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Some Properties of a Subclass of Harmonic Univalent Functions Defined by Fractional Calculus

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Abstract : The purpose of the present paper is to a systematic study of a new class of harmonic univalent functions defined by fractional calculus operator. Apart from coefficient bound, extreme points and distortion theorem, many interesting and useful properties of this class of functions is given, some of these properties concerning fractional calculus, convolution, partial sums and neighborhoods are also indicated. It is worth mentioning that results obtained in [1] appear to be particular cases are of our results. Many of our results are quite new and improves the results of previous authors and not found in the literature so far.

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1 Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply-connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|\,,\ z \in D$. See Clunie and Sheill-Small [2].

Let S_H denotes the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.1}$$

We note that for $g \equiv 0$, the class S_H reduces to the class S of analytic univalent functions for which f can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

Further, we denote the class A of functions of the form (1.2) which are analytic in the open unit disc U.

The following definitions of fractional derivatives are due to Owa [3], Owa and Srivastava [4], (see also [5]).

Definition 1.1. *The fractional derivative of order λ is defined for a function $f(z)$ of the form* (1.2), *by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{1.3}$$

where $0 \le \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ *to be real when* $(z-\zeta) > 0$.

Definition 1.2. *Under the hypothesis of Definition 1.1, the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ of the form* (1.2) *by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \tag{1.4}$$

where $0 \le \lambda < 1$ and $n \in N_0 = \{0, 1, 2, \ldots\}$.

Using these definitions Owa and Srivastava [4] introduced the operator $I^{\lambda} : A \to A$ by

$$I^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), \qquad (\lambda \neq 2, 3, 4, \ldots). \tag{1.5}$$

Now we define the operator I^{λ} for functions f of the form (1.1) as

$$I^{\lambda} f(z) = I^{\lambda} h(z) + \overline{I^{\lambda} g(z)}. \tag{1.6}$$

Recently, Dixit and Porwal [6], by using the Definition 1.1 and 1.2 introduce a new fractional derivative operator for function f of the form (1.2),

$$\Omega^0 f(z) = f(z)$$
$$\Omega^1 f(z) = \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Omega^n f(z) = \Omega(\Omega^{n-1} f(z)).$$

We note that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k, \tag{1.7}$$

where

$$\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.$$

It is interesting to note that for $\lambda = 0$, $\Omega^n f(z)$ reduces to familiar Salagean operator defined by Salagean in [7].

Now for $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n \in N$ and $z \in U$, suppose that $S_H(n, \lambda, \alpha)$ denote the family of harmonic univalent functions f of the form (1.1) such that

$$Re\left\{\frac{\Omega^n h(z) + \Omega^n g(z)}{z}\right\} > \alpha, \tag{1.8}$$

where $\Omega^n f(z)$ defined in (1.7).

Further, let the subclass $\overline{S_H}(n, \lambda, \alpha)$ consisting of harmonic functions $f = h + \overline{g}$ in $S_H(n, \lambda, \alpha)$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \text{ and } g(z) = -\sum_{k=1}^{\infty} |b_k| z^k. \tag{1.9}$$

The classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$ with $b_1 = 0$ will be denoted by $S_H^0(n, \lambda, \alpha)$ and $\overline{S_H^0}(n, \lambda, \alpha)$, respectievely. We note that $S_H(1, 0, \alpha) = HP(\alpha)$, $\overline{S_H}(1, 0, \alpha) = HP^*(\alpha)$ studied by Karpuzoğullari *et al.* in [1], (see also [8]) and for $n = 1, \lambda = 0$ with $g \equiv 0$ the class $S_H(n, \lambda, \alpha)$ reduce to the class $B(\alpha)$. The functions in $B(\alpha)$ are called functions of bounded turning (cf. [9]).

In the present paper, results involving the coefficient inequalities, extreme points, distortion bounds, fractional calculus, convolution, partial sums and neighborhood are determined for the classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$. It is worthy to note that our results not only generalizes the results of Karpuzoğullari *et al.* [1] but also some new results are investigated which are not explored in the literature so far.

2 Main Results

We first mention a sufficient condition for the function f of the form (1.1) belong to the class $S_H(n, \lambda, \alpha)$ given by the following result which can be established easily.

Theorem 2.1. *Let the function $f = h + \overline{g}$ be such that h and g are given by* (1.1). *Furthermore, let*

$$\sum_{k=2}^{\infty} [\phi(k,\lambda)]^n |a_k| + \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n |b_k| \leq 1 - \alpha, \tag{2.1}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n \in N$ *and* $\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}$. *Then* f *is harmonic univalent, sense-preserving in* U *and* $f \in S_H(n,\lambda,\alpha)$.

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f = h + \overline{g}$, where h and g are of the form (1.9).

Theorem 2.2. *Let* $f = h + \overline{g}$ *be given by* (1.9). *Then* $f \in \overline{S_H}(n,\lambda,\alpha)$, *if and only if*

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} |b_k| \leq 1, \tag{2.2}$$

where $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $n \in N$ *and* $\phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}$.

Proof. The if part follows from Theorem 2.1, so we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.9), we notice that the condition

$$Re\left\{\frac{\Omega^n h(z) + \Omega^n g(z)}{z}\right\} > \alpha$$

is equivalent to

$$Re\left\{1 - \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n |a_k| z^{k-1} - \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n |b_k| z^{k-1}\right\} > \alpha.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis and making $z \to 1^-$, we must have

$$1 - \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n |a_k| - \sum_{k=1}^{\infty} [\phi(k,\lambda)]^n |b_k| \geq \alpha$$

which is the required condition. □

Next, we determine the extreme points of closed convex hulls of $\overline{S_H}(n,\lambda,\alpha)$ denoted by clco $\overline{S_H}(n,\lambda,\alpha)$.

Theorem 2.3. *Let the functions* $f = h + \overline{g}$ *be given by* (1.9). *Then* $f \in \overline{S_H}(n,\lambda,\alpha)$, *if and only if*

$$f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)),$$

where $h_1(z) = z$, $h_k(z) = z - \frac{1-\alpha}{[\phi(k,\lambda)]^n} z^k$, $(k = 2,3,4....)$, $g_k(z) = z - \frac{1-\alpha}{[\phi(k,\lambda)]^n} \bar{z}^k$, $(k = 1,2,3....)$, $x_k \geq 0$, $y_k \geq 0$, $\sum_{k=1}^{\infty}(x_k + y_k) = 1$. *In particular, the extreme points of* $\overline{S_H}(n,\lambda,\alpha)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

The following theorem gives the bounds for functions in $\overline{S_H}(n,\lambda,\alpha)$ *which yields a covering result for this class.*

Theorem 2.4. *Let* $f \in \overline{S_H}(n, \lambda, \alpha)$. *Then for* $|z| = r < 1$ *we have*

$$|f(z)| \leq (1 + |b_1|)\, r + \left(\frac{1-\lambda}{2}\right)^n (1 - |b_1| - \alpha)\, r^2,$$

and

$$|f(z)| \geqslant (1 - |b_1|)\, r - \left(\frac{1-\lambda}{2}\right)^n (1 - |b_1| - \alpha)\, r^2.$$

Proof. The proofs of the above Theorems 2.3 and 2.4 are analogues to the corresponding similar Theorems proved in [1] and therefore we omit details involved. □

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.5. *Let* f *of the form* (1.9) *be such that* $f \in \overline{S_H}(n, \lambda, \alpha)$. *Then*

$$\left\{\omega : |\omega| < \frac{2^n - (1-\alpha)(1-\lambda)^n}{2^n} - |b_1| \left(1 - \left(\frac{1-\lambda}{2}\right)^n\right)\right\}.$$

Theorem 2.6. *Let the functions* $f = h + \overline{g}$ *be given by* (1.1) *with* $b_1 = 0$ *satisfies the inequality*

$$\sum_{k=2}^{\infty} \frac{k\,[\phi(k,\lambda)]^n}{1-\alpha} (|a_k| + |b_k|) \leq 2 - \mu, \tag{2.3}$$

for $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \mu < 1$ *and* $n \in N$, *then* $I^\mu f(z)$ *belongs to the class* $S_H(n, \lambda, \alpha)$.

Proof. Using the definition of fractional derivative, we have

$$I^\mu f(z) = I^\mu h(z) + \overline{I^\mu g(z)}, \tag{2.4}$$

$$= z + \sum_{k=2}^{\infty} kM(k,\mu)\, a_k z^k + \sum_{k=2}^{\infty} kM(k,\mu)\, \overline{b_k z^k},$$

where

$$M(k,\mu) = \frac{\Gamma k \Gamma(2-\mu)}{\Gamma(k+1-\mu)}, k \geq 2.$$

Since $M(k,\mu)$ is non-increasing on k, we see that

$$0 < M(k,\mu) \leq M(2,\mu) = \frac{1}{2-\mu}. \tag{2.5}$$

Therefore

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} M(k,\mu)\, k\, (|a_k| + |b_k|) \leq M(2,\mu) \sum_{k=2}^{\infty} \frac{k\,[\phi(k,\lambda)]^n}{1-\alpha} (|a_k| + |b_k|) \leq 1.$$

Hence applying Theorem 2.1, we have $I^\mu f(z) \in S_H(n, \lambda, \alpha)$.

Thus the proof of Theorem 2.6 is established. □

3 Application of the Hadamard Product

Let $f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k$, $(j = 1, 2)$, be analytic in U. Then the convolution (or Hadamard product) of $f_1(z)$ and $f_2(z)$, denoted by $(f_1 * f_2)(z)$, is defined as

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{3.1}$$

Similarly, the convolution of two harmonic functions is defined as:
Let $f_i(z)$ $(i = 1, 2)$ in S_H be given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k,i} z^k + \sum_{k=1}^{\infty} \overline{b_{k,i} z^k}. \tag{3.2}$$

Then the convolution $(f_1 * f_2)(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k + \sum_{k=1}^{\infty} \overline{b_{k,1} b_{k,2} z^k} \tag{3.3}$$

and the quasi-convolution for two harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \bar{z}^k \tag{3.4}$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \bar{z}^k, \tag{3.5}$$

is defined as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k - \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k. \tag{3.6}$$

In order to study some applications of convolution to the classes $S_H(n, \lambda, \alpha)$ and $\overline{S_H}(n, \lambda, \alpha)$, we shall require the following lemma due to Ruscheweyh and Sheil-Small [10], (see also [11]).

Lemma 3.1. *Let $\varphi(z)$ and $q(z)$ be analytic in U and satisfy the condition $\varphi(0) = q(0) = 0$, $\varphi'(0) = 1$, $q'(0) = 1$. Suppose that for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$ we have*

$$\varphi * \frac{1 + \rho\sigma z}{1 - \sigma z} q(z) \neq 0, (0 < |z| < 1).$$

Then for each function $F(z)$ analytic in U, satisfying
$Re\{F(z)\} > 0$, $(z \in U)$, we have

$$Re\left\{\frac{\varphi * Fq(z)}{\varphi * q(z)}\right\} > 0, \ (z \in U).$$

We now obtain the following result:

Theorem 3.2. *Let* $f \in \overline{S_H^0}(n,\lambda,\alpha)$*,* $\delta < 1$ *and*

$$\psi * \frac{1+\rho\sigma z}{1-\sigma z} f(z) \neq 0, \quad (0 < |z| < 1),$$

for each $\sigma\,(|\sigma| = 1)$ *and* $\rho\,(|\rho| = 1)$*, where*

$$\psi(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\,\Gamma(2-\delta)}{\Gamma(k+1-\delta)} z^k.$$

Then $I^\delta f(z)$ *is also in the class* $\overline{S_H^0}(n,\lambda,\alpha)$*.*

Proof. From definition of $I^\delta f(z)$, we have

$$I^\delta f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\,\Gamma(2-\delta)}{\Gamma(k+1-\delta)} |a_k| z^k - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\,\Gamma(2-\delta)}{\Gamma(k+1-\delta)} |b_k| \bar{z}^k.$$

$$= \psi(z) * h(z) + \overline{\psi(z) * g(z)}.$$

Letting $\varphi(z) = \psi(z)$, $q(z) = z$, $F(z) = \frac{\Omega^n h(z) + \Omega^n g(z)}{z} - \alpha$ in Lemma 3.1, we see that

$$Re\left\{\frac{\varphi * Fq(z)}{\varphi * q(z)}\right\} = Re\left\{\frac{\psi * \left(\Omega^n h(z) + \Omega^n g(z)/_z - \alpha\right) z}{\psi * z}\right\}$$

$$= Re\left\{\frac{\Omega^n \psi * h(z) + \Omega^n \psi * g(z)}{z}\right\} - \alpha$$

$$= Re\left\{\frac{\Omega^n I^\delta h(z) + \Omega^n I^\delta g(z)}{z}\right\} - \alpha$$

$$> 0,$$

which implies $I^\delta f \in \overline{S_H^0}(n,\lambda,\alpha)$. □

Several authors such as ([8], [12], [13], [1] and [14]) studied the convolution properties for the functions with negative as well as positive coefficients only. Their result do not say anything for the function of the form (1.1). It is therefore natural to ask whether their results can be improved for function of the form (1.1). In our next theorem, we establish a result on convolution which improves the results of previous authors ([8], [12], [13], [1] and [14]) to the case when f is of the form (1.1). It is worth mentioning that the technique employed by us is entirely different from the previous authors.

For this, we shall require the following definition and lemmas.

Definition 3.3. *A sequence* $\{c_k\}_0^\infty$ *of non-negative numbers is said to be a convex null sequence if* $c_k \to 0$ *as* $k \to \infty$ *and*

$$c_0 - c_1 \geq c_1 - c_2 \geq ... \geq c_k - c_{k+1} \geq ... \geq 0.$$

Lemma 3.4. *Let* $\{c_k\}_0^\infty$ *be a convex null sequence. Then the function*

$$q_1(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$$

is analytic in U *and* $Re\ q_1(z) > 0$, $z \in U$.

Lemma 3.5. *Let* $P(z)$ *be analytic in* U, $P(0) = 1$ *and* $Re\{P(z)\} > \frac{1}{2}$ *in* U. *For functions* F *analytic in* U, *the convolution function* $P * F$ *takes values in the convex hull of the image on* U *under* F.

Lemmas 3.4 is due to Fejěr [15]. The assertion of Lemma 3.5 readily follows by using the Herglotz representation for $P(z)$.

Lemma 3.6. *Let* $f(z) \in S_H^0(n, \lambda, \alpha)$. *Then*

$$Re\left\{\frac{h(z) + g(z)}{z}\right\} > \frac{1}{2}, \quad z \in U.$$

Proof. Let $f(z)$ be given by (1.1) with $b_1 = 0$. Since $f(z) \in S_H^0(n, \lambda, \alpha)$, hence by definition

$$Re\left\{\frac{\Omega^n h(z) + \Omega^n g(z)}{z}\right\} > \alpha, \quad z \in U,$$

which is equivalent to

$$\text{Re}\left\{1 - \alpha + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^{k-1} + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n b_k z^{k-1}\right\} > 0,$$

and hence

$$\text{Re}\left\{1 + \frac{1}{2}\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} (a_k + b_k) z^{k-1}\right\} > \frac{1}{2}. \tag{3.7}$$

We observe that the sequence $\{c_k\}_0^\infty$ defined by $c_0 = 1$, $c_k = \frac{2(1-\alpha)}{[\phi(k+1,\lambda)]^n}$, $k \geq 1$, is a convex null sequence, we have in view of Lemma 3.4

$$\text{Re}\left\{1 + 2\sum_{k=2}^{\infty} \frac{1-\alpha}{[\phi(k,\lambda)]^n} z^{k-1}\right\} > \frac{1}{2}. \tag{3.8}$$

Now

$$\frac{h(z) + g(z)}{z} = \left[1 + \frac{1}{2}\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} (a_k + b_k) z^{k-1}\right] * \left[1 + 2\sum_{k=2}^{\infty} \frac{1-\alpha}{[\phi(k,\lambda)]^n} z^{k-1}\right]$$

and making use of (3.7), (3.8) and Lemma 3.5, we conclude that

$$Re\left\{\frac{h(z)+g(z)}{z}\right\} > \frac{1}{2}.$$

□

Theorem 3.7. *If the functions $f_i(z)$ $(i = 1,2)$ defined by* (3.2) *with $b_{1,i} = 0$ $(i = 1,2)$ are in the classes $S_H(n,\lambda,\alpha_i)$, where $0 \le \alpha_2 \le \alpha_1 < 1$, and satisfy the condition $a_{k,1}b_{k,2} + a_{k,2}b_{k,1} = 0$, $\quad (k = 2,3,\ldots)$, then*

$$P(z) = H(z) + \overline{G(z)} = (f_1 * f_2)(z) \in S_H(n,\lambda,\alpha_1).$$

Proof. To prove that $P(z) \in S_H(n,\lambda,\alpha_1)$ we have to show that

$$\mathrm{Re}\left\{\frac{\Omega^n H(z) + \Omega^n G(z)}{z}\right\} > \alpha_1,$$

which is equivalent to

$$\mathrm{Re}\left\{1 + \frac{1}{2}\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{1-\alpha_1}a_{k,1}a_{k,2}z^{k-1} + \frac{1}{2}\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{1-\alpha_1}b_{k,1}b_{k,2}z^{k-1}\right\} > \frac{1}{2}. \quad (3.9)$$

Since $f_1(z) \in S_H(n,\lambda,\alpha_1)$ from (3.7) we have

$$\mathrm{Re}\left\{1 + \frac{1}{2}\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{1-\alpha_1}a_{k,1}z^{k-1} + \frac{1}{2}\sum_{k=2}^{\infty}\frac{[\phi(k,\lambda)]^n}{1-\alpha_1}b_{k,1}z^{k-1}\right\} > \frac{1}{2}. \quad (3.10)$$

and since $f_2(z) \in S_H(n,\lambda,\alpha_2)$, from Lemma 3.6 we have

$$Re\left\{1 + \sum_{k=2}^{\infty}a_{k,2}z^{k-1} + \sum_{k=2}^{\infty}b_{k,2}z^{k-1}\right\} > \frac{1}{2}. \quad (3.11)$$

From (3.10), (3.11) and Lemma 3.5 we immediately have (3.9).

This establishes the proof of the Theorem 3.7.

□

Motivated with the work of Kumar [16], Porwal *et al.* [17], we improve the result of Theorem 3.7 for functions of the form (1.9). For this we shall require the following lemma which can be established easily.

Lemma 3.8. *$\overline{S_H}(m,\lambda,\beta) \subseteq \overline{S_H}(n,\lambda,\alpha)$, if $0 \le \alpha \le \beta < 1$ and $m \ge n$.*

Theorem 3.9. *Let the functions $f(z)$, $F(z)$ defined by* (3.4), (3.5) *are in the classes $\overline{S_H}(m,\lambda,\beta)$, $\overline{S_H}(n,\lambda,\alpha)$, respectively, where $m,n \in N$, $0 \le \beta < 1$, $0 \le \alpha < 1$, then $f * F$ defined by* (3.6) *is in the class $\overline{S_H}(m+n,\lambda,\eta)$, where $\eta = \alpha+\beta-\alpha\beta$.*

Proof. Since $f(z) \in \overline{S_H}(m, \lambda, \beta)$, then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^m}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^m}{1-\beta} |b_k| \le 1. \tag{3.12}$$

Similarly $F(z) \in \overline{S_H}(n, \lambda, \alpha)$, we have

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} |A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} |B_k| \le 1. \tag{3.13}$$

Therefore

$$\frac{[\phi(k,\lambda)]^n}{1-\alpha} |A_k| \le 1, \quad k = 2, 3, ... \tag{3.14}$$

and

$$\frac{[\phi(k,\lambda)]^n}{1-\alpha} |B_k| \le 1, \quad k = 1, 2, 3, ... \tag{3.15}$$

Now, for the convolution function $f * F$ we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^{m+n}}{1-\eta} |a_k A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^{m+n}}{1-\eta} |b_k B_k| \\
&\le \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^m}{1-\beta} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^m}{1-\beta} |b_k|, \quad (\text{using}(3.14)\text{and}(3.15)) \\
&\le 1, \quad (\text{using}(3.12)).
\end{aligned}$$

Thus the proof of Theorem 3.9 is established. □

Remark 3.10. *From Lemma 3.8, it is easy to see that*

$$\overline{S_H}(m+n, \lambda, \eta) \subseteq \overline{S_H}(m, \lambda, \beta)$$

and

$$\overline{S_H}(m+n, \lambda, \eta) \subseteq \overline{S_H}(n, \lambda, \alpha).$$

Thus the result of Theorem 3.9 provides smaller class in comparison to the class given by Theorem 3.7.

Theorem 3.11. *Let the functions $f_i(z)$ defined as*

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{k,i}| z^k - \sum_{k=1}^{\infty} |b_{k,i}| \overline{z^k},$$

belong to the class $\overline{S_H}(n_i, \lambda, \alpha_i)$ for every $i = 1, 2, ..., q$, then the convolution $f_1 * f_2 * ...f_q$ belongs to the class $\overline{S_H}(\sum_{i=1}^{q} n_i, \lambda, \epsilon)$, where $\epsilon = 1 - \prod_{i=1}^{q}(1-\alpha_i)$.

Proof. The proof of the above theorem is much akin that of Theorem 3.9. Hence we omit the details involved. □

In our next result, we study mapping properties of the integral operator $J_c(f(z))$ on the class $\overline{S_H}(n,\lambda,\alpha)$ in which we show that $J_c(f(z)) \in \overline{S_H}(n,\lambda,\beta)$ if $f(z) \in \overline{S_H}(n,\lambda,\alpha)$, the result is sharp.

For this purpose, we define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k, \qquad (c \neq 0,-1,-2,...), \tag{3.16}$$

where $(a)_k$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$$

$$= \begin{cases} 1 & (k=0) \\ a(a+1)(a+2)......(a+k-1), & (k \in N = \{1,2,3...\}). \end{cases}$$

The function $\phi(a,c;z)$ is an incomplete function related to the Gauss hypergeometric function by

$$\phi(a,c;z) = zF(1,a;c;z). \tag{3.17}$$

Carlson and Shaffer [18] defined a linear operator $L(a,c)$, corresponding to the function $\phi(a,c;z)$ on A via the convolution as:

$$L(a,c)h(z) = \phi(a,c;z) * h(z), \qquad (h(z) \in A). \tag{3.18}$$

If $c > a > 0$, $L(a,c)$ has the integral representation

$$L(a,c)h(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} h(uz)du. \tag{3.19}$$

Clearly, $L(a,a)$ is the identity operator and

$$L(a,c) = L(a,b).L(b,c) = L(b,c).L(a,b), \qquad (b,c \neq 0,-1,-2,...).$$

Moreover if $a \neq 0,-1,-2,...$, then $L(a,c)$ has an inverse $L(c,a)$ and is a one-one mapping of A onto itself, (see Owa and Srivastava [4]).

Bernardi [19] defined the integral operator J_c by

$$J_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \qquad (c > -1)$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k$$

$$= L(c+1,c+2) f(z) \tag{3.20}$$

or

$$J_c(f) = \phi(c+1,c+2;z) * f(z). \tag{3.21}$$

Now, we define the Bernardi integral operator $J_c(f)$ on the class S_H of harmonic univalent functions of the form (1.1) as follows:

$$J_c(f) = J_c(h) + \overline{J_c(g)}$$

$$= z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k + \overline{\sum_{k=1}^{\infty} \frac{c+1}{c+k} b_k z^k} \tag{3.22}$$

$$= L(c+1, c+2)h(z) + \overline{L(c+1, c+2)g(z)} \tag{3.23}$$

$$= \phi(c+1, c+2; z) * h(z) + \overline{\phi(c+1, c+2; z) * g(z)}. \tag{3.24}$$

Theorem 3.12. *If the function $f(z)$ defined by (1.9) with $b_1 = 0$ is in the class $\overline{S_H}(n, \lambda, \alpha)$. Then $J_c(f)$ defined by (3.22) is in the class $\overline{S_H}(n, \lambda, \beta)$, where*

$$\beta = (2\alpha - 1) + 2(1-\alpha)(c+1) \sum_{k=1}^{\infty} \frac{(-1)^k}{c+k+1}. \tag{3.25}$$

The result is sharp.

Proof. By using (3.24), we have

$$\Omega^n J_c(h(z)) + \Omega^n J_c(g(z)) = \phi(c+1, c+2; z) * \{\Omega^n h(z) + \Omega^n g(z)\}.$$

A simple calculation shows that

$$\frac{\Omega^n J_c(h(z)) + \Omega^n J_c(g(z))}{z} = \frac{1}{z}[L(c+1, c+2)(\Omega^n h(z) + \Omega^n g(z))].$$

Using (3.19), we obtain that

$$Re\left\{\frac{\Omega^n J_c(h) + \Omega^n J_c(g)}{z}\right\} = (c+1) \int_0^1 u^c Re\left\{\frac{\Omega^n h(zu) + \Omega^n g(zu)}{zu}\right\} du. \tag{3.26}$$

Since $f(z) \in \overline{S_H}(n, \lambda, \alpha)$, we put

$$\frac{\Omega^n h(z) + \Omega^n g(z)}{z} = G(z),$$

then $G(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U and $Re\{G(z)\} > \alpha$. It is known that [20] if $q(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is analytic in U and $Re\{q(z)\} > \gamma$, $(0 \leq \gamma < 1)$, then

$$Re\{q(z)\} \geq \frac{1 + (2\gamma - 1)r}{1+r}, \quad (|z| = r < 1). \tag{3.27}$$

Hence by using (3.26) and (3.27), we have

$$Re\left\{\frac{\Omega^n J_c(h) + \Omega^n J_c(g)}{z}\right\} \geq (c+1) \int_0^1 u^c \frac{1 + (2\alpha - 1)u}{1+u} du$$

$$= (2\alpha - 1) + 2(1-\alpha)(c+1)\int_0^1 \frac{u^c}{1+u}du$$

$$= (2\alpha - 1) + 2(1-\alpha)(c+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{c+k+1},$$

that is $J_c(f(z)) \in \overline{S_H}(n,\lambda,\beta)$, where β is defined by (3.25).

Further to show that the result is sharp, we consider the function $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are connected by the relation

$$\Omega^n h(z) + \Omega^n g(z) = \frac{z + (1-2\alpha)z^2}{1-z}. \tag{3.28}$$

□

4 Partial Sums of the Libera Integral Operator

For $f(z)$ of the form (1.2), the Libera integral operator $F(z)$ is given by

$$F(z) = \frac{2}{z}\int_0^z f(\varsigma)\,d\varsigma = z + \sum_{k=2}^{\infty}\frac{2}{k+1}a_k z^k. \tag{4.1}$$

For $f = h + \overline{g}$ in S_H, where h and g are given by (1.1), the Libera integral operator led us to define integral operator given by

$$F(z) = \frac{2}{z}\int_0^z h(\varsigma)\,d\varsigma + \overline{\frac{2}{z}\int_0^z g(\varsigma)\,d\varsigma} = z + \sum_{k=2}^{\infty}\frac{2}{k+1}a_k z^k + \sum_{k=1}^{\infty}\overline{\frac{2}{k+1}b_k z^k}. \tag{4.2}$$

The jth partial sums $F_j(z)$ of the integral operator $F(z)$ for functions f of the form (1.1) are given by

$$F_j(z) = z + \sum_{k=2}^{j}\frac{2}{k+1}a_k z^k + \sum_{k=1}^{j}\overline{\frac{2}{k+1}b_k z^k}. \tag{4.3}$$
$$= H_j(z) + \overline{G_j(z)}.$$

The jth partial sums $F_j(z)$ of the Libera integral operator $F(z)$ for analytic univalent functions of the form (1.2) have been studied by various authors in ([21], [22]), see also ([23]). Very recently, motivated with the work of Jahangiri and Farahmand [21], Porwal and Dixit [24] and Porwal *et al.* [25] studied the analogues results on harmonic univalent functions. Here we give a systematically study on the partial sums of harmonic univalent functions for the classes $S_H(n,\lambda,\alpha)$ and $\overline{S_H}(n,\lambda,\alpha)$.

To derive our first main result of this section, we shall require the following lemma which is due to Jahangiri and Farahmand [21].

Lemma 4.1. *For $z \in U$,*

$$Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right) > -\frac{1}{3}. \tag{4.4}$$

Theorem 4.2. *If f of the form (1.1) with $b_1 = 0$ and $f \in S_H(n, \lambda, \alpha)$, then $F_j \in S_H\left(n, \lambda, \frac{4\alpha-1}{3}\right)$, for $\frac{1}{4} \le \alpha < 1$.*

Proof. Let f be of the form (1.1) and belong to $S_H(n, \lambda, \alpha)$ for $\frac{1}{4} \le \alpha < 1$.

Since

$$Re\left\{\frac{\Omega^n h(z) + \Omega^n g(z)}{z}\right\} > \alpha,$$

we have

$$Re\left\{1 + \frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^{k-1} + \sum_{k=2}^{\infty}[\phi(k,\lambda)]^n b_k z^{k-1}\right)\right\} > \frac{1}{2}. \tag{4.5}$$

Applying the convolution properties of power series to $\frac{\Omega^n H_j(z)+\Omega^n G_j(z)}{z}$, we may write

$$\begin{aligned}\frac{\Omega^n H_j(z) + \Omega^n G_j(z)}{z} &= 1 + \sum_{k=2}^{j}\frac{2[\phi(k,\lambda)]^n}{k+1}a_k z^{k-1} + \sum_{k=2}^{j}\frac{2[\phi(k,\lambda)]^n}{k+1}b_k z^{k-1}\\ &= \left(1 + \frac{1}{2(1-\alpha)}\left(\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n(a_k+b_k)z^{k-1}\right)\right) *\\ &\quad \left(1 + (1-\alpha)\sum_{k=2}^{j}\frac{4}{k+1}z^{k-1}\right) \qquad (4.6)\\ &= P(z) * Q(z).\end{aligned}$$

From Lemma 4.1 for $m = j-1$, we obtain

$$Re\left(\sum_{k=2}^{j}\frac{z^{k-1}}{k+1}\right) > -\frac{1}{3}. \tag{4.7}$$

By applying a simple algebra to inequality (4.7) and $Q(z)$ in (4.6), one may obtain

$$Re(Q(z)) = Re\left\{1 + (1-\alpha)\sum_{k=2}^{j}\frac{4}{k+1}z^{k-1}\right\} > \frac{4\alpha-1}{3}.$$

On the other hand, the power series $P(z)$ in (4.6) in conjunction with the condition (4.5) yields

$$Re(P(z)) > \frac{1}{2}.$$

Therefore, by Lemma 3.5, $Re\left\{\frac{\Omega^n H_j(z)+\Omega^n G_j(z)}{z}\right\} > \frac{4\alpha-1}{3}$.

This completes the proof of Theorem 4.2. □

Remark 4.3. *If we put $n = 1, \lambda = 0$ in in Theorem 4.2 then we obtain the corresponding result of Porwal and Dixit [24].*

Next, if f of the form (1.2) with $n = 1, \lambda = 0$ in Theorem 4.2, we obtain the corresponding result of Jahangiri and Farahmand in [21].

Theorem 4.4. *Let f be of the form (1.9) with $b_1 = 0$ and $f \in \overline{S_H}(n, \lambda, \alpha)$, then the functions $F(z)$ defined by (4.2) belongs to $\overline{S_H}(n, \lambda, \rho)$, where $\rho = \frac{1+2\alpha}{3}$. The result is sharp. Further, the converse need not to be true.*

Proof. Since $f \in \overline{S_H}(n, \lambda, \alpha)$, Theorem 2.2 ensures that

$$\sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} (|a_k| + |b_k|) \leq 1. \tag{4.8}$$

Also, from (4.2) we have

$$F(z) = z - \sum_{k=2}^{\infty} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1} |b_k| \bar{z}^k.$$

Let $F(z) \in \overline{S_H}(n, \lambda, \sigma)$, then, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \left(\frac{[\phi(k,\lambda)]^n}{1-\sigma} \right) \left(\frac{2}{k+1} |a_k| + \frac{2}{k+1} |b_k| \right) \leq 1.$$

Thus we have to find largest value of σ so that the above inequality holds. Now this inequality holds if

$$\sum_{k=2}^{\infty} \left(\frac{[\phi(k,\lambda)]^n}{1-\sigma} \right) \left(\frac{2}{k+1} |a_k| + \frac{2}{k+1} |b_k| \right) \leq \sum_{k=2}^{\infty} \frac{[\phi(k,\lambda)]^n}{1-\alpha} (|a_k| + |b_k|).$$

or, if

$$\left(\frac{[\phi(k,\lambda)]^n}{1-\sigma} \right) \frac{2}{k+1} \leq \frac{[\phi(k,\lambda)]^n}{1-\alpha}, \qquad \text{for each } \ k = 2, 3, 4......$$

which is equivalent to

$$\sigma \leq \frac{k-1+2\alpha}{k+1} = \rho_k, \qquad k = 2, 3, 4.......$$

It is easy to verify that ρ_k is an increasing function of k. Therefore, $\rho = \inf_{k \geq 2} \rho_k = \rho_2$ and, hence

$$\rho = \frac{1+2\alpha}{3}.$$

To show the sharpness, we take the function $f(z)$ given by

$$f(z) = z - (1-\alpha) \left(\frac{1-\lambda}{2} \right)^n |x| z^2 - (1-\alpha) \left(\frac{1-\lambda}{2} \right)^n |y| \bar{z}^2, \text{ where } |x| + |y| = 1.$$

Then

$$F(z) = z - \frac{(1-\alpha)(1-\lambda)^n}{3.2^{n-1}} |x| z^2 - \frac{(1-\alpha)(1-\lambda)^n}{3.2^{n-1}} |y| \bar{z}^2$$
$$= H(z) + \overline{G(z)}$$

and therefore

$$\frac{\Omega^n(z) + \Omega^n G(z)}{z} = 1 - \frac{2(1-\alpha)}{3} |x| z - \frac{2(1-\alpha)}{3} |y| z$$
$$= \frac{3 - 2(1-\alpha)(|x|+|y|) z}{3}$$
$$= \frac{1+2\alpha}{3}, \quad \text{for } z \to 1.$$

Hence, the result is sharp.

We now show that the converse of above theorem need not to be true. To this end, we consider the function

$$F(z) = z - (1-\sigma)\left(\frac{(2-\lambda)(1-\lambda)}{6}\right)^n |x| z^3 - (1-\sigma)\left(\frac{(2-\lambda)(1-\lambda)}{6}\right)^n |y| \bar{z}^3,$$

where

$$|x| + |y| = 1, \qquad \sigma = \frac{2\alpha+1}{3}.$$

Theorem 2.2 guarantees that $F(z) \in \overline{S_H}(n, \lambda, \sigma)$.

But the corresponding function

$$f(z) = z - 2(1-\sigma)\left(\frac{(2-\lambda)(1-\lambda)}{6}\right)^n |x| z^3 - 2(1-\sigma)\left(\frac{(2-\lambda)(1-\lambda)}{6}\right)^n |y| \bar{z}^3,$$

does not belong to $\overline{S_H}(n, \lambda, \alpha)$, since, for this the function $f(z)$ does not satisfy the coefficient inequality of Theorem 2.2. □

In our next theorem, we improve the result of Theorem 4.2 for functions f of the form (1.9).

Theorem 4.5. *Let f of the form* (1.9) *with $b_1 = 0$ and $f \in \overline{S_H^0}(n, \lambda, \alpha)$. Then the function $F_j(z)$ defined by* (4.3) *belong to* $\overline{S_H^0}\left(n, \lambda, \frac{2\alpha+1}{3}\right)$.

Proof. Since

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=2}^{\infty} |b_k| \bar{z}^k.$$

Then

$$F(z) = z - \sum_{k=2}^{\infty} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{\infty} \frac{2}{k+1} |b_k| \bar{z}^k.$$

By using Theorem 4.4, we have

$$F(z) \in \overline{S_H^0}(n, \lambda, \sigma), \quad \text{where } \sigma = \frac{2\alpha + 1}{3}.$$

Now

$$F_j(z) = z - \sum_{k=2}^{j} \frac{2}{k+1} |a_k| z^k - \sum_{k=2}^{j} \frac{2}{k+1} |b_k| \bar{z}^k.$$

To show that $F_j(z) \in \overline{S_H}(n, \lambda, \sigma)$, we have

$$\begin{aligned} & \sum_{k=2}^{j} \left(\frac{[\phi(k,\lambda)]^n}{1-\sigma} \right) \left(\frac{2}{k+1} |a_k| + \frac{2}{k+1} |b_k| \right) \\ \leq & \sum_{k=2}^{\infty} \left(\frac{[\phi(k,\lambda)]^n}{1-\sigma} \right) \left(\frac{2}{k+1} |a_k| + \frac{2}{k+1} |b_k| \right) \\ \leq & 1. \end{aligned}$$

Thus $F_j(z) \in \overline{S_H}(n, \lambda, \sigma)$.

□

In the following corollary, we improve a result of Jahangiri and Farahmand in [21] when f has form $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, for this we need the following Lemma.

Lemma 4.6. *If* $0 \leq \alpha_1 \leq \alpha_2 < 1$ *, then*

$$B(\alpha_2) \subseteq B(\alpha_1).$$

Proof. The proof of the above lemma is straightforward, so we omit the details. □

If we put $n = 1, \lambda = 0, g = 0$ in Theorem 4.5 then we obtain the following

Corollary 4.7. *Let* $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$ *. If* $f(z) \in B(\alpha)$ *, then* $F_j(z) = z - \sum_{k=2}^{j} \frac{2}{k+1} |a_k| z^k$ *belongs to*

$$B\left(\frac{2\alpha+1}{3}\right).$$

Remark 4.8. *For* $\frac{1}{4} \leq \alpha < 1$ *,* $f(z) \in B(\alpha)$ *Jahangiri and Farahmand [21] shows that* $F_j(z) \in B\left(\frac{4\alpha-1}{3}\right)$ *and our result states that* $F_j(z) \in B\left(\frac{2\alpha+1}{3}\right)$*.*

Since $\frac{2\alpha+1}{3} > \frac{4\alpha-1}{3}$ *, for* $\frac{1}{4} \leq \alpha < 1$ *, and using Lemma 4.6, we have*

$$B\left(\frac{2\alpha+1}{3}\right) \subset B\left(\frac{4\alpha-1}{3}\right).$$

Hence our result provides a smaller class in comparison to the class given by Jahangiri and Farahmand [21].

5 Inclusion Relationship Involving Neighborhoods

In this section, we study some neighborhood properties for the class $\overline{S_H}(n,\lambda,\alpha)$. The earlier investigation were done by Altintas and Owa [26], Altintas *et al.* [27] and Ruscheweyh [28], (see also [29], [1]). Now we define the δ- neighborhood of f is the set

$$N_\delta(f)=\left\{F_1:\,F_1\left(z\right)=z-\sum_{k=2}^{\infty}\left|A_k\right|z^k-\sum_{k=1}^{\infty}\left|B_k\right|\bar{z}^k \text{ and } \sum_{k=1}^{\infty}k(|a_k-A_k|+|b_k-B_k|)\leq\delta\right\}. \tag{5.1}$$

Theorem 5.1. *Let $f\in\overline{S_H}\,(n,\lambda,\alpha)$. If $\delta\leqslant(1-\alpha-|b_1|)(1-\left(\frac{1-\lambda}{2}\right)^{n-1})$, then $N_\delta(f)\subset HP^*(\alpha)$.*

Proof. Let

$$F_1\left(z\right)=z-\sum_{k=2}^{\infty}\left|A_k\right|z^k-\sum_{k=1}^{\infty}\left|B_k\right|\bar{z}^k \tag{5.2}$$

belongs to $N_\delta(f)$.

Now

$$\begin{aligned}
&|B_1|+\sum_{k=2}^{\infty}k(|A_k|+|B_k|)\\
&\leq|B_1-b_1|+|b_1|+\sum_{k=2}^{\infty}k(|A_k-a_k|+|B_k-b_k|)+\sum_{k=2}^{\infty}k(|a_k|+|b_k|)\\
&\leq\delta+|b_1|+\left(\frac{1-\lambda}{2}\right)^{n-1}\sum_{k=2}^{\infty}\left[\phi(k,\lambda)\right]^n(|a_k|+|b_k|)\\
&\leq\delta+|b_1|+\left(\frac{1-\lambda}{2}\right)^{n-1}(1-\alpha-|b_1|)\leq 1-\alpha,
\end{aligned}$$

if $\delta\leqslant(1-\alpha-|b_1|)(1-\left(\frac{1-\lambda}{2}\right)^{n-1})$.

Thus $F_1(z)\in HP^*\left(\alpha\right)$. □

Theorem 5.2. *Let $f\in\overline{S_H}\,(n,\lambda,\alpha)$. If $\delta\leqslant(1-\alpha-|b_1|)\left(1-\frac{1}{(2-\mu)}\left(\frac{1-\lambda}{2}\right)^{n-2}\right)$, then $N_\delta(I^\mu f)\subset HP^*\left(\alpha\right)$.*

Proof. Let

$$F_1\left(z\right)=z-\sum_{k=2}^{\infty}\left|A_k\right|z^k-\sum_{k=1}^{\infty}\left|B_k\right|\bar{z}^k \tag{5.3}$$

belongs to $N_\delta(I^\mu f)$.

Now

$$|B_1|+\sum_{k=2}^{\infty}k(|A_k|+|B_k|)$$

$$\leq |B_1 - b_1| + |b_1| + \sum_{k=2}^{\infty} k(|A_k - kM(k,\mu)a_k| + |B_k - kM(k,\mu)b_k|)$$

$$+ \sum_{k=2}^{\infty} k(M(k,\mu)k\,|a_k| + M(k,\mu)k\,|b_k|)$$

$$\leq \delta + |b_1| + \frac{1}{(2-\mu)}\left(\frac{1-\lambda}{2}\right)^{n-2}\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n\,(|a_k| + |b_k|)$$

$$\leq \delta + |b_1| + \frac{1}{(2-\mu)}\left(\frac{1-\lambda}{2}\right)^{n-2}(1-\alpha-|b_1|) \leq 1-\alpha,$$

if $\delta \leqslant (1-\alpha-|b_1|)\left(1 - \frac{1}{(2-\mu)}\left(\frac{1-\lambda}{2}\right)^{n-2}\right)$.

Thus $F_1(z) \in HP^*(\alpha)$. □

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