Thai Journal of Mathematics
Volume 13 (2015) Number 1 : 153–163

http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Applying the Reduced Differential Transform Method to Solve the Telegraph and Cahn-Hilliard Equations[1]

Mahmoud S. Rawashdeh[2] and Nazek A. Obeidat

Department of Mathematics and Statistics, Faculty of Science
Jordan University of Science and Technology
Irbid 22110, Jordan
e-mail : msalrawashdeh@just.edu.jo (M. S. Rawashdeh)
nazekobeidat@yahoo.com (N. A. Obeidat)

Abstract : In this paper, we apply the Reduced Differential Transform Method (RDTM) to solve two Physics models of nonlinear partial differential equations (NLPDEs) such as, Telegraph equation, Cahn-Hilliard equation, and two nonhomogeneous NLPDEs equation. The study outlines the significant of the method and the results showed that the method reduces the numerical calculations. The examples we present in this paper reveals that the proposed method is very effective, simple and can be applied to other nonlinear partial differential equations models in the area of Mathematical Physics and Engineering.

Keywords : reduced differential transform method (RDTM); differential transform method (DTM); telegraph equation; Cahn–Hilliard equation.
2010 Mathematics Subject Classification : 35C10; 41A58; 35L05.

1 Introduction

Most of the applications that arises in Mathematical physics and Engineering fields can be described by partial differential equations (PDEs). In Physics for

[1]This research was supported by the Deanship of Research at Jordan University of Science and Technology No. (87/2013).
[2]Corresponding author.

example, the heat flow and the wave propagation phenomena are well described by Partial differential equations see [1, 2]. The standard form of the telegraph equation [3] is given by $u_{xx} = au_{tt} + bu_t + cu$, where $u = u(x,t)$ is the resistance, and $a, b$and c are constants related to the inductance, capacitance and conductance of the cable respectively. Note that the telegraph equation is a linear partial differential equation. The telegraph equation arises in the propagation of electrical signals along a telegraph line. If we set $a = 0$ and $c = 0$, because of electrical properties of the cable, we then obtain $u_{xx} = bu_t$, which is the standard linear heat equation. On the other hand, the electrical properties may lead to $b = 0$ and $c = 0$; hence we obtain $u_{xx} = au_{tt}$, which is the standard linear wave equation. In this paper, we were being able to find approximate and exact solutions for the following NLPDEs: First, the Telegraph equation:

$$u_{xx} = au_{tt} + bu_t + cu, \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = 1 + \sinh(2x);\ u_t(x,0) = -2. \tag{1.2}$$

Second, the Cahn-Hilliard equation:

$$u_t - u_{xx} - u + u^3 = 0, \tag{1.3}$$

subject to the initial condition

$$u(x,0) = \frac{1}{1 + e^{\left(\frac{x}{\sqrt{2}}\right)}}. \tag{1.4}$$

Third, consider the nonhomogeneous equation:

$$u_t - \frac{1}{4}(u_x)^2 = x^2, \tag{1.5}$$

subject to the initial condition

$$u(x,0) = 0;\ u_t(x,0) = x^2. \tag{1.6}$$

Finally, consider the nonhomogeneous equation:

$$u_t + \frac{1}{36}x(u_{xx})^2 = x^3, \tag{1.7}$$

subject to the initial condition

$$u(x,0) = 0;\ u_t(x,0) = x^3. \tag{1.8}$$

The aim of the study is to be able to use the Reduced Differential Transform Method (RDTM) as an alternative method to the existing methods in solving different types of nonlinear partial differential equations (NLPDEs). Many authors used different methods to solve NLPDEs, to name few: The DTM, ADM, VIM, Tanh-Coth method, and Sine-Cosine method. Keskin, in his PhD thesis [4–6], introduced the reduced form of the differential transform method (DTM) as reduced differential transform method (RDTM) and he used the RDTM to solve the Gas Dynamics Equation and linear and nonlinear Klein Gordon Equations. Also, Keskin and Oturanc (2010) used the RDTM to solve linear and nonlinear wave equations and they showed the effectiveness, and the accuracy of the proposed method. Moreover, they showed that the number of iterations it takes to get an approximate solutions is less than the one used by the DTM. The most important advantage of the RDTM is the fact that it provides us with analytic approximate solution and in many cases it gives an exact solution, in a rapidly convergent sequence with less computed terms. İbiş and Bayram [7] used the RDTM to find approximate solutions for the (KdVB) equation, Drinefel'd–Sokolov–Wilson equations, coupled Burgers equations and modified Boussinesq equation. Also, Alquran [6] used the DTM to solve the Cahn-Hilliard equation and Wazwaz [8], find exact solution to the Telegraph equation. Finally, Rawashdeh [9], used the RDTM to find exact and approximate solution for Gardner equation, Variant Nonlinear Water Wave equation (VNWW), and the Fifth-Order Korteweg-de Vries (FKdV) equation.

The rest of this paper is organized as follows: In Section 2, the reduced differential transform method is introduced. Section 3 is devoted to apply the method to three test problems to show the effectiveness of the RDTM. Section 4 discussion and conclusion of this paper.

2 Reduced Differential Transform Method (RDTM)

In this section, we will give the methodology of the RDTM. So let's start with a function of two variables $u(x,t)$ which is analytic and $k-$times continuously differentiable with respect to time t and space x in the domain of our interest. Assume we can represent this function as a product of two single-variable functions $u(x,t) = f(x).g(t)$. From the definitions of the DTM, the function can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)\ t^k. \tag{2.1}$$

where $U_k(x)$ is the transformed function of $u(x,t)$ which can be defined as:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\, u(x,t)\right]_{t=0}. \tag{2.2}$$

From equations (2.1) and (2.2) we can deduce

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}\, u(x,t)\right]_{t=0} t^k = \sum_{k=0}^{\infty} \frac{1}{k!}\, U_k(x)\, t^k. \tag{2.3}$$

Note that from the above discussion, one can realize that the RDTM is derived from the power series expansion.

Now we will use the following theorems which can be deduced from Equations (2.1)–(2.3):

Theorem 2.1. *If $f(x,t) = \alpha\, u(x,t) \pm \beta\, v(x,t)$, then $F_k(x) = \alpha\, U_k(x) \pm \beta\, V_k(x)$, where α and β are constant.*

Theorem 2.2. *If $f(x,t) = u(x,t).\, v(x,t)$, then $F_k(x) = \sum_{i=0}^{k} U_i(x)\; V_{k-i}(x)$.*

Theorem 2.3. *If $f(x,t) = u(x,t).\, v(x,t).\, w(x,t)$ then $F_k(x) = \sum_{i=0}^{k} \sum_{j=0}^{i} U_j(x)$ $V_{i-j}(x)\; W_{k-i}(x)$.*

Theorem 2.4. *If $f(x,t) = \frac{\partial^n}{\partial t^n} u(x,t)$, then $F_k(x) = \frac{(k+n)!}{K!}\, U_{k+n}(x)$.*

Theorem 2.5. *If $f(x,t) = \frac{\partial^n}{\partial x^n} u(x,t)$, then $F_k(x) = \frac{\partial^n}{\partial x^n} U_k(x)$.*

Theorem 2.6. *If $f(x,t) = x^m t^n u(x,t)$, then $F_k(x) = x^m U_{k-n}\,(x)$.*

Theorem 2.7. *If $f(x,t) = x^m t^n$, then $F_k(x) = x^m \delta\,(k-n)$, where $\delta(k-n) = \left\{ \begin{array}{ll} 1, & k=n \\ 0, & k \neq n \end{array} \right\}$.*

The proofs of the above theorems and more can be found in [4–6].

Now, we illustrate the RDTM by using the Cahn–Hilliard equation in standard form:

$$L\,(u(x,t)) + R\,(u(x,t)) + N\,(u(x,t)) = 0 \tag{2.4}$$

with initial conditions

$$u(x,0) = f(x) \tag{2.5}$$

where $L = \frac{\partial}{\partial t}$ is a linear operator, $N\,(u(x,t)) = u^3$ is the remaining linear term. Using the theorems above, we can derive the following recursive relation:

$$(k+1)\, U_k(x) = R\,(U_k(x)) - N\,(U_k(x)) + U_k(x) \tag{2.6}$$

where, $R\,(U_k(x))$, $U_k(x)$ and $N\,(U_k(x))$ are the transformations of $R\,(u(x,t))$, $u(x,t)$ and $N\,(u(x,t))$ respectively. Now from equation (2.5), we can write the initial condition as:

$$U_0(x) = f(x) \tag{2.7}$$

To find all other iterations, we first substitute equation (2.7) into equation (2.6) and then we find the values of $U_k(x)$.

$$\widehat{u}\;(x,t) = \sum_{k=0}^{n} U_k(x)\, t^k \tag{2.8}$$

where n is the number of iterations we use to find the approximate solution. Hence, the exact solution of the problem is given by $u(x,t) = \lim_{n\to\infty} \widehat{u}\;(x,t)$.

3 Numerical Examples

In this section, we will give test problems by applying the RDTM to four numerical examples and then compare our approximate solutions to the exact solutions.

Example 3.1. *First, we consider the Telegraph equation:*

$$u_{xx} = u_{tt} + 4u_t + 4u, \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = 1 + \sinh(2x); u_t(x,0) = -2; u(0,t) = e^{-2t}; u_x(0,t) = 2, \tag{3.2}$$

where the exact solution is

$$u(x,t) = \sinh(2x) + e^{-2t}. \tag{3.3}$$

Applying the RDTM to (3.1) and (3.2), we obtain the recursive relation

$$U_{k+1}(x) = \frac{1}{(k+2)(k+1)}\left(\frac{\partial^2}{\partial x^2}(U_k(x)) - 4(k+1)U_k(x) - 4U_k(x)\right), \tag{3.4}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = 1 + \sinh(2x); \qquad U_1(x) = -2. \tag{3.5}$$

Now, substitute Eq. (3.5) into Eq. (3.4) to obtain the following:

$$U_2(x) = 2,\ U_3(x) = \frac{4}{3}, ... \tag{3.6}$$

And so on. So after few iterations, the differential inverse transform of $\{U_k(x)\}_{k=0}^{\infty}$ will provide us with the following approximate solution:

$$\begin{aligned}\widehat{u}\ (x,t) &= \textstyle\sum_{k=0}^{\infty} U_k(x)\, t^k = U_0(x) + U_1(x)\, t + U_2(x)\, t^2 + ...\\ &= 1 + \sinh(2x) - 2t + \tfrac{4}{3}t^2 + ...\\ &= 1 - 2t + 2t^2 - \tfrac{4t^3}{3} + \tfrac{2t^4}{3} - \tfrac{4t^5}{15} + \tfrac{4t^6}{45} - \tfrac{8t^7}{315} + \tfrac{2t^8}{315} + ... + \sinh(2x)\\ &= e^{-2t} + \sinh(2x).\end{aligned}$$

Note that this is the exact solution of Eq. (3.1).

Example 3.2. *We consider the Cahn-Hilliard equation*

$$u_t - u_{xx} - u + u^3 = 0, \tag{3.7}$$

subject to the initial condition

$$u(x,0) = \frac{1}{1+e^{\left(\frac{x}{\sqrt{2}}\right)}}, \tag{3.8}$$

where the exact solution is

$$u(x,t) = \frac{1}{1+e^{\left(\frac{x}{\sqrt{2}}-\frac{3t}{2}\right)}}. \tag{3.9}$$

Similar to the previous example, using the theorems above applied to Eq.(3.8) and Eq. (3.7) we get

$$U_{k+1}(x) = \frac{1}{k+1}\left(\frac{\partial^2 U_k(x)}{\partial x^2} + U_k(x) - \sum_{i=0}^{k}\sum_{j=0}^{i} U_{i-j}(x)U_j(x)U_{k-i}(x)\right), \tag{3.10}$$

where the $U_k(x)$, is the transform function of the $t-$dimensional spectrum. Note that

$$U_0(x) = \frac{1}{1+e^{\left(\frac{x}{\sqrt{2}}\right)}}. \tag{3.11}$$

Now, substitute Eq. (3.11) into Eq. (3.10) to obtain the following:

$$\begin{aligned} U_1(x) &= \frac{(192e^{\frac{x}{\sqrt{2}}}+576e^{\frac{3x}{\sqrt{2}}}+576e^{\sqrt{2}x}+192e^{2\sqrt{2}x})}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} \\ U_2(x) &= \frac{(-144e^{\frac{x}{\sqrt{2}}}+144e^{\frac{3x}{\sqrt{2}}}-144e^{\sqrt{2}x}+144e^{2\sqrt{2}x})}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}, \ldots. \end{aligned} \tag{3.12}$$

So after the fourth iteration, the differential inverse transform of $\{U_k(x)\}_{k=0}^{4}$ will give the following approximate solution:

$$
\begin{aligned}
\widehat{u}\,(x,t) &= \textstyle\sum_{k=0}^{4} U_k(x)\,t^k \\
&= U_0(x) + U_1(x)\,t + U_2(x)\,t^2 + U_3(x)\,t^3 + U_4(x)\,t^4 \\
&= \frac{128+512e^{\frac{x}{\sqrt{2}}}+512e^{\frac{3x}{\sqrt{2}}}+768e^{\sqrt{2}x}+128e^{2\sqrt{2}x}}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} \\
&\quad + \frac{(192e^{\frac{x}{\sqrt{2}}}+576e^{\frac{3x}{\sqrt{2}}}+576e^{\sqrt{2}x}+192e^{2\sqrt{2}x})t}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} + \frac{(-144e^{\frac{x}{\sqrt{2}}}+144e^{\frac{3x}{\sqrt{2}}}-144e^{\sqrt{2}x}+144e^{2\sqrt{2}x})t^2}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} \\
&\quad + \frac{(72e^{\frac{x}{\sqrt{2}}}-216e^{\frac{3x}{\sqrt{2}}}-216e^{\sqrt{2}x}+72e^{2\sqrt{2}x})t^3}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} + \frac{(-27e^{\frac{x}{\sqrt{2}}}-297e^{\frac{3x}{\sqrt{2}}}+297e^{\sqrt{2}x}+27e^{2\sqrt{2}x})t^4}{128\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5}.
\end{aligned}
$$

Hence the approximate solution converges rapidly to the exact solution of Eq. (3.7).

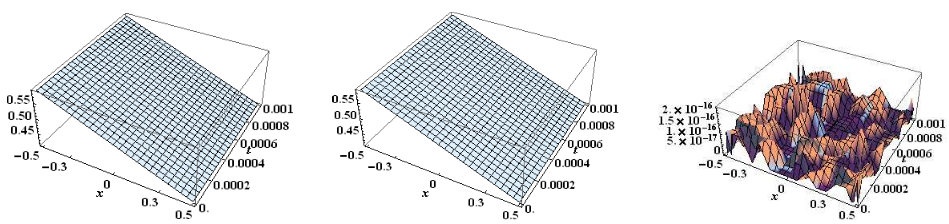

Figure 1: The approximate, exact solutions and absolute error, respectively for Example 3.2 when $-0.5 < x < 0.5$ and $0 < t < 0.001$.

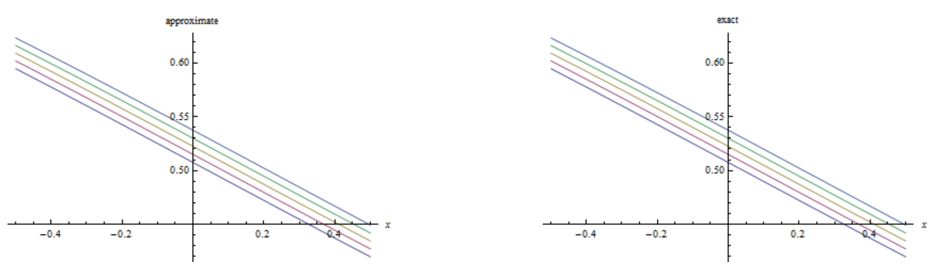

Figure 2: The exact and approximate solutions for Example 3.2 when -0.5< x <0.5 and $t = 0.02, 0.04, 0.06, 0.08, 0.1$.

Example 3.3. *We consider the nonlinear nonhomogeneous PDE*

$$u_t - \frac{1}{4}(u_x)^2 = x^2, \tag{3.13}$$

subject to the initial condition

$$u(x,0) = 0 = U_0(x), \tag{3.14}$$

where the exact solution

$$u(x,t) = x^2 \tan(t). \tag{3.15}$$

Now, we apply the RDTM to Eq. (3.13) and Eq. (3.14) we get

$$U_{k+1}(x) = \left(\frac{1}{k+1}\right)\left(x^2\delta(k) + \frac{1}{4}\sum_{i=0}^{k} \frac{\partial}{\partial x}U_i(x)\frac{\partial}{\partial x}U_{k-i}(x)\right). \tag{3.16}$$

So for $k=0$*, we obtain* $U_1(x) = x^2$*. Now for* $k \geq 1$ *we obtain*

$$U_2(x) = 0,\ U_3(x) = \frac{x^2}{3},\ U_4(x) = 0,\ U_5(x) = \frac{2x^2}{15}, \ldots. \tag{3.17}$$

Thus

$$\begin{aligned} u(x,t) &= tx^2 + \tfrac{t^3x^2}{3} + \tfrac{2t^5x^2}{15} + \tfrac{17t^7x^2}{315} + \tfrac{62t^9x^2}{2835} + \ldots \\ &= x^2\left(t + \tfrac{t^3}{3} + \tfrac{2t^5}{15} + \tfrac{17t^7}{315} + \tfrac{62t^9}{2835} + O[t]^{11}\right) \\ &= x^2\tan(t). \end{aligned}$$

This is the exact solution of Eq. (3.13) as was given in [8, p-324].

Example 3.4. *We consider the nonlinear nonhomogeneous PDE*

$$u_t + \frac{1}{36}x(u_{xx})^2 = x^3, \tag{3.18}$$

subject to the condition

$$u(x,0) = 0, \tag{3.19}$$

where the exact solution is

$$u(x,t) = x^3 \tanh(t). \tag{3.20}$$

Applying the RDTM to equ. (3.19) and equ. (3.18), we obtain the recursive relation

$$U_{k+1}(x) = \left(\frac{-1}{36(k+1)}\right)\left(x^3\delta(k) + x\sum_{i=0}^{k}\frac{\partial^2}{\partial x^2}U_i(x)\frac{\partial^2}{\partial x^2}U_{k-i}(x)\right), \tag{3.21}$$

So for $k = 0$, we obtain $U_1(x) = x^3$. Now for $k \geq 1$ we obtain

$$U_2(x) = 0,\ U_3(x) = -\frac{x^3}{3},\ U_4(x) = 0,\ U_5(x) = \frac{2x^3}{15}, \ldots. \tag{3.22}$$

Thus, the exact solution of Eq. (3.17) is

$$\begin{aligned} u(x,t) &= tx^3 - \tfrac{t^3x^3}{3} + \tfrac{2t^5x^3}{15} - \tfrac{17t^7x^3}{315} + \tfrac{62t^9x^3}{2835} + \ldots \\ &= x^3\left(t - \tfrac{t^3}{3} + \tfrac{2t^5}{15} - \tfrac{17t^7}{315} + \tfrac{62t^9}{2835} + O[t]^{11}\right) \\ &= x^3 \tanh(t). \end{aligned}$$

This is the exact solution of Eq. (3.18) as was given in [8, p-329].

4 Tables of Numerical Calculations

In this section, we shall illustrate the accuracy and efficiency of the RDTM. For this purpose, we can evaluate the approximate solution using the nth-order approximation. Table 1 shows the exact solution, the approximate solution and the absolute error for Cahn–Hilliard equation obtained by the RDTM. We must emphasize here only the fourth-order approximate was used for the same values of x and t, specifically, x = -0.5, -0.3, 0.3, 0.5 and t = 0.0002, 0.0004, 0.0006, 0.001.

Table 1. Comparison of the absolute error of the solution for example (3.2), by RDTM

x	t	Exact Solution	RDTM Solution	Abs-error-RDTM-(n=4)
-0.5	0.0002	0.5875517031584103	0.5875517031584103	$2.01088542E^{-17}$
	0.0004	0.587624401658184	0.587624401658184	$6.87902720E^{-17}$
	0.0006	0.5876970963359607	0.5876970963359608	$7.03708157E^{-17}$
	0.001	0.5878424742136453	0.5878424742136453	$2.80766342E^{-17}$
-0.3	0.0002	0.5529091870441258	0.5529091870441258	$1.76855678E^{-17}$
	0.0004	0.5529833460518396	0.5529833460518396	$5.33603154E^{-17}$
	0.0006	0.5530575027020757	0.5530575027020757	$3.92289032E^{-17}$
	0.001	0.5532058089172163	0.5532058089172162	$8.81393004E^{-17}$
0.3	0.0002	0.4472391380308357	0.44723913803083565	$2.30267789E^{-18}$
	0.0004	0.44731330409163234	0.44731330409163234	$5.22791547E^{-18}$
	0.0006	0.4473874724970048	0.4473874724970048	$3.10629862E^{-17}$
	0.001	0.4475358163285707	0.4475358163285707	$2.06891635E^{-17}$
0.5	0.0002	0.4125937052952466	0.41259370529524664	$1.84332270E^{-17}$
	0.0004	0.41266641524318715	0.4126664152431872	$6.72348714E^{-17}$
	0.0006	0.4127391290012399	0.41273912900123993	$1.75602238E^{-17}$
	0.001	0.41288456793578904	0.41288456793578904	$3.16011515E^{-17}$

5 Conclusion

In this paper, we applied the Reduced Differential Transform Method (RDTM) to all four physical models, namely, the Telegraph equation, Cahn-Hilliard equation, and two nonhomogeneous NLPDEs equation. We successfully found approximate solution for the Cahn-Hilliard equation with only three iterations. Also we were being able to find exact solutions to Examples 3.1, 3.3 and 3.4. The results we obtained in Example 3.2 were in excellent agreement with the exact solution. The RDTM introduces a significant improvement in the fields over existing techniques because it takes less calculations and the number of iteration is less compared by other methods. My goal in the future is to apply this method to other nonlinear PDEs which arise in other areas of science. Computations of this paper have been carried out using the computer package of Mathematica 7.

Acknowledgements : The author would like to express his appreciation and gratitude to the Editor and the anonymous referees' for their comments and suggestions on this paper. This research was supported by the Deanship of Research at Jordan University of Science and Technology No. (87/2013).

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(Received 7 January 2013)
(Accepted 3 September 2013)