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Note on "Modified α-ψ-Contractive Mappings with Applications"

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Abstract : In this short paper, we unexpectedly notice that the modified version of α-ψ-contractive mappings, suggested by Salimi et al. [Modified $\alpha-\psi$-contractive mappings with applications, Fixed Point Theory and Applications 2013, **2013**:151] is not a real generalization.

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1 Introduction and preliminaries

Let Ψ be a set a function defined by

- $\Psi = \{\psi : [0,+\infty) \to [0,+\infty)$ such that ψ is nondecreasing and $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0\}$, where ψ^n is the nth iterate of ψ.

Definition 1.1 (Samet et al., [1]). *Let T be a self-mapping on a metric space (X,d) and let $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if*

$$x, y \in X, \quad \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

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Definition 1.2 (Peyman et al., [2]). *Let T be a self-mapping on a metric space (X, d) and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible with respect to η mapping if*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Theorem 1.3 (Peyman et al., [2]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)),$$

where $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Also, suppose that the following assertions hold:

(a_1) *T is α-admissible mapping with respect to η;*

(a_2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;*

(a_3) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

2 Main results

Karapınar and Samet in [3], have established the following theorem.

Theorem 2.1 (Karapınar and Samet, [3]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Assume that*

$$x, y \in X, \quad \beta(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

where $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Also, suppose that the following assertions hold:

(A_1) *T is β-admissible mapping;*

(A_2) *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$;*

(A_3) *either T is continuous or for any sequence $\{x_n\}$ in X with $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\beta(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Now, we can state our first theorem.

Theorem 2.2. *Theorem 1.3 follows from Theorem 2.1.*

Proof. Define the mapping $\beta : X \times X \to [0, +\infty)$ by

$$\beta(x, y) = \begin{cases} 1 & \text{if } \alpha(x, y) \geq \eta(x, y), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that all conditions of Theorem 1.3 are satisfied, and we have to check that all conditions of Theorem 2.1 are satisfied too.
Clearly, if T is α-admissible with respect to η, then T is β-admissible. Hence the condition (A_1) of Theorem 2.1 is satisfied. Further, if we have

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)),$$

then, we have also

$$x, y \in X, \quad \beta(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

Next, if there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then this x_0 satisfies also $\beta(x_0, Tx_0) \geq 1$, that is, the condition (A_2) of Theorem 2.1 is satisfied. Finally, suppose that for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$, this implies also that for any sequence $\{x_n\}$ in X with $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\beta(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, that is, condition ($A_3$) is satisfied. Consequently, all conditions of Theorem 2.1 are satisfied. This implies the existence of the fixed point of T by Theorem 2.1. □

3 Consequences

We first state the following classes of mappings:

$$\Phi = \{\varphi | \varphi : [0, \infty) \to [0, \infty) \text{ is lower semi-continous}\} \text{ and}$$

$$\Psi_1 = \{\psi | \psi : [0, \infty) \to [0, \infty) \text{ continous, non-decreasing, }\}.$$

where $\phi(t) = 0 \Leftrightarrow t = 0$ for all functions ϕ in the class of either Ψ_1 or Φ.

Theorem 3.1 (Peyman et al., [2]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an α-admissible mapping with respect to η. Assume that for all $\psi \in \Psi_1$ and $\varphi \in \Phi$*

$$x, y \in X, \quad \alpha(x, Tx)\alpha(y, Ty) \geq \eta(x, Tx)\eta(y, Ty) \Longrightarrow \psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)).$$

Also, suppose that the following assertions hold:

(a$_1$) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$*;*

(a$_2$) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$*, we have* $\alpha(x, Tx) \geq \eta(x, Tx)$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

Then T *has a fixed point.*

Instead of Theorem 3.1, we state the following theorem:

Theorem 3.2. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be an* β*-admissible mapping . Assume that for all* $\psi \in \Psi_1$ *and* $\varphi \in \Phi$

$$x, y \in X, \quad \beta(x, Tx)\beta(y, Ty) \geq 1 \Longrightarrow \psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)).$$

Also, suppose that the following assertions hold:

(a$_1$) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(a$_2$) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\beta(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$*, we have* $\beta(x, Tx) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

Then T *has a fixed point.*

Theorem 3.3. *Theorem 3.1 follows from Theorem 3.2.*

Proof. Define the mapping $\beta : X \times X \to [0, +\infty)$ such that

$$\beta(x, y) = \begin{cases} 1 & \text{if } \alpha(x, y) \geq \eta(x, y), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have

$$\beta(x, Tx)\beta(y, Ty) = \begin{cases} 1 & \text{if } \alpha(x, Tx)\alpha(y, Ty) \geq \eta(x, Tx)\eta(y, Ty), \\ 0 & \text{otherwise.} \end{cases}$$

The rest can be derived from the following lines in the proof Theorem 2.2. □

Theorem 3.4 (Peyman et al., [2])**.** *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be an* α*-admissible mapping with respect to* η*. Assume that*

$$x, y \in X, \quad \alpha(x, Tx)\alpha(y, Ty) \geq \eta(x, Tx)\eta(y, Ty) \Longrightarrow \psi(d(Tx, Ty)) \leq c\psi(d(x, y)),$$

where for all $\psi \in \Psi_1$ *and* $0 < c < 1$*. Also, suppose that the following assertions hold:*

(a$_1$) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$*;*

(a$_2$) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$*, we have* $\alpha(x, Tx) \geq \eta(x, Tx)$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

Then T has a fixed point.

Analogously, we state the following theorem instead of Theorem 3.4, :

Theorem 3.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be an β-admissible mapping. Assume that*

$$x, y \in X, \quad \beta(x, Tx)\beta(y, Ty) \geq 1 \Longrightarrow \psi(d(Tx, Ty)) \leq c\psi(d(x, y)),$$

where for all $\psi \in \Psi_1$ and $0 < c < 1$. Also, suppose that the following assertions hold:

(a$_1$) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(a$_2$) *either T is continuous or for any sequence $\{x_n\}$ in X with $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\beta(x, Tx) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

We skipped the proof which can be derived easily following the lines in the proof of Theorem 3.4 in [2].

Theorem 3.6. *Theorem 3.4 follows from Theorem 3.5.*

Proof. By following the lines in the proof of Theorem 3.3. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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