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Parameter Estimation of One-dimensional Itô Processes by LTDRM

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Abstract : Itô processes are processes commonly used as a mathematical model in many fields. In order to estimate the unknown parameters of an Itô process based on the maximum likelihood method, the transitional probability density function (PDF) of the Itô process is needed. In fact, the transitional PDF is the solution of a Fokker-Planck equation subject to an initial condition in which the transitional PDF is set to be coincided with the Dirac delta function at the initial time. In this research, we applied the numerical method called the Laplace transform dual reciprocity method (LTDRM) to approximate the solution of the Fokker-Planck equations, corresponding to a one-dimensional Itô process. The key idea of the LTDRM for solving this type of problems is to transform the Dirac delta function into the Laplace space and then use the dual reciprocity method (DRM) to solve the transformed equation without approximating the Dirac delta function. The Stehfests algorithm is used to convert the solutions back into the transitional PDF. We tested and ran experiments on the OU and CIR models by comparing with exact transitional PDF. The tests show that our results using LTDRM give a very accurate approximation and can be used in the maximum likelihood estimation (MLE).

Keywords : Fokker-Planck equation; Transitional probability density function; Maximum likelihood.

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1 Introduction

An Itô process is a stochastic process which has many applications in various fields such as finance, physics, engineering, biology, medicine, etc. We consider a stochastic process $X = \{X_t, t \geq 0\}$ satisfying one-dimensional time-homogeneous stochastic differential equation (SDE)

$$dX_t = \mu(X_t;\theta)dt + \sigma(X_t;\theta)dW_t, \tag{1.1}$$

where $\{W_t, t \geq 0\}$ is a one-dimensional Wiener process, $\mu : \mathbb{R} \times \Theta \to \mathbb{R}$ is the drift coefficient, $\sigma : \mathbb{R} \times \Theta \to \mathbb{R}^+$ is the diffusion coefficient, and $\theta \in \Theta \subset \mathbb{R}^r$, $r \geq 1$, is an unknown parameter vector. X is called a *one-dimensional time-homogeneous Itô process*. Itô processes without knowing parameter vector θ are less used in real applications, especially for simulations and predictions of consequences. However, the parameters of the processes can be estimated by known information from the past, and one suitable method for obtaining such parameters is the maximum likelihood estimation (MLE).

Suppose that an Itô process $X = \{X_t, t \geq 0\}$ is observed at times $t_0, t_1, t_2, ..., t_n$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$. If the *transitional probability density function (PDF)* of X is known, denoted by $f_X \equiv f_X(t, x; t_0, x_0; \theta)$, then the log-likelihood function of a parameter vector θ can be obtained by

$$l_n(\theta) := \log f_X(t_0, X_0; \theta) + \sum_{k=0}^{n-1} \log f_X(t_{k+1}, X_{k+1}, t_k, X_k; \theta), \tag{1.2}$$

where $f_X(t_0, X_0; \theta)$ is the density of the initial state X_0 at the initial time t_0 and $f_X(t_{k+1}, X_{k+1}; t_k, X_k; \theta)$ is the value of the transitional PDF at (X_{k+1}, t_{k+1}) for a process starting at (X_k, t_k) and evolving to (X_{k+1}, t_{k+1}) according to SDE (1.1). According to MLE, the parameter vector θ that maximizes the log likelihood function is the maximum likelihood estimator, denoted by $\hat{\theta}_n$. The SDE corresponding to $\hat{\theta}_n$ is likely to be the best one that represents the observed data, and can be further applied in real applications.

The transitional PDF plays a significant role in order to estimate the maximum likelihood estimator $\hat{\theta}_n$ for $l_n(\theta)$. By a property of an Itô process, the transitional PDF of a one-dimensional Itô process X is the solution of the Fokker-Planck equation corresponding to the SDE (1.1), i.e.,

$$\frac{\partial f_X}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial(\sigma^2(x;\theta)f_X)}{\partial x} - \mu(x;\theta)f_X\right), \tag{1.3}$$

for $x \in S$, the state space. For a given parameter θ and the starting state (X_k, t_k), the transitional PDF $f_X(t, x; t_k, x_k; \theta)$ satisfies (1.3) with initial condition

$$f_X(t_k, x; t_k, X_k; \theta) = \delta(x - X_k). \tag{1.4}$$

For the state space $S = [A, B]$, the boundary conditions required to have unit density within S are the zero-flux conditions

$$\lim_{x \to A^+} \left(\mu f_X - \frac{1}{2} \frac{\partial(\sigma^2 f_X)}{\partial x} \right) = 0, \qquad \lim_{x \to B^-} \left(\mu f_X - \frac{1}{2} \frac{\partial(\sigma^2 f_X)}{\partial x} \right) = 0. \tag{1.5}$$

Since the transitional PDF at the initial state involves the Dirac delta function which may cause problems in direct substitution and it is not practical to get a usable closed form solution; therefore, the numerical approximations are often used instead. The first result for using the transitional PDF numerically for MLE is by Lo [1]. Hurn and Lindsay [2] applied the spectral method, Jensen and Poulsen [3] employed the finite difference method (FDM) via the approximation of the Dirac delta function using the normal distribution, and also, Hurn *et al.* [4] applied the FDM without approximating the Dirac delta function by reconsidering as the problem of the transitional cumulative distribution function (CDF) where the initial condition was transformed into a step function.

In this paper we suggest the Laplace transform dual reciprocity method (LTDRM) to approximate the transitional PDF numerically without approximating the Dirac delta function absorbed by the sifting property. The results of testing from this method shown experimentally that the LTDRM is very accurate for approximating the transitional PDF compared with other approximation such as Shoji and Ozaki [5]. In addition with testing for parameter estimation using MLE, the results confirm that the parameters obtained by using the transitional PDF from the LTDRM is as accurate as the one using the known transitional PDF.

The paper is organized as follows. Section 2 presents techniques required for approximating the transitional PDF: the transformation of a one-dimensional SDE to unit diffusion; the LTDRM; and, the Stehfest's algorithm for the inversion of Laplace transformation. Section 3 presents results of the testing of the method for approximating the transitional PDF via two models, the Ornstein-Uhlenbeck (OU) and the Cox, Ingersoll and Ross (CIR) models, and also comparing with other approximations obtained by Shoji and Ozaki [5]. In addition, we provided here the results of parameter estimation via MLE comparing between the approximating and the exact transitional PDF. Finally, the conclusion is given in Section 4.

2 Method

2.1 Unit diffusion

For a one-dimensional Itô process, we obtain a unit diffusion by using the transformation [4],

$$Y = \int^X \frac{du}{\sigma(u)}, \tag{2.1}$$

where the lower bound of integration can be chosen as any point in the domain of an Itô process X. Hence, an Itô process X in SDE (1.1) can be transformed into an Itô process Y by the change of variable based on the Itô formula. We then obtain the transformed SDE in the form

$$dY = \hat{\mu}(Y;\theta)dt + dW, \tag{2.2}$$

on the domain of an Itô process Y where

$$\hat{\mu}(Y;\theta) = \frac{\mu(X;\theta)}{\sigma(X;\theta)} - \frac{1}{2}\frac{d\sigma(X;\theta)}{dX}.$$

The Fokker-Planck equation for the Itô process Y is, therefore,

$$\frac{\partial f_Y}{\partial t} = \frac{1}{2}\frac{\partial^2 f_Y}{\partial y^2} - \hat{\mu}\frac{\partial f_Y}{\partial y} - f_Y\frac{\partial \hat{\mu}}{\partial y}, \tag{2.3}$$

for $f_Y(t, y; t_k, Y_k; \theta)$ where Y_k is the transform of X_k according to (2.1) with the initial and boundary conditions

$$f_Y(t_k, y; t_k, Y_k; \theta) = \delta(y - Y_k), \tag{2.4}$$

$$\lim_{y\to a^+}\left(\hat{\mu} f_Y - \frac{1}{2}\frac{\partial f_Y}{\partial y}\right) = 0, \quad \lim_{y\to b^-}\left(\hat{\mu} f_Y - \frac{1}{2}\frac{\partial f_Y}{\partial y}\right) = 0, \text{ for } t \geq 0 \tag{2.5}$$

In this paper we solve the Fokker-Planck equations (2.3)-(2.5). By solving for f_Y, and we can obtain the transitional PDF f_X by the relation

$$f_X = f_Y\frac{dY}{dX} = \frac{f_Y}{\sigma(X;\theta)}. \tag{2.6}$$

2.2 Laplace Transform Dual Reciprocity Method (LTDRM)

The Laplace transform dual reciprocity method (LTDRM) is a numerical technique which is an approximation method for obtaining an accurate solution of a partial differential equation involving the Dirac delta function [6, 7]. By taking the Laplace transform of the equations (2.3)-(2.5) where $F(s, y; \theta)$ denotes the Laplace transform of $f_Y(t, y; t_k, y_k; \theta)$ in variable t we obtain a boundary value problem

$$\frac{1}{2}\frac{\partial^2 F}{\partial y^2} + \delta(y - Y_k) = \hat{\mu}\frac{\partial F}{\partial y} + \left(s + \frac{\partial \hat{\mu}}{\partial y}\right)F \tag{2.7}$$

with the boundary conditions

$$\lim_{y\to a^+}\left(\hat{\mu} F - \frac{1}{2}\frac{\partial F}{\partial y}\right) = 0, \quad \lim_{y\to b^-}\left(\hat{\mu} F - \frac{1}{2}\frac{\partial F}{\partial y}\right) = 0. \tag{2.8}$$

Let L be the number of the interior nodes of the interval $[a, b]$ and $y_l \in [a, b]$, $l = 1, 2, 3, \ldots, L+2$. In this paper we assume that $[a, b]$ is subdivided to have equidistant subintervals, but it is not required in general.

Multiplying the equation (2.7) by the test function $w_l(y) := w(y; y_l)$ with property

$$\frac{\partial^2 w(y;y_l)}{\partial y^2} = \delta(y - y_l),$$

integrating by parts twice, and using the sifting property of the Dirac delta function

$$\int_a^b F(y)\delta(y-y_l)dy = c_l F(y_l), \quad y_l \in [a,b], \tag{2.9}$$

where

$$c_l = \begin{cases} 1 & \text{if } y_l \in (a,b), \\ \frac{1}{2} & \text{if } y_l = a \text{ or } y_l = b, \end{cases}$$

we have

$$c_l F(y_l) + [F' w_l]_a^b - [F w_l']_a^b + 2w_l(Y_k) = 2\int_a^b w_l\left(\hat{\mu}F' + (s+\hat{\mu}')F\right), \tag{2.10}$$

where $\hat{\mu}', F', w'$ denote the derivatives of $\hat{\mu}, F, w$ in variable y.
Representing the terms on the right hand side of the equation (2.10) by interpolation functions f_j with unknowns α_j as

$$\hat{\mu}F' + (s+\hat{\mu}')F = \sum_{j=1}^{L+2} \alpha_j f_j, \tag{2.11}$$

where basis f_j can be chosen as

$$f_j(y) = 1 + \sum_{i=1}^{r} |y - y_j|^i, \qquad j = 1,2,3,\ldots,L+2,$$

for $r = 1,2,3,\ldots$. In this paper we use the piecewise-linear basis for the approximation, $r = 1$. In theory, one can improve the accuracy of the approximation by using $r > 1$. Furthermore, we select functions $F_j(y)$ such that

$$\frac{d^2 F_j}{dy^2} = f_j, \qquad j = 1,2,3,\ldots,L+2.$$

This gives, for example,

$$F_j(y) = \begin{cases} \frac{1}{2}(y-y_j)^2 + \frac{1}{6}(y-y_j)^3 & \text{if } y \geq y_j, \\ \frac{1}{2}(y_j-y)^2 + \frac{1}{6}(y_j-y)^3 & \text{if } y < y_j, \end{cases}$$

and

$$F_j'(y) = \begin{cases} (y-y_j) + \frac{1}{2}(y-y_j)^2 & \text{if } y \geq y_j, \\ -(y_j-y) + \frac{1}{2}(y_j-y)^2 & \text{if } y < y_j. \end{cases}$$

Substituting (2.11) into (2.10) and integrating by parts twice again on the right hand side, we obtain the linear system ($l = 1, 2, 3, \ldots, L+2$)

$$c_l F(y_l) + \left[F' w_l\right]_a^b - \left[F w_l'\right]_a^b + 2w_l(Y_k) = \\ 2\sum_{j=1}^{L+2} \alpha_j \left(c_l F_j(y_l) + \left[F_j' w_l\right]_a^b - \left[F_j w_l'\right]_a^b \right). \tag{2.12}$$

Using the relations for α_j in (2.11) for $y = y_i$, $i = 1, 2, 3, \ldots, L+2$, we obtain a system for α_j

$$\hat{\mu}(y_i)F'(y_i) + (s + \hat{\mu}'(y_i))F(y_i) = \sum_{j=1}^{L+2} \alpha_j f_j(y_i). \tag{2.13}$$

The boundary conditions give

$$2\hat{\mu}(a)F(a) = F'(a), \qquad 2\hat{\mu}(b)F(b) = F'(b). \tag{2.14}$$

The linear system (2.12) has $L+2$ equations with $3L+6$ variables from $F(y_i), F'(y_i)$ and α_j. The $L+2$ variables α_j can be solved from (2.13) in terms of $F(y_i)$ and $F'(y_i)$. Moreover, the $L+2$ variables for $F'(y_i)$ can be approximated in terms of $F(y_i)$ via the boundary conditions (2.14) and the central different approximation: $F'(y_i) \approx (F(y_{i+1}) - F(y_{i-1}))/2h$ where $h = (b-a)/(L+1)$. This implies that $3L+6$ variables can be reduced to $L+2$ variables for $F(y_i)$ in the linear system (2.12), which can be solved linearly. The approximation of the transitional PDF can be obtained by taking the inverse Laplace transform where we employ the Stehfest's algorithm explained in the next section.

2.3 Stehfest's algorithm

The Stehfest's algorithm is a numerical inversion of Laplace transform [8], which is a well-known accurate method for the inverse Laplace transform. The inverse Laplace transform $f(t)$ of $F(s)$ can be approximated by, for $t > 0$,

$$f(t) \approx \frac{\ln 2}{t} \sum_{\nu=1}^{N_p} W_\nu F(p_\nu),$$

where N_p is an even positive integer, $p_\nu = \dfrac{\ln 2}{t}\nu$, and

$$W_\nu = (-1)^{(\sigma+\nu)} \sum_{k=K}^{M} \frac{k^\sigma (2k)!}{(M-k)!k!(k-1)!(\nu-k)!(2k-\nu)!},$$

$\sigma = \dfrac{N_p}{2}$, $M = \min\{\sigma, \nu\}$, and $K = \lfloor 0.5(\nu+1) \rfloor$ when $\lfloor \cdot \rfloor$ denotes the floor function.

3 Experiments

3.1 Ornstein-Uhlenbeck (OU) Model

The OU model has been used for modelling the interest rates, currency exchange rates, and commodity prices. The OU process established by Vasicek in 1997 is defined by, in a form of SDE,

$$dX_t = \alpha(\beta - X_t)dt + \sigma dW_t, \tag{3.1}$$

where $\alpha > 0$ is the speed of adjustment, $\beta > 0$ is the mean interest rate, and $\sigma > 0$ is the volatility control. The α, β, σ are parameters to be estimated, that is, $\theta = (\alpha, \beta, \sigma)$. The domain of the state variable is $S = \mathbb{R}$. The transitional PDF of the OU process is known exactly as

$$f_X(t_{k+1}, X_{k+1}; t_k, X_k; \theta) = \frac{1}{\sqrt{2\pi V}} \exp\left(-\frac{(x - \bar{x})^2}{2V}\right), \tag{3.2}$$

where

$$V = \frac{\sigma^2 \left(1 - \exp(-2\alpha(t_{k+1} - t_k)\right)}{2\alpha}, \qquad \bar{x} = \beta + (X_k - \beta)\exp(-\alpha(t_{k+1} - t_k)).$$

Assuming the parameters (α, β, σ) are given together with initial state (t_k, X_k), we obtain the approximated transitional PDF via the procedures in the previous section, denoted by $f_Y^{DRM}(t, Y; \theta)$. The comparisons with the exact $f_Y(t, Y; \theta)$ which can be obtain via (2.6) and (3.2), for various t and with different parameters are shown in Figure 1.

In the Figure 1, the approximations of the transitional PDF appear to be the same as the exact solutions. The relative error for the approximation $t = 0.15, 0.25, 0.35$ are shown in Figure 2.

The results of testing in Figures 1 and 2 show that the obtained transitional PDF f_Y^{DRM} is as accurate as the exact f_Y.

3.2 Cox, Ingersoll and Ross (CIR) Model

The CIR process proposed by Cox, Ingersoll and Ross in 1985 is also known as the square-root process and has been used for modeling the instantaneous short term interest rate. The CIR model is defined by

$$dX_t = \alpha(X_t - \beta)dt + \sigma\sqrt{X_t}dW_t, \tag{3.3}$$

where $\alpha > 0$ is the speed of adjustment, $\beta > 0$ is the mean interest rate, and $\sigma > 0$ is the volatility control with condition $2\alpha\beta \geq \sigma^2$ to have $X_t > 0$ for all t. The domain of the state variable is positive real number, $S = \mathbb{R}^+$. The transitional PDF of the CIR process is known exactly as

$$f_X(t_{k+1}, X_{k+1}; t_k, X_k; \theta) = c\left(\frac{v}{u}\right)^{\frac{q}{2}} \exp(-u - v) I_q(2\sqrt{uv}), \tag{3.4}$$

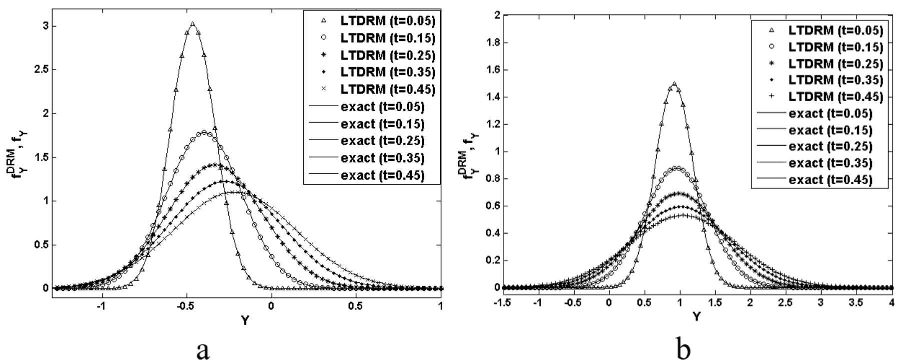


Figure 1: (a) Numerical approximation of transitional PDF of OU Model on $[-1.3, 1]$ with $Y_k = -0.5$, $\theta_a^{OU} = (0.5, 0.9, 0.6)$ and $L + 2 = 41$, (b) Numerical approximation of transitional PDF of OU Model on $[-1.5, 4]$ with $Y_k = 0.9$, $\theta_b^{OU} = (0.3, 2.0, 1.2)$ and $L + 2 = 66$

where

$$c = \frac{2\alpha}{\sigma^2\left(1 - \exp(-\alpha(t_{k+1} - t_k))\right)}, \;\; u = cX_k \exp(-\alpha(t_{k+1} - t_k)), \;\; v = cx, \;\; q = \frac{2\alpha\beta}{\sigma^2} - 1,$$

and $I_q(x)$ is the modified Bessel function of the first kind of order q. Similarly, we followed steps in the Section 2 and obtained the transitional PDF of the CIR process by assuming the parameters α, β, σ. The Figures 3 (a) and (b) show the transitional PDF of the CIR process using the LTDRM compared with Shoji [5] and with the exact solutions. The result from Shoji [5] is obtained by applying the Taylor series expansion to the drift function $\hat{\mu}(Y;\theta)$ in the equation (2.2) about the state Y_k, i.e.,

$$dY = \left(\hat{\mu}(Y_k;\theta) + \hat{\mu}'(Y_k;\theta)(Y - Y_k) + \frac{\triangle}{2}\hat{\mu}''(Y_k;\theta)\right) dt + dW, \tag{3.5}$$

where $\triangle = (t_{k+1} - t_k)$, to have the OU model for (3.5), with three parameters,

$$\alpha = -\hat{\mu}'(Y_k;\theta), \;\; \beta = Y_k - \frac{2\hat{\mu}(Y_k;\theta) + \triangle\hat{\mu}''(Y_k;\theta)}{2\hat{\mu}'(Y_k;\theta)}, \;\; \sigma = 1$$

so that the exact solution can be obtained via (3.2).

The comparison of approximated transitional PDF f_Y^{DRM} are shown in the Figure 3 for various times t with the exact f_Y and with Shoji and Ozaki f_Y^{Shoji}.

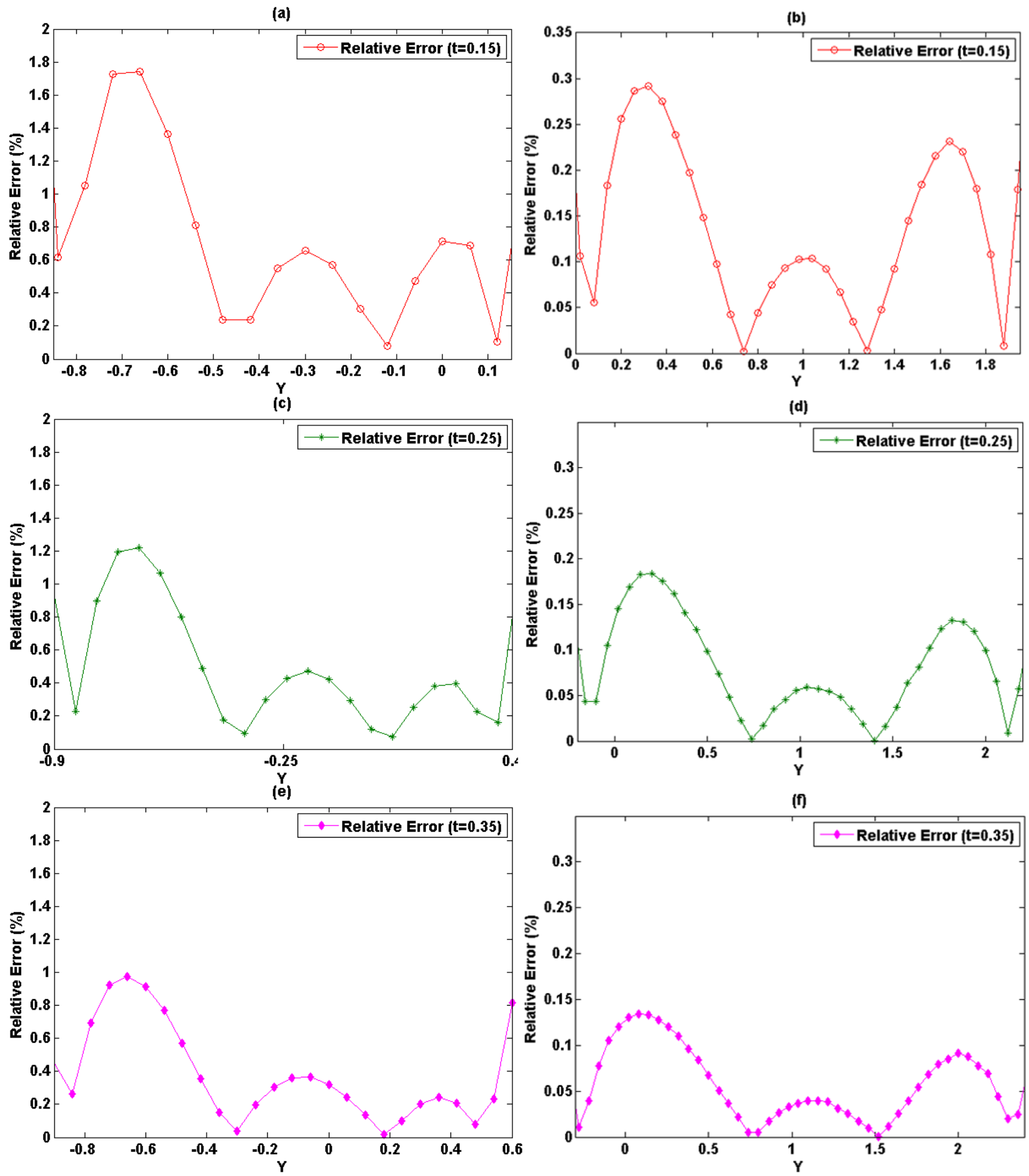
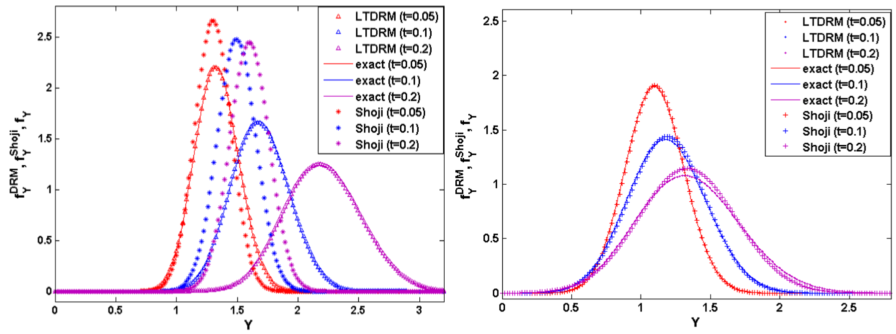


Figure 2: (a,c,e) Relative error of approximated transitional PDF by using LTDRM on Figure 1 (a), (b,d,f) Relative error of approximated transitional PDF by using LTDRM on Figure 1 (b)



a b

Figure 3: (a) Numerical approximation of transitional PDF of CIR Model on $[0, 3.2]$ with $Y_k = 0.8$, $\theta_a^{CIR} = (1, 3, 0.7)$ and $L + 2 = 181$, (b) Numerical approximation of transitional PDF of CIR Model on $[0, 2.8]$ with $Y_k = 0.9$, $\theta_b^{CIR} = (0.7, 3.5, 1.3)$ and $L + 2 = 141$

In the Figure 3 (a), the approximations of the transitional PDF by using LTDRM appear to be the same as the exact solutions, however, the approximations of the transitional PDF by Shoji and Ozaki are quite different. From the Figure 3 (b), f_Y^{DRM} and f_Y^{Shoji} appear to be very closed to f_Y but f_Y^{DRM} is better when comparing using the relative errors. The relative errors for the approximation for $t = 0.5, 0.10, 0.20$ are shown in Figure 4.

The results of testing in Figures 3 and 4 show that the obtained transitional PDFs, f_Y^{DRM}, are much more accurate than the approximations of the transitional PDF by Shoji and Ozaki, f_Y^{Shoji}, when comparing with the exact solution.

3.3 Experiments with maximum likelihood estimation

We tested the parameter estimation using the MLE for the OU model based on the obtained transitional PDF from the LTDRM. In this paper, the sample data is generated using the Euler-Maruyama scheme; for OU model (3.1)

$$x_{j+1} = x_j + \alpha(\beta - x_j)\Delta + \sigma\varepsilon_j, \qquad j = 0, 1, \ldots, n-1, \tag{3.6}$$

where x_j and x_{j+1} are consecutive states of X separated by a time interval of duration $\Delta = t_{k+1} - t_k$, $\varepsilon_j \sim N(0, \Delta t)$, and assuming that parameters $x_0, \alpha, \beta, \sigma$ are randomly given.

3.3.1 Estimating one parameter

In this setup we assumed that two parameters from the set $\{\alpha, \beta, \sigma\}$ are given and then estimated the remaining one. The test is preformed with 10 samples

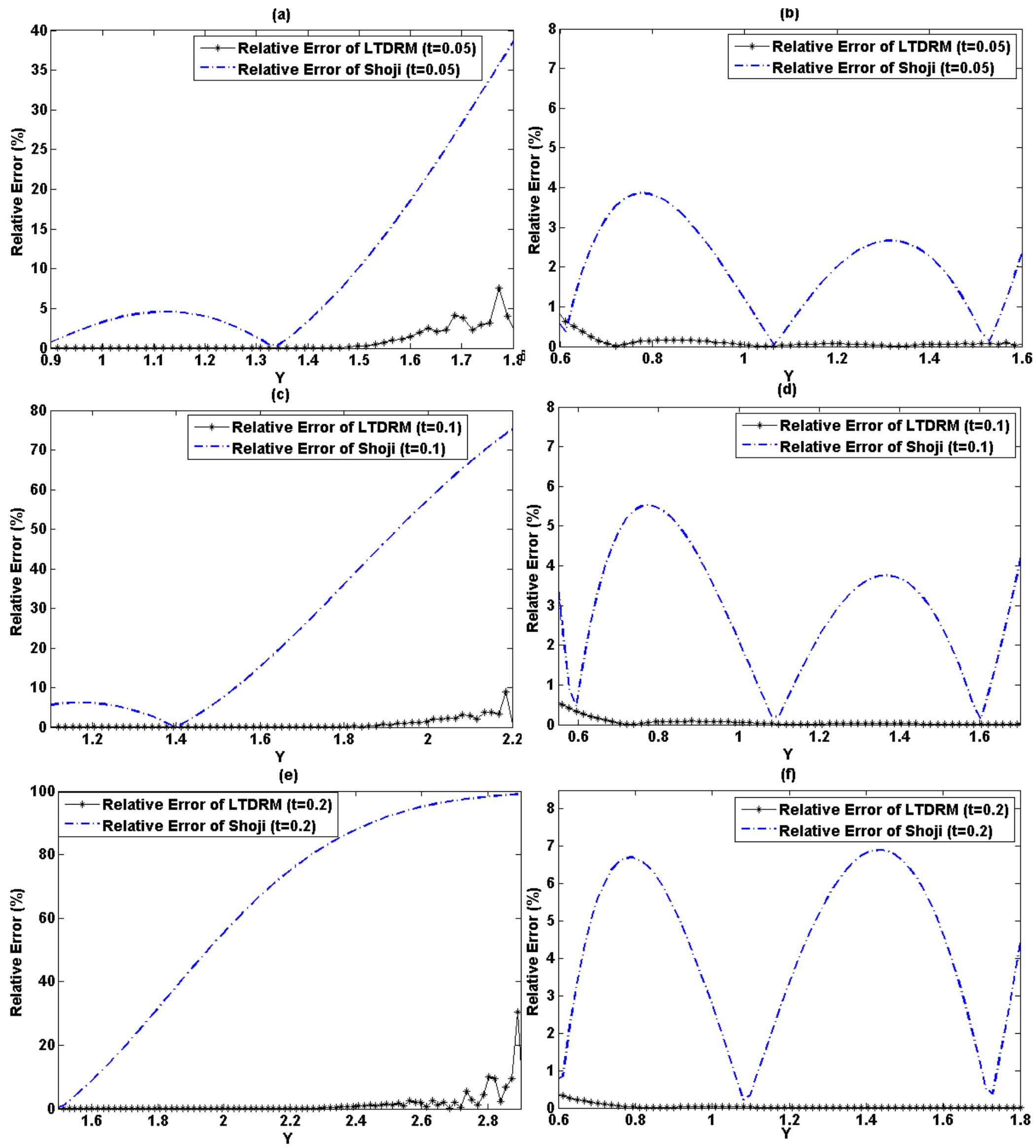


Figure 4: (a,c,e) Relative error of approximated transitional PDF by using LTDRM on Figure 3 (a), (b,d,f) Relative error of approximated transitional PDF by using LTDRM on Figure 3 (b)

of observed data sets generated by (3.6) with $n = 60, t = 1/12$ in three groups depending on the remaining parameter.

	fixed β, σ estimate α	fixed α, σ estimate β	fixed α, β estimate σ
Sample No.	$(\hat{\alpha}_{LTDRM}, \hat{\alpha}_{exact})$	$(\hat{\beta}_{LTDRM}, \hat{\beta}_{exact})$	$(\hat{\sigma}_{LTDRM}, \hat{\sigma}_{exact})$
1	(1.50,1.50)	(0.75,0.75)	(1.75,1.75)
2	(1.25,1.25)	(2.00,2.00)	(1.25,1.25)
3	(0.25,0.25)	(0.75,0.75)	(1.75,1.75)
4	(1.25,1.25)	(0.50,0.50)	(1.50,1.50)
5	(0.50,0.50)	(0.50,0.50)	(1.50,1.50)
6	(1.25,1.25)	(0.50,0.50)	(1.50,1.50)
7	(0.25,0.25)	(0.50,0.50)	(1.50,1.50)
8	(1.50,1.50)	(0.50,0.50)	(1.25,1.25)
9	(0.75,0.75)	(1.00,1.00)	(2.00,2.00)
10	(1.25,1.25)	(1.00,1.00)	(1.75,1.75)

Table 1: *Optimal parameters with MLE for 10 times from using f_y^{DRM} and f_Y with $n = 60, t = 1/12$*

The results comparing between the estimated transitional PDF by the LTDRM and the exact one are shown in Table 1. It shown that the estimated parameters for MLE based on the LTDRM appear to be almost the same as ones obtained using the exact known transitional PDF.

3.3.2 Estimating all three parameters

In this set up we test the parameter estimation of all three parameters (α, β, σ) using 3 samples of observed data sets generated by (3.6) with $n = 60, t = 1/12$.

The results are shown in Table 2, the parameters obtained based on the LTDRM ($\hat{\theta}_{LTDRM}$) appear to be the same as the ones using the exact transitional PDF ($\hat{\theta}_{exact}$).

Based on the results of estimating of one parameter in Section 3.3.1 and three parameters in Section 3.3.2, the estimated parameters with the MLE from the transitional PDFs obtained by the LTDRM are as accurate as the ones obtained using the exact transitional PDFs. This shows that the approximated transitional PDFs from the LTDRM is applicable for parameter estimations with MLE and gives the results that is as accurate as the ones using the exact transitional PDF.

Sample No.	1	2	3
$\hat{\theta}_{LTDRM}$	(0.65,1.75,1.9)	(0.65,1.75,1.9)	(0.65,1.75,1.9)
$\hat{\theta}_{exact}$	(0.65,1.75,1.9)	(0.65,1.75,1.9)	(0.65,1.75,1.9)

Table 2: *Optimal parameters with MLE for 3 times from using* f_y^{DRM} *and* f_Y *with* $n = 60, t = 1/12$

4 Conclusions

The LTDRM can be employed to approximate the transitional PDF without worrying about the Dirac Delta function in the initial data of the problem (2.3)-(2.5) for estimating unknown parameters of a one-dimensional time-homogeneous Itô process based on the MLE. In this paper we tested the approximated transitional PDF from the LTDRM of two Itô processes; the OU model and the CIR model, and used the approximated transitional PDF from the LTDRM of the OU model for estimating parameters using the MLE.

From testing in Sections 3.1 and 3.2, it is sufficient to conclude that the LTDRM can be applied to approximate the transitional PDF with good accuracy, and better than other method such as Shoji and Ozaki, see Figures 1-4. As seen in testing for parameter estimation with MLE in Section 3.3, see Tables 1-2, the obtained parameters based on the approximation of the transitional PDF by the LTDRM is almost the same as the ones obtained when using the exact transitional PDF. This shown that the approximated of the transitional PDF based on the LTDRM can be used in the parameter estimation for MLE without knowing the exact ones, especially when Itô processes have no closed-form transitional PDF.

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