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An Iterative Shrinking Metric f-Projection Method for Finding a Common Fixed Point of Two Quasi Strict f-Pseudo-contractions and Applications in Hilbert Spaces

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Abstract : In this paper, we establish the significant inequality related to quasi strict f-pseudo contractions in the framework of Hilbert spaces. By using the ideas of metric f-projection, we propose an iterative shrinking metric f-projection method for finding a common fixed point of two quasi strict f- pseudo contractions. Moreover, we also provide some applications of the main theorem as well as other related results.

Keywords : Quasi strict f-pseudo-contraction; Metric f-projection; Common fixed point; Hybrid algorithm; Hilbert space.
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1 Introduction

Metric projection operators in Hilbert spaces and Banach spaces play an important role in several fields of mathematics such as functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality and complementarity problem. (see, for example, [1, 2]). In 1994, Alber [3] proposed the generalized projections from Hilbert spaces to uniformly

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convex and uniformly smooth Banach spaces. Moreover, Alber [1] presented some applications of the generalized projections to approximately solving variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [2] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [4] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by [3] and proved some properties of the generalized f-projection operator. Fan et al. [5] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

In this paper, unless otherwise specified, I stands for an identity mapping. The mapping T is said to be a strict pseudo-contraction if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in D(T).$$

In this case, T may be called as k-strict pseudo-contraction. We use $F(T)$ to denote the set of fixed point of T (i.e. $F(T) = \{x \in D(T) : Tx = x\}$). T is said to be a quasi-strict pseudo-contraction if the set of fixed point $F(T)$ is nonempty and if there exists a constant $0 \le k < 1$ such that

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2, \ \forall x \in D(T) \text{ and } p \in F(T).$$

There are several attempts to establish an iteration method to find a fixed point of some well-known nonlinear mappings, for instant, nonexpansive mapping. We note that Mann's iterations [6] have only weak convergence even in a Hilbert space (see e.g., [7]). Nakajo and Takahashi [8] modified the Mann iteration method so that strong convergence is guaranteed, later well known as a hybrid projection method. Since then, the hybrid method has received rapid developments. For the details, the readers are referred to papers [9, 12, 20, 22, 21, 19, 17, 16, 15, 18, 13, 11, 14, 10] and the references therein.

In 2007, Takahashi et al. [18] studied a strong convergence theorem by the hybrid method for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1}x_0$, and let

$$\begin{cases} y_n &= \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)}x_0$.

In 2011, Saewan and Kumam [24] introduced a new hybrid projection method based on modified Mann iterative scheme by the generalized f-projection operator for a countable family of relatively quasi-nonexpansive mappings and the solutions of the system of generalized mixed equilibrium problems. Later, they [25] also studied the new hybrid Ishikawa iteration process by the generalized f-projection operator for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space.

Recently, Li et al. [27] studied the following hybrid iterative scheme for a relatively nonexpansive mapping by using the generalized f-projection operator in Banach spaces as follows:

$$\begin{cases} x_0 \in C, \ C_0 = C, \\ y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) JTx_n\right), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \le G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \ge 1. \end{cases}$$

Under some appropriate assumptions, they obtained strong convergence theorems in Banach spaces.

Motivated and inspired by the work mentioned above, in this paper, we establish the significant inequality related to quasi strict f-pseudo contractions in the framework of Hilbert spaces. By using the ideas of metric f-projection, we propose an iterative shrinking metric f-projection method for finding a common fixed point of two quasi strict f- pseudo contractions. Moreover, we also provide some applications of the main theorem as well as other related results.

2 Preliminaries

In this section, we provide some definitions and some relevant lemmas which are useful to prove in the next section. Most of them are known others are not hard to find and understand the proof. Throughout this paper, we will use the notation $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Lemma 2.1 (Takahashi [28]). *Let $\{a_n\}$ be a sequence of real numbers. Then, $\lim\limits_{n\to\infty} a_n = 0$ if and only if for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\left\{a_{n_{i_j}}\right\}$ of $\{a_{n_i}\}$ such that $\lim\limits_{j\to\infty} a_{n_{i_j}} = 0$.*

Let H be a real Hilbert space and C be nonempty, closed and convex subset of H. Let $(\cdot,\cdot)_f : C \times H \to (-\infty, +\infty]$ be a functional defined as follows (see [27] (see also [4])):

$$(y,x)_f := \|y\|^2 - 2\langle y, x\rangle + \|x\|^2 + 2\rho f(y) = \|y-x\|^2 + 2\rho f(y) \tag{2.1}$$

where $y \in C$, $x \in H$, ρ is positive number and $f : C \to (-\infty, +\infty]$ is proper, convex and lower semicontinous. From the definitions of $(\cdot,\cdot)_f$ and f, it is easy to see the following properties:

(1) $(y,x)_f$ is convex and continuous with respect to x when y is fixed;

(2) $(y,x)_f$ is convex and lower semicontinuous with respect to y when x is fixed.

Definition 2.2 (Li et al. [27] (see also [4])). *Let H be a real Hilbert space and C be nonempty, closed and convex subset of H. We say that $P_C^f : H \to 2^C$ is a metric f-projection operator if*

$$P_C^f x = \left\{ u \in C \,\middle|\, (u,x)_f = \inf_{\xi \in C} (\xi, x)_f \right\}, \quad \forall x \in H.$$

Lemma 2.3 (Li et al. [27, Lemma 3.1(ii)]). *Let H be a real Hilbert space and let $\emptyset \neq C \subset H$. Then for every $x \in H$, $\hat{x} = P_C^f x$ if and only if*

$$\langle \hat{x} - y, x - \hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \ \forall y \in C.$$

Lemma 2.4 (Li et al. [27, Lemma 3.2]). *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let $x \in H$, $\hat{x} = P_C^f x$. Then*

$$\|y - \hat{x}\|^2 + (\hat{x}, x)_f \leqslant (y,x)_f, \ \forall y \in C. \tag{2.2}$$

Lemma 2.5 (Deimling [29]). *Let H be a real Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exists $z \in H$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, z \rangle + \alpha, \ \forall x \in X. \tag{2.3}$$

Due to the properties of f, we have the motivation and ideas to create a new type of mappings which is general and covers a quasi-strict pseudo-contraction as follows.

Definition 2.6. *Let H be a real Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper a lower semicontinuous convex funtional, a mapping T with domain $D(T)$ and range $R(T)$ in H is called* quasi strict f-pseudo-contraction *if the set of fixed point $F(T)$ is nonempty and if there exists a constant $0 \leq k < 1$ such that for each $p \in F(T)$*

$$(p, Tx)_f \leqslant (p,x)_f + k\Big((x, Tx)_f - 2\rho f(p)\Big), \ \forall x \in D(T). \tag{2.4}$$

It is obvious from above definition that (2.4) equivalent to

$$\|p - Tx\|^2 \leq \|p - x\|^2 + k\|x - Tx\|^2 + 2k\rho\,(f(x) - f(p)), \ \forall x \in C \text{ and } p \in F(T). \tag{2.5}$$

Definition 2.7. *A mapping $T : C \to C$ is said to be* closed *if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.*

Example 2.1 ([23, Example 9]). *Let $T : H \to H$. be a mapping defined by $Tx = \frac{3}{2}x$ for all $x \in H$. Then, $F(T) = \{x \in H : Tx = x\} = \{0\}$ and T is closed and quasi-strict $\|\cdot\|^2$-pseudo contraction.*

Example 2.2 (Saewan and Kumam [24, Example 1.5]). *Let $X = \mathbb{R}^3$ be provided with the norm $\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2} + \sqrt{x_2^2 + x_3^2}$.*

This is smooth strictly convex Banach space and $C = \{x \in \mathbb{R}^3 | x_2 = 0,\ x_3 = 0\}$ is a closed and convex subset of X. It is a simple computation; we get

$$P_C(1,1,1) = (1,0,0),\ \ \Pi_C(1,1,1) = (2,0,0). \tag{2.6}$$

We set $\rho = 1$ is positive number and define $f : C \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} 2 + 2\sqrt{5}, & x < 0; \\ -2 - 2\sqrt{5}, & x \geq 0. \end{cases}$$

Then, f is proper, convex, and lower semicontinuous. Simple computations show that

$$\Pi_C^f(1,1,1) = (4,0,0). \tag{2.7}$$

Lemma 2.8 ([23, Lemma 15]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a quasi strict f-pseudo-contraction. Then the fixed point set $F(T)$ of T is closed and convex.*

The following lemma provides the significant inequality related to two quasi strict f-pseudo-contractions in the framework of Hilbert spaces.

Lemma 2.9. *Let C be a nonempty closed convex subset of a real Hilbert spaces H. Let $S, T : C \to C$ be two quasi strict f-pseudo contractions such that $\Omega := F(T) \cap F(S) \neq \emptyset$. Then*

$$\begin{aligned} &\|x - Tx\|^2 + \|x - Sx\|^2 \\ &\quad \leq \frac{2}{1-k_T} \langle x - p, x - Tx \rangle + \frac{2}{1-k_S} \langle x - p, x - Sx \rangle \\ &\quad\quad + 2\rho \left(\frac{k_T}{1-k_T} + \frac{k_S}{1-k_S} \right) (f(x_n) - f(p)) \end{aligned} \tag{2.8}$$

for all $x \in C$ and $p \in \Omega$.

Proof Let $x \in C$ and $p \in \Omega$. Since T is a quasi strict f-pseudo-contraction, we have

$$\begin{aligned}
&\|p-Tx\|^2 \le \|p-x\|^2 + k_T\|x-Tx\|^2 + 2k_T\rho(f(x)-f(p))\\
\Leftrightarrow&\|p-x\|^2 + 2\langle p-x, x-Tx\rangle + \|x-Tx\|^2 \\
&\quad \le \|p-x\|^2 + k_T\|x-Tx\|^2 + 2k_T\rho\left(f(x)-f(p)\right)\\
\Leftrightarrow&(1-k_T)\|x-Tx\|^2 \le 2\langle x-p, x-Tx\rangle + 2k_T\rho\left(f(x)-f(p)\right)\\
\Leftrightarrow&\|x-Tx\|^2 \le \frac{2}{1-k_T}\langle x-p, x-Tx\rangle + \frac{2k_T\rho}{1-k_T}\left(f(x)-f(p)\right).
\end{aligned} \tag{2.9}$$

Similarly, we can show that

$$\|x-Sx\|^2 \le \frac{2}{1-k_S}\langle x-p, x-Sx\rangle + \frac{2k_S\rho}{1-k_S}\left(f(x)-f(p)\right). \tag{2.10}$$

It follows from (2.9) and (2.10) that we obtain

$$\begin{aligned}
&\|x-Tx\|^2 + \|x-Sx\|^2\\
&\qquad \le \frac{2}{1-k_T}\langle x-p, x-Tx\rangle + \frac{2}{1-k_S}\langle x-p, x-Sx\rangle\\
&\qquad\quad + 2\rho\left(\frac{k_T}{1-k_T} + \frac{k_S}{1-k_S}\right)(f(x_n)-f(p)).
\end{aligned}$$

This completes the proof.

3 Main result

In this section, an iterative shrinking metric f-projection method is provided in order to find a common fixed point of two quasi strict f-pseudo-contractions.

Theorem 3.1. *Let H be a real Hilbert space, C a nomempty closed convex subset of H, let S and T be closed and quasi strict f-pseudo-contractions from C into itself. Suppose that $\Omega := F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases}
x_0 \in H, \text{ chosen arbitrarily,}\\
C_1 = C,\\
x_1 = P^f_{C_1}x_0,\\
C_{n+1} = \left\{ z \in C_n \;\middle|\; \begin{aligned} &\|x_n - Tx_n\|^2 + \|x_n - Sx_n\|^2\\ &\quad \le \tfrac{2}{1-k_T}\langle x_n - z, x_n - Tx_n\rangle + \tfrac{2}{1-k_S}\langle x_n - z, x_n - Sx_n\rangle\\ &\quad\quad + 2\rho\left(\tfrac{k_T}{1-k_T} + \tfrac{k_S}{1-k_S}\right)(f(x_n) - f(z)) \end{aligned} \right\},\\
x_{n+1} = P^f_{C_{n+1}}x_0.
\end{cases} \tag{3.1}$$

Then, the sequence $\{x_n\}$ converges strongly to $P^f_{\Omega}x_0$.

Proof The proof is divided into six steps.

Step 1. Show that C_n is closed and convex for all $n \geq 1$.
For $n = 1$, $C_1 = C$ is closed and convex. Assume that C_i is closed and convex for some $i \in \mathbb{N}$. For $z \in C_{i+1}$, we have

$$\begin{aligned}
&\|x_i - Tx_i\|^2 + \|x_i - Sx_i\|^2 \\
&\qquad \leq \tfrac{2}{1-k_T} \langle x_i - z, x_i - Tx_i \rangle + \tfrac{2}{1-k_S} \langle x_i - z, x_i - Sx_i \rangle \\
&\qquad + 2\rho \left(\tfrac{k_T}{1-k_T} + \tfrac{k_S}{1-k_S} \right) (f(x_i) - f(z)).
\end{aligned}$$

It is not hard to see that the continuity and linearity of $\langle \cdot, x_i - Tx_i \rangle$ and $\langle \cdot, x_i - Sx_i \rangle$ together with the lower semicontinuity and convexity of f, allow C_{i+1} to be closed and convex. Then, for all $n \geq 1$, C_n is closed and convex.

Step 2. Show that $\Omega \subset \bigcap_{n=1}^{\infty} C_n := D$.

It is obvious that $\Omega := F(T) \cap F(S) \subset C = C_1$. Suppose that $\Omega \subset C_i$ for some $i \in \mathbb{N}$. For any $p \in \Omega$, we have $p \in C_i$ and by Lemma 2.9 we obtain

$$\begin{aligned}
&\|x_i - Tx_i\|^2 + \|x_i - Sx_i\|^2 \\
&\qquad \leq \tfrac{2}{1-k_T} \langle x_i - p, x_i - Tx_i \rangle + \tfrac{2}{1-k_S} \langle x_i - p, x_i - Sx_i \rangle \\
&\qquad + 2\rho \left(\tfrac{k_T}{1-k_T} - \tfrac{k_S}{1-k_S} \right) (f(x_i) - f(p)).
\end{aligned}$$

This means that $p \in C_{i+1}$. By mathematical induction, $\Omega \subset C_n$ for all $n \geq 1$. Therefore $\Omega \subset \bigcap_{n=1}^{\infty} C_n := D \neq \emptyset$.

Step 3. Show that $\{x_n\}$ is bounded and the $\lim\limits_{n\to\infty} (x_n, x_0)_f$ exists.

Since $f : X \to \mathbb{R}$ is a convex and lower semicontinuous mapping, applying Lemma 2.5, we see that there exist $z \in H$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, z \rangle + \alpha, \quad \forall y \in H.$$

It follows that

$$\begin{aligned}
(x_n, x_0)_f &= \|x_n\|^2 - 2\langle x_n, x_0 \rangle + \|x_0\|^2 + 2\rho f(x_n) \\
&\geq \|x_n\|^2 - 2\langle x_n, x_0 \rangle + \|x_0\|^2 + 2\rho \langle x_n, z \rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n, x_0 - \rho z \rangle + \|x_0\|^2 + 2\rho\alpha \qquad (3.2) \\
&\geq \|x_n\|^2 - 2\|x_0 - \rho z\|\|x_n\| + \|x_0\|^2 + 2\rho\alpha \\
&= (\|x_n\| - \|x_0 - \rho z\|)^2 + \|x_0\|^2 - \|x_0 - \rho z\|^2 + 2\rho\alpha.
\end{aligned}$$

Since $x_n = P^f_{C_n} x_0$, it follows from (3.2) that

$$\begin{aligned} \|x_0\|^2 - \|x_0 - \rho z\| + 2\rho\alpha &\le (\|x_n\| - \|x_0 - \rho z\|)^2 + \|x_0\|^2 - \|x_0 - \rho z\| + 2\rho\alpha \\ &\le (x_n, x_0)_f = \left(P^f_{C_n}(x_0), x_0\right)_f \\ &= \inf_{\xi \in C_n} (\xi, x_0)_f \\ &\le (u, x_0)_f \end{aligned}$$

for each $u \in \Omega$. This implies that $\{x_n\}$ and $(x_n, x_0)_f$ are bounded. By the fact that $x_{n+1} \in C_{n+1} \subset C_n$ and Lemma 2.4 we obtain

$$\|x_{n+1} - x_n\|^2 + (x_n, x_0)_f \le (x_{n+1}, x_0)_f.$$

Since $\|x_{n+1} - x_n\|^2 \ge 0$, $\left\{(x_n, x_0)_f\right\}$ is nondecreasing. Therefore, the limit of $\left\{(x_n, x_0)_f\right\}$ exists.

Step 4. Show that $x_n \to p$ as $n \to \infty$, where $p = P^f_D x_0$.

Let $\{x_{n_i}\} \subset \{x_n\}$. From the boundedness of $\{x_{n_i}\}$ there exists $\left\{x_{n_{i_j}}\right\} \subset \{x_{n_i}\}$ such that

$$x_{n_{i_j}} \rightharpoonup p \quad \text{as } j \to \infty. \tag{3.3}$$

Write $\tilde{x}_j := x_{n_{i_j}}$, it is easy to see that $p \in \widetilde{C}_j$ where $\widetilde{C}_j := C_{n_{i_j}}$. Note that

$$(\tilde{x}_j, x_0)_f = \left(P^f_{\widetilde{C}_j}(x_0), x_0\right)_f = \min_{\xi \in \widetilde{C}_j} (\xi, x_0)_f \le (p, x_0)_f \tag{3.4}$$

On the other hand, since $\tilde{x}_j \rightharpoonup p$, so $\tilde{x}_j - x_0 \rightharpoonup p - x_0$ and then by weakly lower semicontinuity of $\|\cdot\|^2$ and f we obtain

$$\|p - x_0\|^2 \le \liminf_{j \to \infty} \|\tilde{x}_j - x_0\|^2 \tag{3.5}$$

and

$$f(p) \le \liminf_{j \to \infty} f(\tilde{x}_j). \tag{3.6}$$

Combine (3.5) and (3.6), we obtain

$$\begin{aligned} (p, x_0)_f = \|p - x_0\|^2 + 2\rho f(p) &\le \liminf_{j \to \infty} \|\tilde{x}_j - x_0\|^2 + 2\rho \liminf_{j \to \infty} f(\tilde{x}_j) \\ &\le \liminf_{j \to \infty} \left(\|\tilde{x}_j - x_0\|^2 + 2\rho f(\tilde{x}_j)\right) \\ &= \liminf_{j \to \infty} (\tilde{x}_j, x_0)_f. \end{aligned} \tag{3.7}$$

It follows from (3.4) and (3.7), we have

$$(p, x_0)_f \le \liminf_{j\to\infty} (\tilde{x}_j, x_0)_f \le \limsup_{j\to\infty} (\tilde{x}_j, x_0)_f \le (p, x_0)_f$$

and then

$$\lim_{j\to\infty} (\tilde{x}_j, x_0)_f = (p, x_0)_f.$$

Next, we consider

$$\begin{aligned}
\limsup_{j\to\infty} \|\tilde{x}_j - x_0\|^2 &= \limsup_{j\to\infty} \Big((\tilde{x}_j, x_0)_f - 2\rho f(\tilde{x}_j) \Big) \\
&\le \limsup_{j\to\infty} (\tilde{x}_j, x_0)_f + \limsup_{j\to\infty} \left(-2\rho f(\tilde{x}_j)\right) \\
&= (p, x_0)_f - 2\rho \liminf_{j\to\infty} f(\tilde{x}_j) \le (p, x_0)_f - 2\rho f(p) \qquad (3.8) \\
&= \|p - x_0\|^2.
\end{aligned}$$

Combine (3.5) and (3.8), we obtain

$$\|p - x_0\|^2 \le \liminf_{j\to\infty} \|\tilde{x}_j - x_0\|^2 \le \limsup_{j\to\infty} \|\tilde{x}_j - x_0\|^2 \le \|p - x_0\|^2$$

and then

$$\lim_{j\to\infty} \|\tilde{x}_j - x_0\|^2 = \|p - x_0\|^2. \qquad (3.9)$$

Note that

$$f(\widetilde{x}_j) = \frac{1}{2\rho} \Big((\tilde{x}_j, x_0)_f - \|\tilde{x}_j - x_0\|^2 \Big).$$

Then we have

$$\begin{aligned}
\limsup_{j\to\infty} f(\tilde{x}_j) &= \frac{1}{2\rho} \limsup_{j\to\infty} \Big((\tilde{x}_j, x_0)_f - \|\tilde{x}_j - x_0\|^2 \Big) = \frac{1}{2\rho} \Big((p, x_0)_f - \|p - x_0\|^2 \Big) \\
&= f(p). \qquad (3.10)
\end{aligned}$$

Combine (3.6) and (3.10), we obtain

$$f(p) \le \liminf_{j\to\infty} f(\tilde{x}_j) \le \limsup_{j\to\infty} f(\tilde{x}_j) = f(p)$$

and then

$$\lim_{j\to\infty} f(x_{n_{i_j}}) = \lim_{j\to\infty} f(\tilde{x}_j) = f(p).$$

By Lemma 2.1, it implies that

$$\lim_{n\to\infty} f(x_n) = f(p). \qquad (3.11)$$

On the other hand, we note that

$$\|\tilde{x}_j - p\|^2 = \|(\tilde{x}_j - x_0) - (p - x_0)\|^2 = \|\tilde{x}_j - x_0\|^2 - 2\langle \tilde{x}_j - x_0, p - x_0\rangle + \|p - x_0\|^2$$

It follows from (3.3) and (3.9), we obtain

$$\begin{aligned}\lim_{j\to\infty} \|\tilde{x}_j - p\|^2 &= \lim_{j\to\infty} \left(\|\tilde{x}_j - x_0\|^2 - 2\langle \tilde{x}_j - x_0, p - x_0\rangle + \|p - x_0\|^2\right) \\ &= \lim_{j\to\infty} \|\tilde{x}_j - x_0\|^2 - 2\lim_{j\to\infty} \langle \tilde{x}_j - x_0, p - x_0\rangle + \|p - x_0\|^2 \\ &= \|p - x_0\|^2 - 2\langle p - x_0, p - x_0\rangle + \|p - x_0\|^2 \\ &= 0.\end{aligned}$$

Therefore $x_{n_{i_j}} = \tilde{x}_j \to p$ as $j \to \infty$. This implies by Lemma 2.1 that

$$x_n \to p \quad \text{as} \quad n \to \infty.$$

Thus,

$$\omega_w(x_n) = \{p\}.$$

It is easy to show that $p \in C_n$ for all $n \geq 1$. Hence $p \in \bigcap_{n=1}^{\infty} C_n =: D$. Since $x_n = P^f_{C_n} x_0$, so by Lemma 2.3 we have

$$\langle x_n - y, x_0 - x_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in D,$$

letting $n \to \infty$, so we obtain

$$\langle p - y, x_0 - p\rangle + \rho f(y) - \rho f(p) \geq 0, \quad \forall y \in D,$$

which implies that $p = P^f_D x_0$.

Step 5. Show that $p \in \Omega$.

Firstly, we prove that $\{Tx_n\}$ and $\{Sx_n\}$ are bounded. Indeed, taking $v \in \Omega = F(T) \cap F(S)$ and then by (2.9) we have

$$\begin{aligned}&\|v\|^2 - 2\|v\|\|Tx_n\| + \|Tx_n\|^2 = (\|v\| - \|Tx_n\|)^2 \leq \|v - Tx_n\|^2 \\ &\leq \|v - x_n\|^2 + k_T\|x_n - Tx_n\|^2 + 2k_T\rho(f(x_n) - f(v)) \\ &\leq \|v - x_n\|^2 + k_T\left(\frac{2}{1-k_T}\langle x_n - v, x_n - Tx_n\rangle + \frac{2k_T\rho}{1-k_T}(f(x_n) - f(v))\right) \\ &\quad + 2k_T\rho(f(x_n) - f(v)) \\ &\leq \|v - x_n\|^2 + \frac{2k_T}{1-k_T}\langle x_n - v, x_n - Tx_n\rangle + 2k_T\rho\left(\frac{k_T}{1-k_T} + 1\right)(f(x_n) - f(v)) \\ &\leq \|v - x_n\|^2 + \frac{2k_T}{1-k_T}\|x_n - v\|\|x_n\| + \frac{2k_T}{1-k_T}\|x_n - v\|\|Tx_n\| + \frac{2k_T\rho}{1-k_T}(f(x_n) - f(v)).\end{aligned}$$

By a simple calculation we get

$$\begin{aligned}
\|Tx_n\|^2 &\le \left(\|v-x_n\|^2 + \frac{2k_T}{1-k_T}\|x_n-v\|\,\|x_n\| - \|v\|^2 + \frac{2k_T\rho}{1-k_T}\left(f(x_n)-f(v)\right)\right) \\
&\quad + \left(\frac{2k_T}{1-k_T}\|x_n-v\| + 2\|v\|\right)\|Tx_n\| \\
&\le M + \widetilde{M}\,\|Tx_n\| = M + \frac{1}{2}\left(2\widetilde{M}\,\|Tx_n\|\right) \\
&\le M + \frac{1}{2}\left(\widetilde{M}^2 + \|Tx_n\|^2\right) = M + \frac{1}{2}\widetilde{M}^2 + \frac{1}{2}\|Tx_n\|^2,
\end{aligned}$$

where $M := \sup\left\{ \|v-x_n\|^2 + \frac{2k_T}{1-k_T}\|x_n-v\|\,\|x_n\| - \|v\|^2 + \frac{2k_T\rho}{1-k_T}\left(f(x_n)-f(v)\right) \,\middle|\, n\in\mathbb{N}\right\}$ and
$\widetilde{M} := \sup\left\{ \frac{2k_T}{1-k_T}\|x_n-v\| + 2\|v\| \,\middle|\, n\in\mathbb{N}\right\}$. So, we have

$$\|Tx_n\|^2 \le 2M + \widetilde{M}^2$$

for all $n\in\mathbb{N}$. Therefore, $\{Tx_n\}$ is bounded. By the same argument, $\{Sx_n\}$ is also bounded. Moreover, we note that

$$\|x_{n+1}-x_n\| \le \|x_{n+1}-p\| + \|p-x_n\| \to 0 \quad \text{as} \quad n\to\infty.$$

Thus, by the fact that $x_{n+1} = P^f_{C_{n+1}}x_0 \in C_{n+1}$ and (3.11), we obtain

$$\begin{aligned}
&\|x_n-Tx_n\|^2 + \|x_n-Sx_n\|^2 \\
&\quad \le \tfrac{2}{1-k_T}\langle x_n-x_{n+1}, x_n-Tx_n\rangle + \tfrac{2}{1-k_S}\langle x_n-x_{n+1}, x_n-Sx_n\rangle \\
&\qquad + 2\rho\left(\tfrac{k_T}{1-k_T} + \tfrac{k_S}{1-k_S}\right)\left(f(x_n)-f(x_{n+1})\right).
\end{aligned}$$

This means that

$$\|x_n-Tx_n\| \to 0 and \|x_n-Sx_n\| \to 0 \quad \text{as} \quad n\to\infty.$$

Next, we have that

$$\|Tx_n-p\| \le \|Tx_n-x_n\| + \|x_n-p\| \to 0 \quad \text{as} \quad n\to\infty.$$

That is $Tx_n\to p$ as $n\to\infty$. It follows from the closedness of T, we obtain $Tp=p$, thus $p\in F(T)$. Similarly, $p\in F(S)$. Therefore, $p\in F(T)\cap F(S)=\Omega$.

Step 6. Show that $p = P^f_\Omega x_0$.

Notice by step 2 that $\Omega\subset D$, so we have $P^f_\Omega x_0 \in D$ and then by Step 5 it yields

$$\begin{aligned}
(p,x_0)_f &= \left(P^f_D x_0, x_0\right)_f = \inf_{\xi\in D}(\xi,x_0)_f \\
&\le \left(P^f_\Omega x_0, x_0\right)_f = \inf_{\zeta\in\Omega}(\zeta,x_0)_f \\
&\le (p,x_0)_f.
\end{aligned}$$

This shows that $\left(P^f_\Omega x_0, x_0\right)_f = (p, x_0)_f$. It follows from the uniqueness we have $p = P^f_\Omega x_0$. Then $\{x_n\}$ converges strongly to $p = P^f_\Omega x_0$. This completes the proof.

4 Deduced theorems and Applications

In this section, some applications of the main theorem are provided in order to find some fixed points or common fixed points. Furthermore, it can be applied to find zeros of some monotone operators as well.

If $S = T$, then $k_T = k_S =: k$ and by Theorem 3.1 we obtain the following corollary.

Corollary 4.1. *Let C, H and T be the same as in Theorem 3.1. Suppose that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P^f_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \|x_n - Tx_n\|^2 \leq \frac{2}{1-k}\langle x_n - z, x_n - Tx_n\rangle + \frac{2k\rho}{1-k}\left(f(x_n) - f(z)\right) \right\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P^f_\Omega x_0$.

If $f = \|\cdot\|^2$, then $(x, y)_{\|\cdot\|^2} = \|x - y\|^2 + 2\rho\|x\|^2$ for all $(x, y) \in C \times H$ and $P^f_{C_n} x_0 = P^{\|\cdot\|^2}_{C_n} x_0$ for all $n \in \mathbb{N}$ then by Theorem 3.1, we obtain the following corollary.

Corollary 4.2. *Let C and H be the same as in Theorem 3.1, let T and S be two closed and quasi strict $\|\cdot\|^2$-pseudo-contractions from C into itself such that $\Omega := F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P^{\|\cdot\|^2}_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \begin{array}{l} \|x_n - Tx_n\|^2 + \|x_n - Sx_n\|^2 \\ \quad \leq \frac{2}{1-k_T}\langle x_n - z, x_n - Tx_n\rangle + \frac{2}{1-k_S}\langle x_n - z, x_n - Sx_n\rangle \\ \quad\quad + 2\rho\left(\frac{k_T}{1-k_T} + \frac{k_S}{1-k_S}\right)\left(\|x_n\|^2 - \|z\|^2\right) \end{array} \right\}, \\ x_{n+1} = P^{\|\cdot\|^2}_{C_{n+1}} x_0. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P^{\|\cdot\|^2}_\Omega x_0$.

If f is a constant function say $f = a \in \mathbb{R}$, then $(x, y)_a = \|x - y\|^2 + 2\rho a$ and not hard to see that T coincide with a quasi strict a-pseudo-contraction. Thus by Theorem 3.1, we obtain the following corollary.

Corollary 4.3. *Let C and H be the same as in Theorem 3.1, let T and S be two closed and quasi strict a-pseudo-contractions from C into itself such that $\Omega := F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P^a_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \ \middle| \ \begin{array}{l} \|x_n - Tx_n\|^2 + \|x_n - Sx_n\|^2 \\ \qquad \leq \frac{2}{1-k_T} \langle x_n - z, x_n - Tx_n \rangle + \frac{2}{1-k_S} \langle x_n - z, x_n - Sx_n \rangle \end{array} \right\}, \\ x_{n+1} = P^a_{C_{n+1}} x_0. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P^a_\Omega x_0$.

If $f = a = 0$, then Corollary 4.3 reduces to the following corollary.

Corollary 4.4. *Let C and H be the same as in Theorem 3.1, let T and S be two closed and quasi strict pseudo-contractions from C into itself such that $\Omega := F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \ \middle| \ \begin{array}{l} \|x_n - Tx_n\|^2 + \|x_n - Sx_n\|^2 \\ \qquad \leq \frac{2}{1-k_T} \langle x_n - z, x_n - Tx_n \rangle + \frac{2}{1-k_S} \langle x_n - z, x_n - Sx_n \rangle \end{array} \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Recall that a mapping $A : D(A) \subset H \to H$ is said to be *monotone* if, for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0.$$

A is said to be r-*inverse strongly monotone* if there exists a positive real number r such that

$$\langle x - y, Ax - Ay \rangle \geq r\|Ax - Ay\|^2, \quad \forall x, y \in D(A).$$

A is said to be r-*quasi inverse strongly monotone* if $A^{-1}(0) = \{z \in D(A) : Az = 0\} \neq \emptyset$ and there exists a positive real number r such that for each $p \in A^{-1}(0)$, the following inequality holds:

$$\langle x - p, Ax \rangle \geq r\|Ax\|^2, \quad \forall x \in D(A).$$

Without loss of generality we can assume that the constant $r \in \left(0, \frac{1}{2}\right]$ since if A is $\hat{r}$-quasi inverse strongly monotone (or $\hat{r}$-inverse strongly monotone), then we can find $r \in \left(0, \frac{1}{2}\right]$ such that $\hat{r} \geq r$ and then $\hat{r}\|Ax\|^2 \geq r\|Ax\|^2$ (or $\hat{r}\|Ax - Ap\|^2 \geq r\|Ax - Ap\|^2$) for all $\hat{r} > 0$ (i.e., A is r-quasi inverse strongly monotone).

Next, A mapping $A : C \to H$ is said to be *r-quasi inverse strongly f-monotone* if $A^{-1}(0) = \{z \in D(A) : Az = 0\} \neq \emptyset$ and there exists a positive real number $r \in \left(0, \frac{1}{2}\right]$ such that for each $p \in A^{-1}(0)$, the following inequality holds:

$$\langle x - p, Ax\rangle + \left(\tfrac{1}{2} - r\right) \rho(f(x) - f(p)) \geq r\|Ax\|^2, \quad \forall x \in D(A),$$

where ρ is positive number and $f : C \to (-\infty, +\infty]$ is proper, convex and lower semicontinuous.

Finally, we provide some applications of the main theorem to find a common zero point problems of a closed and two quasi inverse strongly monotone operators and via an iterative shrinking metric f-projection method in the framework of Hilbert spaces.

Theorem 4.5. *Let C, H be the same as in Theorem 3.1, and $A, B : H \to H$ be r_A and r_B-quasi inverse strongly f-monotone and closed operators, respectively, such that $p \in \Omega := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P^f_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \ \middle| \ \begin{array}{l} \|Ax_n\|^2 + \|Bx_n\|^2 \\ \qquad \leqslant \frac{1}{r_A}\langle x_n - z, Ax_n\rangle + \frac{1}{r_B}\langle x_n - z, Bx_n\rangle \\ \qquad\quad + 2\rho\left(\frac{1}{r_A} + \frac{1}{r_B}\right)(f(x_n) - f(z)) \end{array} \right\}, \\ x_{n+1} = P^f_{C_{n+1}} x_0, \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P^f_{\Omega} x_0$.

Proof Since A is r_A-quasi inverse strongly f-monotone, we obtain

$$\begin{aligned} &\langle x - p, Ax\rangle + \left(\tfrac{1}{2} - r_A\right)\rho(f(x) - f(p)) \geq r_A\|Ax\|^2 \\ &\Leftrightarrow 2\langle x - p, Ax\rangle + (1 - 2r_A)\,\rho(f(x) - f(p)) \geq 2r_A\|Ax\|^2 \\ &\Leftrightarrow \|x - p\|^2 + \|Ax\|^2 - \|(I - A)\,x - p\|^2 + (1 - 2r_A)\,\rho(f(x) - f(p)) \geq 2r_A\|Ax\|^2 \\ &\Leftrightarrow \|(I - A)\,x - p\|^2 \leq \|x - p\|^2 + (1 - 2r_A)\,\|(I - A)\,x - x\|^2 + (1 - 2r_A)\,\rho(f(x) - f(p)). \end{aligned}$$

Similarly, by the assumption of B we can show that

$$\|(I - B)\,x - p\|^2 \leqslant \|x - p\|^2 + (1 - 2r_B)\,\|(I - B)\,x - x\|^2 + (1 - 2r_B)\,\rho(f(x) - f(p)).$$

Let $T := (I - A)$ and $S := (I - B)$. Then we have T and S are closed and quasi strict f-pseudo-contractions such that $k_T = 1-2r_A \Leftrightarrow \frac{2}{1-k_T} = \frac{1}{r_A}$ and $\frac{2}{1-k_S} = \frac{1}{r_B}$. By applying Theorem 3.1, we have the desired results.

If $f = 0$, then Theorem 4.5 reduces to the following corollary.

Corollary 4.6. *Let C, H be the same as in Theorem 3.1, and $A, B : H \to H$ be r_A and r_B-quasi inverse strongly monotone and closed operators, respectively, such that $p \in \Omega := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in H, \ \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ C_{n+1} = \left\{ z \in C_n \ \middle| \ \begin{array}{l} \|Ax_n\|^2 + \|Bx_n\|^2 \\ \qquad \leqslant \frac{1}{r_A} \langle x_n - z, Ax_n \rangle + \frac{1}{r_B} \langle x_n - z, Bx_n \rangle \end{array} \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

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