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All Maximal Clones of a Majority Reflexive Graph

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Abstract : A reflexive graph is extensively investigated not only in graph theory but also in the context of universal algebra. In this paper, we characterize all maximal clones containing the clone of all operations preserving edges of a majority reflexive graph. Our results together with NU-duality Theorem [1] imply a duality for a tolerance-primal algebra having a majority term operation.

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1 Introduction

A *clone* on a finite set is a set of finitary nonnullary operations which contains all projection maps and is closed under compositions. The set of all clones over a finite set is a complete lattice with the co-atoms being *maximal clones*. There are only finitely many maximal clones over a finite set and every proper subclone of the full clone is contained in a maximal one. I.G. Rosenberg [2] has classified all maximal clones over a finite set by determining six classes of relations such that maximal clones are just the clones of operations preserving one of the following relations (for details, let see in [2]): (1) bounded orders, (2) prime permutations, (3) prime affine relations, (4) non-trivial equivalence relations, (5) k-regularly generated relations for some $3 \leq k \leq |M|$ and (6) central relations.

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A *tolerance relation* on a set is a reflexive and symmetric binary relation on the set. A *reflexive (undirected) graph* $\mathbf{M} = (M; \Theta)$ is a finite set M whose elements are called the *vertices* of $\mathbf{M}$ equipped with a tolerance relation on M whose elements are called the *edges* of $\mathbf{M}$. For a positive number n, an operation $m : M^n \longrightarrow M$ is called an n-ary edge-preserving if $(x_i, y_i) \in \Theta$ for all $1 \leq i \leq n$ implies $(m(x_1, \ldots, x_n), m(y_1, \ldots, y_n)) \in \Theta$. Especially for $n = 3$, m is called a *majority* if it satisfies

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

for all $x, y \in M$. A reflexive graph $\mathbf{M} = (M; \Theta)$ is called a *majority reflexive graph* if $\mathrm{Pol}\,(\Theta)$, the clone of all $\mathbf{M}$ edge-preservings, contains a majority operation. It was proved in [3] that a reflexive graph $\mathbf{M}$ is majority if and only if $\mathbf{M}$ is an absolute retract of bipartite graphs.

Davey et al. proved in [4] that if a finite ordered set $\mathbf{P}$ is disconnected then the non-trivial equivalence relation $\cup_{1 \leq i \leq m} C_i \times C_i$ such that $\mathbf{C_1}, \ldots, \mathbf{C_m}$ are all components of $\mathbf{P}$ will give a maximal clone containing the monotone clone of $\mathbf{P}$. C. Ratanaprasert [5] has shown that the monotone clone of a finite unbounded connected ordered set is a subclone of a maximal clone preserving only either k-regularly generated relations or central relations with arity more than 1. It is a well-known fact that a maximal clone preserving a k-regularly generated relation contains no majority functions. Hence, the monotone clone containing a majority operation is a subclone of a maximal clone preserving only either non-trivial equivalence relations or central relations. C. Ratanaprasert and U. Chotwattakawanit [6] described all possibilities of maximal clones of a majority ordered set using the distant function. Although the definitions of ordered set and reflexive graph are different, they can be represented as pictures. It is interesting whether all maximal clones of a majority reflexive graph can be described by the distant function. In this paper, we will show the affirmative answer. Moreover, a monotone clone and a clone of a majority reflexive graph are subclones of maximal clones preserving relations in the same classes.

A natural duality for a reflexive graph was constructed by Johansen in [7]. NU-duality Theorem [1] implies that the structure $\mathrm{M} := \big(M; \mathbb{S}(\underline{\mathrm{M}}^2), \mathscr{T}\big)$ yields a duality on $\mathbb{ISP}(\underline{\mathrm{M}})$ where $\underline{\mathrm{M}} = (M; F)$ is an algebra whose $\langle F \rangle$ contains a majority operation and $\mathscr{T}$ is the discrete topology on M. If $\underline{\mathrm{M}}$ is a *tolerance primal-algebra*; i.e., $\langle F \rangle = \mathrm{Pol}\,(\Theta)$ for some tolerance relation Θ then $\mathbb{S}(\underline{\mathrm{M}}^2)$ is the set of all binary relations ρ such that $\mathrm{Pol}\,(\rho) \supseteq \mathrm{Pol}\,(\Theta)$. We will start the works by describing all elements in $\mathbb{S}(\underline{\mathrm{M}}^2)$ in section 2 and then in section 3, we will characterize all maximal clones of a majority reflexive graph.

2 A Duality for a Tolerance-primal Algebra Admitting a Majority Operation

In this section, let $\mathbf{M} = (M;\Theta)$ be a reflexive graph and $\underline{M} = (M;F)$ be an algebra whose $\langle F\rangle = \mathrm{Pol}\,(\Theta)$. Denote $\Theta^0 = \triangle_{M\times M}$ and for each natural number k, denote $\Theta^k = \underbrace{\Theta\circ\ldots\circ\Theta}_{k}$. We recall the basic definitions in the graph theory as the followings.

1. For each $a, b \in M$, a *path* from a to b (of length n) in $\mathbf{M}$ is the subgraph $\{\{a_0,\ldots,a_n\};\{(a_i,a_{i+1})\in\Theta\,|\,0\le i\le n-1\}\}$ of $\mathbf{M}$ with $a_0 = a$, $a_n = b$ and $a_i \neq a_j$ for all $i \neq j$ which we will denote it by $a = a_0\Theta a_1\Theta\ldots\Theta a_n = b$.
2. The *distant* between two vertices a and b is the length of a shortest path from a to b and is denoted by $d(a, b)$.
3. $d(\mathbf{M}) = \max\{d(a,b)\,|\,a,b\in M \text{ such that } d(a,b) \text{ exists}\}$.

C. Ratanaprasert and U. Chotwattakawanit [6] described all elements in $\mathbb{S}(\underline{P}^2)$ using the distant function for all order-primal algebras $\underline{P}$. We will show in the following proposition that those above relations Θ^k can be also described by the distant function and its proof follows from the definitions.

Proposition 2.1. $\Theta^k = \{(a,b)\in M\times M\,|\,d(a,b)\le k\}$ *for all non-negative integers k.*

Theorem 2.2. *The set $\mathbb{S}(\underline{M}^2)$ of all binary relations whose clones containing* $\mathrm{Pol}\,(\Theta)$ *is* $\{\Theta^0,\ldots,\Theta^{d(\mathbf{M})}, M\times M\}$.

Proof Let $0 \le k \le d(\mathbf{M})$ and let $f \in \mathrm{Pol}\,(\Theta)$ be m-ary for a positive number m and $(a_i, b_i) \in \Theta^k$ for all $1 \le i \le m$. Then for each $1 \le i \le m$, $a_i = a_i^0\Theta a_i^1\Theta\ldots\Theta a_i^k = b_i$ for some $a_i^0,\ldots,a_i^k \in M$; so,

$$f(a_1,\ldots,a_m) = f(a_1^0,\ldots,a_m^0)\Theta f(a_1^1,\ldots,a_m^1)\Theta\ldots\Theta f(a_1^k,\ldots,a_m^k) = f(b_1,\ldots,b_m).$$

Thus, $(f(a_1,\ldots,a_m), f(b_1,\ldots,b_m)) \in \Theta^k$. Hence, $\Theta^k \in \mathbb{S}(\underline{M}^2)$. Conversely, let $\rho \in \mathbb{S}(\underline{M}^2)$ and $\mathbf{C_1},\ldots,\mathbf{C_m}$ be all components of $\mathbf{M}$.

Suppose that there is $(a, b) \in \rho$ such that $a \in C_i$ but $b \notin C_i$ for some $1 \le i \le m$. For $c, d \in M$, we define $f : M \to M$ by

$$f(x) = \begin{cases} c & \text{if } x \in C_i,\\ d & \text{if } x \notin C_i \end{cases}$$

for all $x \in M$. For each $x, y \in M$, if $(x, y) \in \Theta$ then either $x, y \in C_i$ or $x, y \notin C_i$ which implies $f(x) = f(y)$. By the reflexivity of Θ, $f \in \mathrm{Pol}\,(\Theta) \subseteq \mathrm{Pol}\,(\rho)$; so, f preserves ρ. From $(a, b) \in \rho$, we get $(c, d) = (f(a), f(b)) \in \rho$. Therefore, $\rho = M \times M$.

But, if $(a, b) \in \rho$ then $a, b \in C_i$ for some $1 \le i \le n$. We choose $k = \max\{d(x,y)\,|\,(x,y)\in\rho\}$ and let $(a, b) \in \rho$ with $d(a, b) = k$. So, $\rho \subseteq \Theta^k$. Conversely, let $(c, d) \in \Theta^k$. Then $d(c, d) = p \le k$. So, there is a path $c = c_0\Theta c_1\Theta\ldots\Theta c_p = d$ from c to d. Define $f : M \to M$ by

$$f(x) = \begin{cases} c_i & \text{if } d(x,a) = i \text{ for some } 1 \le i \le p,\\ d & \text{otherwise} \end{cases}$$

for all $x \in M$. Let $(x, y) \in \Theta$. Then x and y are in the same component and $d(x, y) \leq 1$. If neither x nor y are in the component of a then $f(x) = d = f(y)$. We assume that both x and y are in the component of a. By the triangle inequality property of d, $d(a, x) - 1 \leq d(a, y) \leq d(a, x) + 1$. If $d(a, x) = i < p$ then $f(x) = c_i$ and $f(y) \in \{c_{i-1}, c_i, c_{i+1}\}$; and if $i \geq p$ then $f(x) = c_p$ and $f(y) \in \{c_{p-1}, c_p\}$. In either cases, $(f(x), f(y)) \in \Theta$. Thus, $f \in \mathrm{Pol}\,(\Theta) \subseteq \mathrm{Pol}\,(\rho)$; hence, f preserves ρ. From $(a, b) \in \rho$, we have $(c, d) = (f(a), f(b)) \in \rho$. Therefore, $\rho = \Theta^k$.

Corollary 2.3. *If $\rho \in \mathbb{S}(\underline{\mathrm{M}}^2)$ and there is $(a, b) \in \rho$ such that a, b are in different components then $\rho = M \times M$.*

NU-duality Theorem [1] and Theorem 2.2 imply the following corollary.

Corollary 2.4. *If $\mathbf{M}$ is a majority reflexive grahp, $\mathrm{M} = \left(M; \left\{\Theta^k \mid 1 \leq k \leq d(\mathbf{M})\right\}, \mathscr{T}\right)$ yields a duality on $\mathbb{ISP}(\underline{\mathrm{M}})$.*

3 All Maximal Clones of a Majority Reflexive Graph

We described all binary relations whose clones contain the clone preserving a tolerance relation in Section 2. It is interesting whether some of them are maximal and the converse is also true. In this section, we study some conditions which prove the questions.

We refer all definitions and notations from Section 2. One can see that $\mathrm{Pol}\,(\Theta)$ is the full clone if and only if Θ is the trivial relations and the lattice of all clones on a singleton set has exactly one element. We will consider a set M with $|M| \geq 2$ and $\triangle_{M \times M} \subset \Theta \subset M \times M$. Then $\mathrm{Pol}\,(\Theta)$ is a subclone of a maximal clone preserving a relation from one of the classes described by I.G. Rosenberg [2]. The following theorem shows all possible classes of relations whose the clones are maximal containing $\mathrm{Pol}\,(\Theta)$.

Theorem 3.1. *The clone $\mathrm{Pol}\,(\Theta)$ is a subclone of a maximal clone $\mathrm{Pol}\,(\rho)$ whose ρ is a non-trivial equivalence relation, a k-regularly generated relation or a central relation.*

Proof If ρ is in the classes (1) or (2), ρ is binary; so, $\rho = M \times M$ or $\rho = \Theta^k$ for some $1 \leq k \leq d(\mathbf{M})$ by Theorem 2.2. But, reflexivity and symmetricity of Θ^k for all $k \geq 0$ imply that Θ^k is an order or permutation if and only if $k = 0$. Hence, $\mathrm{Pol}\,(\rho)$ is the full clone, a contradiction.

Suppose that ρ is an affine relation corresponding to a group $(M; +, -, 0)$. Let $(a, b) \in \Theta$ with $a \neq b$. We may assume that $a \neq 0$. Define $f : M \times M \to M$ by

$$f(x, y) = \begin{cases} a & \text{if } x = y = a, \\ b & \text{otherwise} \end{cases}$$

for all $x, y \in M$. Since $\mathrm{Im} f = \{a, b\}$, we have $f \in \mathrm{Pol}\,(\Theta) \subseteq \mathrm{Pol}\,(\rho)$; i.e., f preserves ρ. From $(a, 0, a, 0), (a, -a, 0, 0) \in \rho$, we get $(a, b, b, b) = (f(a, a), f(0, -a), f(a, 0), f(0, 0)) \in \rho$; thus, $a + b = b + b$ which implies $a = b$, a contradiction.

Corollary 3.2. *If* $\mathrm{Pol}\,(\Theta)$ *contains a majority operation, it is a subclone of a maximal clone* $\mathrm{Pol}\,(\rho)$ *whose* ρ *is only a non-trivial equivalence relation or a central relation. Moreover, if* ρ *is a central relation then* ρ *is binary.*

Proof By the result in [6], if ρ is a central relation then ρ is at most binary. Reflexivity of Θ implies that all constants are in $\mathrm{Pol}\,(\Theta) \subseteq \mathrm{Pol}\,(\rho)$; so, ρ is not unary.

From now, we consider Θ whose $\mathrm{Pol}\,(\Theta)$ contains a majority operation. Theorem 2.2 and Corollary 3.2 imply that all relations ρ whose $\mathrm{Pol}\,(\rho)$ is a maximal clone containing $\mathrm{Pol}\,(\Theta)$ are in the forms Θ^k for some $1 \leq k \leq d(\mathbf{M})$. For each $a, b \in M$ with $d(a, b) = d(\mathbf{M})$ and $1 \leq k \leq d(\mathbf{M})$, if Θ^k is transitive then $(a, b) \in \Theta^k$; so, $d(\mathbf{M}) = d(a, b) \leq k \leq d(\mathbf{M})$.

Remark 3.3. *For each* $1 \leq k \leq d(\mathbf{M})$*,* Θ^k *is an equivalence relation if and only if* $k = d(\mathbf{M})$*.*

Theorem 3.4. *Suppose that* $\mathrm{Pol}\,(\rho)$ *is a maximal clone containing* $\mathrm{Pol}\,(\Theta)$*.*

1. *If the graph* $\mathbf{M}$ *is connected then* ρ *is precisely central relations in the forms* Θ^k *for some* $\lceil d(\mathbf{M})/2 \rceil \leq k < d(\mathbf{M})$*.*
2. *If the graph* $\mathbf{M}$ *is disconnected then* ρ *is the non-trivial equivalence relation* $\Theta^{d(\mathbf{M})} = \cup_{1 \leq i \leq m} C_i \times C_i$ *where* $\mathbf{C_1} \ldots, \mathbf{C_m}$ *are all components of* $\mathbf{M}$*.*

Proof (1). Connectedness of $(M; \Theta)$, we get $\Theta^{d(\mathbf{M})} = M \times M$. By Remark 3.3, ρ is not non-trivial equivalence relations. So, ρ is a central relation. One can see that Θ^k is central if and only if $\lceil d(\mathbf{M})/2 \rceil \leq k < d(\mathbf{M})$.

(2). It is easily shown that $\Theta^{d(\mathbf{M})} = \cup_{1 \leq i \leq m} C_i \times C_i$. If ρ is central, the center elements will be related to all elements of M; so by Corollary 2.3, $\rho = M \times M$ which is impossible. Hence, ρ is a non-trivial equivalence relation which implies that $\rho = \Theta^{d(\mathbf{M})}$.

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