0 - x_0\| \\
&\leq f(\|x_0 - x^*\|) \|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned}
\tag{2.20}
$$

which shows (2.11) for $n = 0$ and that $x_1 \in U(x^*, r)$. To complete the induction, simply replace in all preceding estimates x_0, y_0, x_1 by x_k, y_k, x_{k+1}, respectively to arrive at

$$
\|x_{k+1} - x_0\| \leq f(\|x_k - x^*\|) \|x_k - x^*\| < \|x_k - x^*\| < r
$$

and

$$
\|y_k - x^*\| \leq \frac{L \|x_k - x^*\|^{1+p}}{(1 + p)(1 - L_0 \|x_k - x^*\|^p)} \leq \|x_k - x^*\| \leq r
$$

which complete the induction. Finally, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\|$, we deduce that $\lim_{k \to \infty} x_k = x^*$.   □

Remark 2.2.   (a) Condition $(\mathcal{A}_2)$ can be dropped, since this condition follows from $(\mathcal{A}_3)$. Notice, however that

$$
L_0 \leq L
\tag{2.21}
$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [6]–[22].

(b) In view of condition $(\mathcal{A}_2)$ and the estimate

$$
\begin{aligned}
\|F'(x^*)^{-1} F'(x)\| &= \|F'(x^*)^{-1}[F'(x) - F'(x^*)] + I\| \\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\
&\leq 1 + L_0 \|x - x^*\|^p,
\end{aligned}
$$

condition $(\mathcal{A}_4)$ can be dropped and K can be replaced by

$$
K(r) = 1 + L_0 r^p.
\tag{2.22}
$$

(c) It is worth noticing that r is such that

$$
r < r_A \text{ for } \alpha \neq 0.
\tag{2.23}
$$

The convergence ball of radius r_A was given by us in [22, 23, 5] for Newton's method under conditions $(\mathcal{A}_1)$- $(\mathcal{A}_3)$. Estimate (2.23) shows that the convergence ball of higher than two (JTM) methods is smaller than the convergence ball of the quadratically convergent Newton's method. The convergence ball given by Rheinboldt [4] (p=1)for Newton's method is

$$r_R = \frac{2}{3L} < r_A \tag{2.24}$$

if $L_0 < L$ and $\frac{r_R}{r_A} \to \frac{1}{3}$ as $\frac{L_0}{L} \to 0$. Hence, we do not expect r to be larger than r_A no matter how we choose L_0, L and K. Finally note that if $\alpha = 0$, then (JTM) reduces to Newton's method and $r = r_A$.

(d) *The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1, 2, 4].*

(e) *The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [1, 2, 3, 4]:*

$$F'(x) = T(F(x)), \tag{2.25}$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

In order for us to present the local results for (INM), let us suppose the $(\mathcal{A}')$ conditions: $(\mathcal{A}'_i)=(\mathcal{A}_i)$, $i = 1, 2, 3, 4$;
$(\mathcal{A}'_5)$ There exists a sequence $\{z_n\}$ in X and $\varphi : [0, \bar{r}) \to [0, \infty)$ continuous, nondecreasing with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \bar{r}^-$ for some $\bar{r} \in [0, R)$ such that

$$\|z_n\| \le \varphi(\|x_n - x^*\|)\|x_n - x^*\| \text{ for each } n = 0, 1, 2, \cdots.$$

A possible choice for sequence $\{z_n\}$ is given by (1.4). In this case we can choose

$$\varphi(t) = f(t) - \frac{Lt^p}{(1 + p)(1 - L_0 t^p)}.$$

Define functions ψ and ψ_1 on $[0, \bar{r})$ by

$$\psi(t) = \frac{Lt^p}{(1 + p)(1 - L_0 t^p)} + \varphi(t)$$

and

$$\psi_1(t) = \psi(t) - 1.$$

We have that $\psi_1(0) = \psi(0) - 1 = -1 < 0$ and $\psi_1(t) \to \infty$ as $t \to \bar{r}^-$. Then, function ψ_1 has zero in $(0, \bar{r})$. Denote by r_2 the smallest such zero. Set

$$\bar{r}^* = \min\{r_2, r_A\}$$

and choose

$$r \in [0, \bar{r}^*). \tag{2.26}$$

Then, we have that

$$\psi(t) < 1 \text{ for each } t \in [0, r).$$

As in the proof of Theorem 2.1 using induction, the conditions $(\mathcal{A}'_i)$ and the estimate

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|y_n - x^*\| + \|z_n\| \\
&\leq \frac{L\|x_n - x^*\|^p}{(1+p)(1 - L_0\|x_n - x^*\|^p)} + \varphi(\|x_n - x^*\|)\|x_n - x^*\| \\
&= \psi(\|x_n - x^*\|)(\|x_n - x^*\|
\end{aligned}
$$

we arrive at the following analog of Theorem 2.1 but for (INM) under the $(\mathcal{A}'_i)$ conditions.

Theorem 2.3. *Suppose that the $(\mathcal{A}')$ conditions and $\overline{U}(x^*, r) \subseteq D$, hold, where r is given by (2.26). Then, sequence $\{x_n\}$ generated by (INM) method (1.3) for any $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$.*

$$\|y_n - x^*\| \leq \frac{L\|x_n - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_n - x^*\|^p)} \leq \|x_n - x^*\|$$

and

$$\|x_{n+1} - x^*\| \leq \psi(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|.$$

Notice that it follows from the definition of function ψ and the properties of function ψ that $r < r_A$ (and $r = r_A$ if $\varphi = 0$).

3   Numerical Examples

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1. *Let $X = Y = \mathbb{R}$. Define function F on $D = [1, 3]$ by*

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x. \tag{3.1}$$

Then, $x^* = \frac{9}{4} = 2.25$, $F'(x^*)^{-1} = 2$, $L_0 = 1 < L = 2$, $p = 1$ and $K = 2(\sqrt{3} - 1)$, $r_1 = 0.2144$, $r \in [0, 0.2144)$ and $r_A = 0.6840$.

Example 3.2. *Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0, 1)$. Define F on D for $v = x, y, z)$ by*

$$F(v) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z). \tag{3.2}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e-1 < L = K = e$, $p = 1$. $r_1 = 0.1399$, $r \in [0,0.1399)$ *and* $r_A = 0.3599$.

Example 3.3. *Let* $X = Y = C[0,1]$, *the space of continuous functions defined on* $[0,1]$*be and equipped with the max norm. Let* $D = \overline{U}(0,1)$. *Define function* F *on* D *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.3}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15, p = 1$ *and* $K = K(r) = 1 + 7.5r$.

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