Thai Journal of Mathematics
Volume 13 (2015) Number 1 : 109–121
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Properties of Attractive Points in CAT(0) Spaces

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Abstract : We study attractive points of nonlinear mappings in $CAT(0)$ spaces. We prove the attractive point theorem for normally generalized hybrid mappings in a $CAT(0)$ space satisfying $(\mathbb{S})$ property.

Keywords : $CAT(0)$ space; attractive point; $(\mathbb{S})$ property; $(\overline{\mathbb{Q}}_4)$ condition; normally generalized hybrid mapping.
2010 Mathematics Subject Classification : 47H09; 47H10.

1 Introduction

In 2011, Takahashi and Takeuchi [1] introduced the notion of attractive points of nonlinear mappings in a Hilbert space: let H be a Hilbert space and C be a nonempty subset of H. Let T be a mapping form C into H. Let $A(T)$ denote the set of all attractive points of T, i.e.,

$$A(T) = \{z \in H : ||z - Ty|| \leq ||z - y||, \text{ for all } y \in C\}.$$

It is known that $A(T)$ is a closed convex subset of H [1, Lemma 2.3].

In 2012, Takahashi, Wong and Yao [2] introduced the class of normally generalized hybrid mappings in a Hilbert space which covers the classes of generalized

[0]The first author is supported by the Development and Promotion of Science and Technology Talents Project (DPST)
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hybrid, hybrid, nonspreading and nonexpansive mappings. A mapping $T : C \to H$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(1) $\alpha + \beta + \gamma + \delta \geq 0$;

(2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and

(3) $\alpha||Tx - Ty||^2 + \beta||x - Ty||^2 + \gamma||Tx - y||^2 + \delta||x - y||^2 \leq 0$, for all $x, y \in C$.

Such a mapping T can be called an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping. The authors also proved the attractive point theorem for normally generalized hybrid mappings in a Hilbert space.

Theorem 1.1 ([2], Theorem 3.1)**.** *Let C be a nonempty subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping. Then $A(T) \neq \emptyset$if and only if there exists $x \in C$ such that the sequence $\{T^n x\}$ is bounded. Additionally, if C is closed and convex, then $F(T) \neq \emptyset$ if and only if there exists $x \in C$ such that the sequence $\{T^n x\}$ is bounded. In particular, a fixed point is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).*

In this paper, we generalize basic properties of attractive points of nonlinear mappings in Hilbert spaces to $CAT(0)$ spaces. Moreover, we prove attractive point theorem for normally generalized hybrid mappings in $CAT(0)$ spaces satisfying $(\mathbb{S})$ property.

2 Preliminaries

Let H be a Hilbert space and C be a nonempty subset of H. A mapping $T : C \to H$ is called nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \text{for all } x, y \in C.$$

In 2008, Kohsaka and Takahashi [3] introduced the class of nonspreading mappings in Hilbert spaces. A mapping $T : C \to H$ is called nonspreading if

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2 \ , \quad \text{for all } x, y \in C.$$

In 2010, Takahashi [4] introduced the class of hybrid mappings in Hilbert spaces. A mapping $T : C \to H$ is called hybrid if

$$3||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2 + ||Ty - x||^2 \ , \quad \text{for all } x, y \in C.$$

In 2010, Kocourek, Takahashi and Yao [5] introduced the class of generalized hybrid mappings in Hilbert spaces. A mapping $T : C \to H$ is called generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that for any $x, y \in C$,

$$\alpha||Tx - Ty||^2 + (1 - \alpha)||x - Ty||^2 \leq \beta||Tx - y||^2 + (1 - \beta)||x - y||^2.$$

We can see that any $(1, 0)$, $(2, 1)$ and $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings are nonexpansive, nospreading and hybrid mappings, respectively. Moreover, If $\alpha +$

$\beta = -\gamma - \delta = 1$, then an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping is a generalized hybrid mapping.

Let (X, d) be a metric space. A geodesic path (or a geodesic) joining x to y in X is a mapping c from a closed interval $[0, \ell] \subseteq \mathbb{R}$ to X such that $c(0) = x, c(\ell) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, \ell]$. In particular, the mapping c is an isometry and $d(x, y) = \ell$. The image of c is called geodesic segment joining x and y which when unique is denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ and $d(y, z) = (1 - \alpha)d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$.

A metric space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic segment joining x and y for each $x, y \in X$.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.

A geodesic space X is called a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle in $\mathbb{R}^2$. Then the geodesic triangle $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

It is well known that any complete simply connected Riemannian manifold of nonpositive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, $\mathbb{R}$-trees (see [6]), Euclidean buildings (see [7]), the complex Hilbert ball with a hyperbolic metric (see [8]), and many others.

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the geodesic segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies the so-called (CN) inequality of Bruhat and Tits [9], i.e.,

$$d^2(x, y_0) \leq \tfrac{1}{2}d^2(x, y_1) + \tfrac{1}{2}d^2(x, y_2) - \tfrac{1}{4}d^2(y_1, y_2).$$

It is known that a uniquely geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [6] for more details).

The following results are important to our work.

Lemma 2.1 ([6]). *Let X be a CAT(0) space, $x_1, x_2, y_1, y_2 \in X$ and $\alpha \in [0, 1]$. Then*

$$d(\alpha x_1 \oplus (1 - \alpha)y_1, \alpha x_2 \oplus (1 - \alpha)y_2) \leq \alpha d(x_1, x_2) + (1 - \alpha)d(y_1, y_2).$$

From the above lemma, it is easy to see that for any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$d(\alpha x \oplus (1 - \alpha)y, z) \leq \alpha d(x, z) + (1 - \alpha)d(y, z).$$

Lemma 2.2 ([10])**.** *Let X be a CAT(0) space, $x, y, z \in X$ and $\alpha \in [0, 1]$. Then*

$$d^2(\alpha x \oplus (1-\alpha)y, z) \leq \alpha d^2(x, z) + (1-\alpha)d^2(y, z) - \alpha(1-\alpha)d^2(x, y).$$

Lemma 2.3 ([6])**.** *Let C be a closed convex subset of a complete CAT(0) space X. For any $x \in X$ there exists a unique point $p \in C$ such that*

$$d(x, p) = \inf_{y \in C} d(x, y).$$

The mapping $P_C : X \to C$ defined by $P_C x = p$ is called the metric projection from X onto C.

In 2008, Kirk and Panyanak [11] specialized Lim's concept [12] of $\triangle$-convergence in a general metric space to a $CAT(0)$ space and showed that many results in Banach space involving weak convergence have precise analogs in this setting.

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space (X, d). For any $x \in X$, we set

$$r(x, \{x_n\}) := \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) := \inf_{x \in X} r(x, \{x_n\}).$$

The asymptotic center of $\{x_n\}$ is given by

$$A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from [13] that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_{n\to\infty} x_n = x$. The uniqueness of asymptotic center implies that $CAT(0)$ space X satisfies Opial's property, i.e., for a given sequence $\{x_n\}$ in X such that $\{x_n\}$ $\triangle$-converges to x and for any $y \in X$ with $y \neq x$, one has

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

We also know the following results.

Lemma 2.4 ([11])**.** *Every bounded sequence in a complete CAT(0) space always has a $\triangle$-convergent subsequence.*

Now, we recall the concept of Banach limit which plays a major role in our results. Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of the dual metric space $(\ell^\infty)^*$ of the space ℓ^∞. We denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, ...) \in \ell^\infty$. We denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ is called a mean if $\mu(e) = ||\mu|| = 1$, where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$, where $\mu_n(x_{n+1}) = \mu_n(x_2, x_3, x_4, ...)$. We know that there exist a Banach limit on ℓ^∞ (see [14] for more details). If μ is a Banach limit on ℓ^∞, then for any $(x_1, x_2, x_3, ...) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in \ell^\infty$ and $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then we have

$$\mu(f) = x = \mu(x_n).$$

In 2010, Kakavandi and Amini [15] introduced the dual metric space (X^*, D) for a complete $CAT(0)$ space (X, d) based on the work of Berg and Nikolaev [16]. We know that $CAT(0)$ inequality and parallelogram identity are equivalent on a norm linear space. Then Berg and Nikolaev [16] introduced the concept of quasilinearization which is an inner product-like notion in complete $CAT(0)$ spaces.

Let $\overrightarrow{ab}$ denote a pair $(a, b) \in X \times X$ and it is called a vector. Then the quasilinearization map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{uv} \rangle = \frac{1}{2}\left(d^2(a, v) + d^2(b, u) - d^2(a, u) - d^2(b, v)\right)$$

for any $a, b, u, v \in X$. We say that (X, d) satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{uv} \rangle \leq d(a, b)d(u, v)$$

for all $a, b, u, v \in X$. The authors also proved the following lemma.

Lemma 2.5 ([16], Corollary 3)**.** *A geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.*

Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X; \mathbb{R})$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle \qquad \text{for all } x \in X,$$

where $C(X; \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is the Lipchitz function with the Lipchitz semi-norm $L(\Theta(t, a, b)) = td(a, b) \quad (t \in \mathbb{R}, a, b \in X)$, where $L(\varphi) = \sup\left\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x, y \in X, \quad x \neq y\right\}$ is the Lipchitz semi-norm for any function $\varphi : X \to \mathbb{R}$.

Kakavandi and Amini [15] defined a pseudometric D on $\mathbb{R} \times X \times X$ by

$$D((t, a, b), (s, u, v)) = L(\Theta(t, a, b) - \Theta(s, u, v)).$$

They also proved the following lemma.

Lemma 2.6 ([15], Lemma 2.1)**.** *Let X be a complete CAT(0) space. Let $(t, a, b), (s, u, v) \in \mathbb{R} \times X \times X$. Then $D((t, a, b), (s, u, v)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle$, for all $x, y \in X$.*

Therefore, D defines an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalent class of (t, a, b) is

$$[t\overrightarrow{ab}] = \Big\{ s\overrightarrow{uv} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle, \text{ for all } x, y \in X \Big\}.$$

Let $X^* = \Big\{ [t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X \Big\}$. Then (X^*, D) is a metric space and it is called the dual metric space of (X, d).

In 2013, Kakavandi [17] introduced the concept of $(\mathbb{S})$ property for a complete $CAT(0)$ space as follows.

Definition 2.7 ([17]). *A complete $CAT(0)$ space (X, d) satisfies $(\mathbb{S})$ property if for any $(x, y) \in X \times X$ the exists a point $y_x \in X$ such that $[\overrightarrow{xy}] = [\overrightarrow{y_x x}]$.*

There are many $CAT(0)$ spaces satisfy the $(\mathbb{S})$ property; for example, Hilbert spaces [17] and symmetric Hadamard manifolds [17]. Moreover, Kakavandi also gave the characterization of $\triangle$-convergence for $CAT(0)$ spaces satisfying the $(\mathbb{S})$ property as follows.

Lemma 2.8 ([17], Lemma 2.8). *Let (X, d) be a complete $CAT(0)$ space, $\{x_n\}$ be a bounded sequence in X and let $x \in X$. If X satisfies the $(\mathbb{S})$ property, then $\triangle - \lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$ for all $y \in X$.*

In 2008, Kirk and Panyanak [11] introduced a geometric condition on a CAT(0) space called a four point condition $(\mathbb{Q}_4)$.
$(\mathbb{Q}_4)$: A $CAT(0)$ space (X, d) is said to satisfy $(\mathbb{Q}_4)$ condition if for any $x, y, p, q \in X$, one has

$$d(p, x) < d(x, q) \ \& \ d(p, y) < d(y, q) \Rightarrow d(p, m) < d(m, q), \quad \text{for all } m \in [x, y].$$

The authors mentioned that this condition holds in many $CAT(0)$ spaces including Hilbert spaces and $\mathbb{R}$-trees. Since then this condition has been studied very deeply by Espínola and Fernández-León [18], who proved that any $CAT(0)$ space of constant curvature satisfies the $(\mathbb{Q}_4)$ condition but any $CAT(0)$ gluing space containing two spaces of constant but different curvatures does not.

In 2013, Kakavandi [17] modified the $(\mathbb{Q}_4)$ condition as follows.
$(\overline{\mathbb{Q}}_4)$: A $CAT(0)$ space (X, d) is said to satisfy $(\overline{\mathbb{Q}}_4)$ condition if for any $x, y, p, q \in X$, one has

$$d(p, x) \le d(x, q) \ \& \ d(p, y) \le d(y, q) \Rightarrow d(p, m) \le d(m, q), \quad \text{for all } m \in [x, y].$$

Since $(\overline{\mathbb{Q}}_4)$ implies $(\mathbb{Q}_4)$, there are some $CAT(0)$ spaces that do not satisfy $(\overline{\mathbb{Q}}_4)$ condition. However, Hilbert spaces, $\mathbb{R}$-trees and every $CAT(0)$ spaces of constant curvature satisfy the $(\overline{\mathbb{Q}}_4)$ condition.

3 Properties of Attractive Points in CAT(0) Spaces

In this section, we collect some properties of attractive points in a $CAT(0)$ space. First of all, we consider the notion of attractive points of any mapping

$T : C \to X$, where X is a metric space and C is a nonempty subset of X. The set of all attractive points of T can be defined analogously to Hilbert spaces, i.e.,

$$A(T) = \{z \in X : d(z, Ty) \leq d(z, y), \text{ for all } y \in C\}.$$

Moreover, in a metric space, a normally generalized hybrid mapping can be defined in analogous way as follows.
A mapping $T : C \to X$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(1) $\alpha + \beta + \gamma + \delta \geq 0$;
(2) $\alpha + \beta > 0$ or $\alpha + \gamma > 0$; and
(3) $\alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \leq 0$, for all $x, y \in C$.

Lemma 3.1. *Let C be a nonempty subset of a complete $CAT(0)$ space (X, d) and let T be a mapping from C into X. Then, $A(T)$ is closed. Moreover, if X satisfies the $(\overline{\mathbb{Q}_4})$ condition, then $A(T)$ is convex.*

Proof. Let $\{x_n\}$ be a sequence in $A(T)$ such that $x_n \to x \in X$ as $n \to \infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have $d(x_n, Ty) \leq d(x_n, y)$. The continuity of the metric d allows us to conclude that

$$\begin{aligned} \lim_{n\to\infty} d(x_n, Ty) &\leq \lim_{n\to\infty} d(x_n, y) \\ d(x, Ty) &\leq d(x, y). \end{aligned}$$

This implies that $A(T)$ is a closed subset of X.

Suppose that X satisfies the $(\overline{\mathbb{Q}_4})$ condition. To show that $A(T)$ is convex, we let $z_1, z_2 \in A(T)$. Then for any $y \in C$, we have $d(z_1, Ty) \leq d(z_1, y)$ and $d(z_2, Ty) \leq d(z_2, y)$. It follows that $d(m, Ty) \leq d(m, y)$ for all $m \in [z_1, z_2]$. Therefore, $A(T)$ is a convex subset of X. □

According to the definition of attractive points, for any mapping, a fixed point need not be an attractive point but except for normally generalized hybrid mappings.

Proposition 3.2. *Let (X, d) be a complete $CAT(0)$ space. Let C be a nonempty subset of X. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping. Then $A(T) \cap C = F(T)$.*

Proof. If $z \in A(T) \cap C$, then $d(z, Tz) \leq d(z, z)$. This implies that $Tz = z$. Then $A(T) \cap C \subseteq F(T)$. To show that $F(T) \subseteq A(T) \cap C$, we let $z \in F(T)$. Since T is a normally generalized hybrid mapping, we have

$$\alpha d^2(Tx, Ty) + \beta d^2(x, Ty) + \gamma d^2(Tx, y) + \delta d^2(x, y) \leq 0\ , \text{ for all } x, y \in C.$$

If $\alpha+\beta>0$, for any $y\in C$, we have

$$\begin{aligned}\alpha d^2(Tz,Ty)+\beta d^2(z,Ty)+\gamma d^2(Tz,y)+\delta d^2(z,y) &\leq 0.\\ \alpha d^2(z,Ty)+\beta d^2(z,Ty)+\gamma d^2(z,y)+\delta d^2(z,y) &\leq 0\\ (\alpha+\beta)d^2(z,Ty) &\leq -(\gamma+\delta)d^2(z,y)\\ d^2(z,Ty) &\leq -\frac{\gamma+\delta}{\alpha+\beta}d^2(z,y)\\ d^2(z,Ty) &\leq d^2(z,y).\end{aligned}$$

This implies that $z\in A(T)$. We can obtain the similar result in the case of $\alpha+\gamma>0$. Therefore, $A(T)\cap C=F(T)$. □

The following corollary is obtained directly from Lemma 3.1 and Proposition 3.2.

Corollary 3.3. *Let (X,d) be a complete $CAT(0)$ space satisfying $(\overline{\mathbb{Q}}_4)$ condition. Let C be a nonempty closed convex subset of X. Let $T:C\to C$ be a normally generalized hybrid mapping. Then $F(T)$ is a closed convex subset of X.*

The following results show that, under some appropriate conditions, the existence of an attractive point implies the existence of a fixed point. We can prove the analogous result to Lemma 2.2 of [1] in a $CAT(0)$ space.

Proposition 3.4. *Let (X,d) be a $CAT(0)$ space. Let C be a nonempty closed convex subset of X. Let $T:C\to C$ be a mapping with $A(T)\neq\emptyset$. Then, $F(T)\neq\emptyset$.*

Proof. Let $z\in A(T)$. Since C is a nonempty closed convex subset of X, from Lemma 2.3 we can find $y=P_Cz\in C$, where P_C is the metric projection from X onto C. Since $z\in A(T)$, we have $d(z,Ty)\leq d(z,y)$. Since $d(z,y)=\inf_{x\in C}d(z,x)$ and $Ty\in C$, we have $Ty=y$. Then $y\in F(T)$ and hence $F(T)\neq\emptyset$. □

Next, we prove the analogous result to Proposition 10 of [19] in a complete $CAT(0)$ space satisfying the $(\overline{\mathbb{Q}}_4)$ condition.

Proposition 3.5. *Let (X,d) be a complete $CAT(0)$ space satisfying the $(\overline{\mathbb{Q}}_4)$ condition. Let C be a closed subset of X. Let T be a continuous mapping from C into X with $A(T)\neq\emptyset$. Suppose that there exists $x_0\in C$ such that $T^nx_0\in C$ for all $n\geq 0$. If $\lim_{n\to\infty}d(T^nx_0,A(T))=0$, then T has a fixed point x^* and the sequence of Picard iterates $\{T^nx_0\}$ converges strongly to x^*.*

Proof. Firstly, we show that $\{T^nx_0\}$ is a Cauchy sequence. Let $\epsilon>0$. Since $\lim_{n\to\infty}d(T^nx_0,A(T))=0$, there exists $N\in\mathbb{N}$ such that

$$d(T^nx_0,A(T))<\frac{\epsilon}{4}\quad\text{for all } n\geq N.$$

Then

$$\inf_{z\in A(T)} d(T^N x_0, z) = d(T^N x_0, A(T)) < \frac{\epsilon}{4}.$$

We can find $z \in A(T)$ such that $d(T^N x_0, z) < \frac{\epsilon}{2}$. Since $z \in A(T)$, for any $m, n \geq N$, we have

$$\begin{aligned} d(T^m x_0, T^n x_0) &\leq d(T^m x_0, z) + d(z, T^n x_0) \\ &\leq d(T^N x_0, z) + d(T^N x_0, z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Then $\{T^n x_0\}$ is a Cauchy sequence. Since C is a closed subset of complete $CAT(0)$ space X, there exists $x^* \in C$ such that $T^n x_0 \to x^*$. To complete the proof, we will show that $x^* \in F(T)$. Since T is continuous, we can see that

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(T^n x_0, A(T)) \\ &= d(\lim_{n\to\infty} T^n x_0, A(T)) \\ &= d(x^*, A(T)). \end{aligned}$$

This implies that $x^* \in \overline{A(T)}$ (the closure of $A(T)$). By Lemma 3.1, we know that $A(T)$ is a closed subset of X. Then $x^* \in A(T) \cap C$. Therefore, $x^* \in F(T)$. □

4 Attractive Point Theorems

In this section we prove the attractive point theorem for normally generalized hybrid mappings in a $CAT(0)$ space satisfying the $(\mathbb{S})$ property. First, we need the following lemma.

Lemma 4.1. *Let C be a nonempty subset of a $CAT(0)$ space X. Let $\{x_n\}$ be a bounded sequence in C. Let T be a mapping from C into itself such that $d(x_n, Tx_n) \to 0$. Then,*
(1) the sequences $\{d(x_n, y)\}$ and $\{d(Tx_n, y)\}$ are bounded for all $y \in C$;
(2) $\mu_n d(x_n, y) = \mu_n d(Tx_n, y)$ for any Banach limit μ on l^∞.

Proof. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$d(x_n, y) \leq d(x_n, x_1) + d(x_1, y).$$

Since $\{x_n\}$ is bounded, we see that the sequence $\{d(x_n, y)\}$ is also bounded. Note that

$$d(Tx_n, y) \leq d(Tx_n, x_n) + d(x_n, y).$$

Since $d(x_n, Tx_n) \to 0$ and $\{d(x_n, y)\}$ is bounded, we see that the sequence $\{d(Tx_n, y)\}$ is bounded. Next, we show that $\mu_n d(x_n, y) = \mu_n d(Tx_n, y)$ for all $y \in C$. By the triangle inequality, for any $y \in C$, we have

$$d(x_n, y) \leq d(x_n, Tx_n) + d(Tx_n, y) \tag{4.1}$$

and

$$d(Tx_n, y) \leq d(Tx_n, x_n) + d(x_n, y). \tag{4.2}$$

Since $d(x_n, Tx_n) \to 0$, by applying a Banach limit μ to both sides of (4.1) and (4.2), we see that

$$\mu_n d(x_n, y) = \mu_n d(Tx_n, y).$$

□

Now, we can prove the attractive point theorem for normally generalized hybrid mappings in a $CAT(0)$ space satisfying the $(\mathbb{S})$ property.

Theorem 4.2. *Let X be a complete $CAT(0)$ space which satisfies the $(\mathbb{S})$ property and let C a nonempty subset of X. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping with $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in C$. Then $A(T) \neq \emptyset$ if and only if there exists $x \in C$ such that the sequence $\{T^n x\}$ is bounded. Additionally, if C is closed and convex, then $F(T) \neq \emptyset$ if and only if there exists $x \in C$ such that the sequence $\{T^n x\}$ is bounded. Moreover, a fixed point is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).*

Proof. Suppose that $z \in A(T)$. Then $d(z, Tx) \leq d(z, y)$ for all $x \in C$. We see that

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, z) + d(z, T^m x) \\ &\leq d(z, x) + d(z, x) \\ &= 2d(z, x). \end{aligned}$$

Then $\{T^n x\}$ is bounded for some $x \in X$.

To prove the converse, suppose that there exists $x \in C$ such that the sequence $\{T^n x\}$ is bounded. For any $n \in \mathbb{N}$, put $x_n := T^n x$. By Lemma 2.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\triangle - \lim_{k\to\infty} x_{n_k} = z$ for some $z \in X$.

Claim 1. We will show that $\mu_k d(x_{n_k}, Ty) \leq \mu_k d(x_{n_k}, y)$, for all $y \in C$.

Since $0 = \lim_{n\to\infty} d(T^{n+1}x, T^n x) = \lim_{n\to\infty} d(Tx_n, x_n)$ and the sequence $\{x_n\}$ is bounded, $d(Tx_{n_k}, x_{n_k}) \to 0$ and the sequence $\{x_{n_k}\}$ is bounded. By Lemma 4.1, for all $y \in C$, we have
(1) the sequences $\{d(x_{n_k-1}, y)\}$ and $\{d(x_{n_k}, y)\}$ are bounded ;
(2) $\mu_n d(x_{n_k-1}, y) = \mu_n d(x_{n_k}, y)$ for any Banach limit μ on l^∞.

Since T is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping,

$$\alpha d^2(Tx, Ty) + \beta d^2(x, Ty) + \gamma d^2(Tx, y) + \delta d^2(x, y) \leq 0, \quad \text{for all } x, y \in C.$$

If $\alpha+\beta>0$, for all $x, y \in C$, we have

$$\alpha d^2(x_{n_k}, Ty) + \beta d^2(x_{n_k-1}, Ty) + \gamma d^2(x_{n_k}, y) + \delta d^2(x_{n_k-1}, y) \leq 0.$$

By applying a Banach limit μ, we have

$$(\alpha+\beta)\mu_k d^2(x_{n_k}, Ty) \leq -(\gamma+\delta)\mu_k d^2(x_{n_k}, y).$$

Since $\alpha+\beta+\gamma+\delta \geq 0$ and $\alpha+\beta>0$, we see that

$$\begin{aligned} \mu_k d^2(x_{n_k}, Ty) &\leq \frac{-(\gamma+\delta)}{\alpha+\beta}\mu_k d^2(x_{n_k}, y) \\ &\leq \mu_k d^2(x_{n_k}, y). \end{aligned}$$

We can obtain the similar result in the case of $\alpha+\gamma>0$. Therefore,

$$\mu_k d(x_{n_k}, Ty) \leq \mu_k d(x_{n_k}, y) \tag{4.3}$$

for all $y \in C$.

Claim 2. We will show that $z \in A(T)$.

Since X satisfies the $(\mathbb{S})$ property and $\triangle - \lim_{k\to\infty} x_{n_k} = z$, we have from Lemma 2.8 that

$$\lim_{k\to\infty}\left(d^2(x_{n_k}, z) - d^2(x_{n_k}, y) + d^2(z, y)\right) = 0, \quad \text{for all } y \in X.$$

Then, for all $y \in X$, we have

$$\mu_k\left(d^2(x_{n_k}, z) - d^2(x_{n_k}, y) + d^2(z, y)\right) = 0. \tag{4.4}$$

By (4.3), for all $y \in X$, we have

$$-\mu_k d(x_{n_k}, y) \leq -\mu_k d(x_{n_k}, Ty).$$

By adding $\mu_k\left(d^2(x_{n_k}, z) + d^2(x_{n_k}, y) + d^2(z, y)\right)$ to both sides of the above inequality, we get

$$\begin{aligned} \mu_k\left(d^2(x_{n_k}, z) + d^2(x_{n_k}, y) + d^2(z, y)\right) - \mu_k d^2(x_{n_k}, y) &\leq \mu_k\left(d^2(x_{n_k}, z) + d^2(x_{n_k}, y) + d^2(z, y)\right) \\ &\quad -\mu_k d^2(x_{n_k}, Ty) \\ d^2(z, Ty) + \mu_k\left(d^2(x_{n_k}, z) - d^2(x_{n_k}, y) + d^2(z, y)\right) &\leq d^2(z, y) + \\ &\quad \mu_k\left(d^2(x_{n_k}, z) - d^2(x_{n_k}, Ty) + d^2(z, Ty)\right). \end{aligned}$$

By (4.4), we have

$$\begin{aligned} d^2(z, Ty) &\leq d^2(z, y) \\ d(z, Ty) &\leq d(z, y). \end{aligned}$$

Therefore, $z \in A(T)$.

Additionally, suppose that C is a closed convex subset of X. By Proposition 3.4, $A(T) \neq \emptyset$ implies $F(T) \neq \emptyset$. Therefore, $F(T) \neq \emptyset$ if and only if $\{T^n x\}$ is bounded for some $x \in C$. Moreover, if $\alpha + \beta + \gamma + \delta + > 0$ and $p_1, p_2 \in F(T)$, then

$$\begin{aligned}
\alpha d^2(Tp_1, Tp_2) + \beta d^2(p_1, Tp_2) + \gamma d^2(Tp_1, p_2) + \delta d^2(p_1, p_2) &\leq 0 \\
\alpha d^2(p_1, p_2) + \beta d^2(p_1, p_2) + \gamma d^2(p_1, p_2) + \delta d^2(p_1, p_2) &\leq 0 \\
(\alpha + \beta + \gamma + \delta) d^2(p_1, p_2) &\leq 0 \\
d^2(p_1, p_2) &\leq 0.
\end{aligned}$$

This implies that $d(p_1, p_2) = 0$ and hence $p_1 = p_2$. This completes the proof. □

Acknowledgements : The authors would like to thank Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand. In particular, the first author is supported by the Development and Promotion of Science and Technology Talents Project (DPST).

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(Received 17 Septerber 2014)
(Accepted 11 February 2015)