



Coupled Coincidence Point Theorems for a (β, g) - ψ -Contractive Mapping in Partially Ordered G -Metric Spaces

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Abstract : In this paper, we introduce the notion (β) -admissible and (β, g) -admissible for mapping $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We showed the existence of a coupled coincidence point theorem for a (β, g) - ψ -contractive mapping in G -metric spaces. We also show the uniqueness of a coupled common fixed point for such mappings and give some examples to show the validity of our result.

Keywords : Fixed point; Coupled coincidence; Invariant set; Admissible; G -metric.

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1 Introduction

The first result in the existence of a fixed point for contraction type of mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [1] in 2004. Following this work, Nieto and Lopez [2, 3] extended the results in [1] for non-decreasing mapping. Later, Agarwal et al. [4] presented some new results for contractions in partially ordered metric spaces.

A notion of coupled fixed point theorem was defined by Guo and Lakshmikantham [5]. After that, Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone property. Furthermore, they proved the existence and uniqueness

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of a coupled fixed point theorems for mappings which satisfy the mixed monotone property in partially ordered metric space. Since 2006, many authors have studied coupled fixed point theorems in partially ordered metric space and their applications have been established. The results in [6] were extend by Lakshimikantam and Ćirić in [7] by defining the mixed g -monotone and to study the existence and uniqueness of coupled coincidence point for such mapping which satisfy the mixed monotone property in partially ordered metric space. As a continuation of this work, several coupled fixed point and coupled coincidence point results have appeared in the recent literature ([8, 9, 10, 11]). Work noted in ([12, 13, 14, 15, 16, 7, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]) are some examples of these works.

In 2004, Mustafa and Sims [28] introduce a G -metric spaces which is a generalization of the concept of a metric space . They developed and introduced a new fixed point theory. After the publication of this work, many authors proved the existence and uniqueness of a fixed point and a coupled fixed point theorem dealing with G -metric space. For coupled fixed point theorem, Choudhury and Maily [29] proved the existence of a coupled fixed point theorem of nonlinear contraction mappings with mixed monotone property in partially ordered G -metric space. Abbas and et al. [30] extended the results of coupled fixed point theorem for a mixed monotone mapping obtained by Choudhury and Maily [29] .

In the case of the coupled coincidence point theory has developed in partially ordered G -metric space, Aydi and et al. [31] establish coupled coincidence and coupled common fixed point results for a mixed g -monotone mapping satisfying nonlinear contractions in partially ordered G -metric space. They generalize the results obtained by Choudhury and Maily [29]. Later, Karapinar and et al. [32] extend the results of coupled coincidence and coupled common fixed point theorem for a mixed g -monotone mapping obtained by Aydi and et al.[31] and several fixed point theorems have appeared in recent literatures (see, e.g., [33, 34, 35, 32, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]).

In 2012, Samet et al. [47] defined the concept of α - ψ contractive mappings and proved that including the Banach fixed point theorems. Later, Alghamdi and karapinar [48] introduce a β -admissible and a G - β - ψ contractive mapping which is a characterization α - ψ contractive mappings in the context of G -metric spaces. They also proved the existence and uniqueness of a fixed point for such mappings in the complete G -metric spaces.

In this work, We introduce the notion (β) -admissible and (β, g) -admissible for mapping $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We also prove the existence of a coupled coincidence point theorem and a coupled common fixed point theorem for a (β, g) - ψ -contractive mapping in G -metric spaces.

2 Preliminaries

In this section, we give some definitions, proposition, examples and remarks which are useful for main results in this paper. Throughout this paper, (X, \leq)

denotes a partially ordered set with the partial order \leq . By $x \leq y$, we mean $y \geq x$. A mapping $f : X \rightarrow X$ is said to be non-decreasing (resp., non-increasing) if for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$ (resp. $f(y) \geq f(x)$).

Definition 2.1 ([28]). *Let X be a non-empty set, and $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:*

- (G1) $G(x, y, z) = 0$ if $x = y = z$.
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables).
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G -metric on X , and the pair (X, G) is called a G -metric space.

Example 2.2. *Let (X, d) be a metric space. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, for all $x, y, z \in X$, is a G -metric space on X .*

Definition 2.3 ([28]). *Let (X, G) be a G -metric space, and let (x_n) be a sequence of point of X . We say that (x_n) is G -convergent to $x \in X$ if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$.*

Proposition 2.4 ([28]). *Let (X, G) be a G -metric space, the following are equivalent:*

- (1) (x_n) is G -convergent to x .
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$.
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$.
- (4) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Definition 2.5 ([28]). *Let (X, G) be a G -metric space. A sequence (x_n) is called a G -Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$. That is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.*

Proposition 2.6 ([28]). *Let (X, G) be a G -metric space, the following are equivalent*

- (1) the sequence (x_n) is G -Cauchy
- (2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 2.7 ([28]). *Let (X, G) be a G -metric space. A mapping $f : X \rightarrow X$ is G -continuous at $x \in X$ if and only if it is G -sequentially continuous at x , that is, whenever (x_n) is G -convergent to x , $(f(x_n))$ is G -convergent to $f(x)$.*

Definition 2.8 ([28]). *A G -metric space (X, G) is called G -complete if every G -Cauchy sequence is G -convergent in (X, G) .*

Definition 2.9 ([29]). *Let (X, G) be a G -metric space. A mapping $F : X \times X \rightarrow X$ is said to be continuous if for any two G -convergent sequences (x_n) and (y_n) converging to x and y respectively, $(F(x_n, y_n))$ is G -convergent to $F(x, y)$.*

Let (x, \leq) is a partially ordered set, the partial order \leq_2 for the product set $X \times X$ defined in the following way

$$(x, y), (u, v) \in X \times X, \quad (x, y) \leq_2 (u, v) \iff x \leq u \quad \text{and} \quad y \geq v.$$

We say that (x, y) is comparable to (u, v) if either $(x, y) \leq_2 (u, v)$ or $(x, y) \geq_2 (u, v)$.

The concept of a mixed monotone property and a coupled fixed point have been introduced by Bhaskar and Lakshmikantham in [6].

Definition 2.10 ([6]). *Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$. We say F has the mixed monotone property if for any $x, y \in X$*

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \text{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Definition 2.11 ([6]). *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Lakshmikantham and Ćirić in [7] introduced the concept of a mixed g -monotone mapping and a coupled coincidence point.

Definition 2.12 ([7]). *Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F has the mixed g -monotone property if for any $x, y \in X$*

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \text{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Definition 2.13 ([7]). *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2.14 ([7]). *Let X be a non-empty set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.*

The following class of functions was considered by Lakshmikantham and Ćirić in [7].

Let Φ denote the set of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying

1. $\phi^{-1}(\{0\}) = \{0\}$,
2. $\phi(t) < t$ for all $t > 0$,
3. $\lim_{r \rightarrow t^+} \phi(r) < t$ for all $t > 0$.

Lemma 2.15 ([7]). *Let $\phi \in \Phi$. For all $t > 0$, we have $\lim_{n \rightarrow \infty} \phi^n(t) = 0$.*

Karapinar and et al. [32] proved the following theorem.

Theorem 2.16 ([32]). *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space. Suppose that there exists $\phi \in \Phi$, $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that*

$$G(F(x, u), F(y, v), F(z, w)) \leq \phi(G(gx, gy, gz) + G(gu, gv, gw))$$

for all $x, y, z, u, v, w \in X$ for which $gx \geq gy \geq gz$ and $gu \leq gv \leq gw$.

Suppose also that F is continuous and has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 2.17 ([31]). *Let (X, \leq) be a partially ordered set and G be a G -metric on X . We say that (X, G, \leq) is regular if the following conditions hold :*

1. if a non-decreasing sequence $(x_n) \rightarrow x$, then $x_n \leq x$ for all n ,
2. if a non-increasing sequence $(y_n) \rightarrow y$, then $y \leq y_n$ for all n .

Theorem 2.18 ([32]). *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G, \leq) is regular. Suppose that there exists $\phi \in \Phi$, $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that*

$$G(F(x, u), F(y, v), F(z, w)) \leq \phi(G(gx, gy, gz) + G(gu, gv, gw))$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \geq g(y) \geq g(z)$ and $g(u) \leq g(v) \leq g(w)$.

Suppose also that $(g(X), G)$ is complete, F has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Alghamdi and Karapinar [48] introduced an β -admissible as follows.

Definition 2.19 ([48]). Let $T : X \rightarrow X$ and $\beta : X \times X \times X \rightarrow [0, \infty)$. We say that T is β -admissible if and only if, for all $x, y, z \in X$, we have

$$\beta(x, y, z) \geq 1 \Rightarrow \beta(Tx, Ty, Tz) \geq 1.$$

Now, we give the notion of (β) -admissible and (β, g) -admissible which is useful for our main results.

Definition 2.20. Let $F : X \times X \rightarrow X$ and $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$. We say that F is (β) -admissible if and only if, for all $x, y, z, u, v, w \in X$, we have

$$\beta((x, u), (y, v), (z, w)) \geq 1 \Rightarrow \beta((F(x, u), F(u, x)), (F(y, v), F(v, y)), (F(z, w), F(w, z))) \geq 1.$$

Definition 2.21. Let $F : X \times X \rightarrow X$, $g : X \rightarrow X$ and $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$. We say that F is (β, g) -admissible if and only if, for all $x, y, z, u, v, w \in X$, we have

$$\beta((gx, gu), (gy, gv), (gz, gw)) \geq 1 \Rightarrow \beta((F(x, u), F(u, x)), (F(y, v), F(v, y)), (F(z, w), F(w, z))) \geq 1.$$

Remark

1. Every (β) -admissible is (β, I_X) -admissible when I_X denote identity map on X .
2. Notice that Definition 2.20 is exactly the same with Definition 2.19 by choosing X^2 and $T(x, y) = (F(x, y), F(y, x))$.

Example 2.22. Let $F : X \times X \rightarrow X$, $g : X \rightarrow X$. Consider a mapping $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$ be such that

$$\beta((x, u), (y, v), (z, w)) = \begin{cases} 1 & \text{if } x \geq y \geq z \text{ and } u \leq v \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that, if F has the mixed monotone property on X , then F is (β) -admissible and if F has the mixed g -monotone property on X , then F is (β, g) -admissible.

Next example, we will show that F is (β, g) -admissible but not (β) -admissible.

Example 2.23. Let $F : X \times X \rightarrow X$, $g : X \rightarrow X$. Define by $F(x, y) = 1 - x^2$ and $g(x) = 1 - x$. Consider a mapping $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$ be such that

$$\beta((x, u), (y, v), (z, w)) = \begin{cases} 1 & \text{if } u = v = y = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that F is (β, g) -admissible.

Let $\beta((gx, 0), (0, 0), (0, gw)) = \beta((gx, g1), (g1, g1), (g1, gw)) \geq 1$ implies that $\beta((F(x, 1), F(1, x)), (F(1, 1), (F(1, 1))), (F(1, w), F(w, 1))) = \beta((1 - x^2, 0), (0, 0), (0, 1 - w^2)) \geq 1$.

Next, we show that F not (β) -admissible. Consider $\beta((1, 0), (0, 0), (0, 1)) \geq 1$ but $\beta((F(1, 0), F(0, 1)), (F(0, 0), (F(0, 0))), (F(1, 0), F(0, 1))) = \beta((0, 1), (1, 1), (1, 0)) = 0$.

3 Main Results

Let Ψ denote the family non-decreasing and right continuous functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n th iterate of ψ satisfying the following conditions:

1. $\psi^{-1}(\{0\}) = \{0\}$,
2. $\psi(t) < t$ for all $t > 0$,
3. $\lim_{r \rightarrow t^+} \psi(r) < t$ for all $t > 0$.

Theorem 3.1. *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mapping having the mixed g -monotone property on X . Assume that there exists $\psi \in \Psi$ and $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$, such that for all $x, y, z, u, v, w \in X$, the following holds*

$$\begin{aligned} \beta((gx, gu), (gy, gv), (gz, gw)) [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ \leq \psi(G(gx, gy, gz) + G(gu, gv, gw)) \end{aligned} \quad (3.1)$$

for all $gx \geq gy \geq gz$ and $gu \leq gv \leq gw$.

Suppose also that

- (i) F is (β, g) -admissible.
- (ii) F is continuous.
- (iii) $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F .
- (iv) there exist $x_0, y_0 \in X$ such that

$$\beta((F(x_0, y_0), F(y_0, x_0)), (gx_0, gy_0), (gx_0, gy_0)) \geq 1.$$

If there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is F has a coupled coincidence point.

Proof Let $(x_0, y_0) \in X \times X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that

$$gx_1 = F(x_0, y_0) \text{ and } gy_1 = F(y_0, x_0).$$

Again from $F(X \times X) \subseteq g(X)$ we can choose $x_2, y_2 \in X$ such that

$$gx_2 = F(x_1, y_1) \text{ and } gy_2 = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{(gx_n)\}$ and $\{(gy_n)\}$ in X such that

$$gx_n = F(x_{n-1}, y_{n-1}) \text{ and } gy_n = F(y_{n-1}, x_{n-1}) \text{ for all } n \geq 1. \quad (3.2)$$

Since F has the mixed g -monotone property, we have

$$gx_0 \leq gx_1 \leq gx_2 \quad \text{and} \quad gy_0 \geq gy_1 \geq gy_2.$$

Continuing this process, we have

$$gx_n \leq gx_{n+1} \quad \text{and} \quad gy_n \geq gy_{n+1}. \quad (3.3)$$

Since

$$\beta((F(x_0, y_0), F(y_0, x_0)), (gx_0, gy_0), (gx_0, gy_0)) = \beta((gx_1, gy_1), (gx_0, gy_0), (gx_0, gy_0)) \geq 1$$

and F is (β, g) -admissible, we get

$$\beta((F(x_1, y_1), F(y_1, x_1)), (F(x_0, y_0), F(y_0, x_0)), (F(x_0, y_0), F(y_0, x_0))) = \beta((gx_2, gy_2), (gx_1, gy_1), (gx_1, gy_1)) \geq 1$$

Again, using the fact that F is (β, g) -admissible, we have

$$\beta((F(x_2, y_2), F(y_2, x_2)), (F(x_1, y_1), F(y_1, x_1)), (F(x_1, y_1), F(y_1, x_1))) = \beta((gx_3, gy_3), (gx_2, gy_2), (gx_2, gy_2)) \geq 1$$

By repeating this argument, we get

$$\begin{aligned} & \beta((F(x_{n+1}, y_{n+1}), F(y_{n+1}, x_{n+1})), (F(x_n, y_n), F(y_n, x_n)), (F(x_n, y_n), F(y_n, x_n))) \\ & = \beta((gx_n, gy_n), (gx_{n-1}, gy_{n-1}), (gx_{n-1}, gy_{n-1})) \geq 1. \end{aligned} \quad (3.4)$$

If there exists $k \in N$ such that $(gx_{k+1}, gy_{k+1}) = (gx_k, gy_k)$ then $gx_k = gx_{k+1} = F(x_k, y_k)$ and $gy_k = gy_{k+1} = F(y_k, x_k)$. Thus, (x_k, y_k) is a coupled coincidence point of F . This finishes the proof. Now we assume that $(gx_{k+1}, gy_{k+1}) \neq (gx_k, gy_k)$ for all $n \geq 0$. Thus, we have either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$ for all $n \geq 0$. From (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{aligned} & [G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n)] \\ & \leq \beta((gx_n, gy_n), (gx_{n-1}, gy_{n-1}), (gx_{n-1}, gy_{n-1})) [G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n)] \\ & = \beta((gx_n, gy_n), (gx_{n-1}, gy_{n-1}), (gx_{n-1}, gy_{n-1})) \\ & [G(F(x_n, y_n), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) + G(F(y_n, x_n), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}))] \\ & \leq \psi(G(gx_n, gx_{n-1}, gx_{n-1}) + G(gy_n, gy_{n-1}, gy_{n-1})). \end{aligned} \quad (3.5)$$

From (3.5), we get

$$G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) \leq \psi(G(gx_n, gx_{n-1}, gx_{n-1}) + G(gy_n, gy_{n-1}, gy_{n-1})). \quad (3.6)$$

Since $\psi(t) < t$ for all $t > 0$, by repeating (3.6), we get

$$G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) \leq \psi^n(G(gx_1, gx_0, gx_0) + G(gy_1, gy_0, gy_0)). \quad (3.7)$$

for all $n \in \mathbb{N}$.

For $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n (G(gx_1, gx_0, gx_0) + G(gy_1, gy_0, gy_0)) < \epsilon.$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\epsilon)$. Then, by using the triangle inequality, we have

$$\begin{aligned} G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\ &\quad + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + G(y_{n+2}, y_{n+3}, y_{n+3}) \\ &\quad + \dots + G(x_{m-1}, x_m, x_m) + G(y_{m-1}, y_m, y_m) \\ &= \sum_{k=n}^{m-1} G(gx_k, gx_{k+1}, gx_{k+1}) + G(gy_k, gy_{k+1}, gy_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k (G(gx_1, gx_0, gx_0) + G(gy_1, gy_0, gy_0)) \\ &\leq \sum_{n \geq n(\epsilon)} \psi^n (G(gx_1, gx_0, gx_0) + G(gy_1, gy_0, gy_0)) \\ &< \epsilon. \end{aligned} \tag{3.8}$$

This implies that $G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \epsilon$. Since

$$G(gx_n, gx_m, gx_m) \leq G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \epsilon$$

and

$$G(gy_n, gy_m, gy_m) \leq G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \epsilon$$

This show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequence in the G -metric space (X, G) . Since (X, G) is complete, $\{gx_n\}$ and $\{gy_n\}$ are G -convergent, there exist $x, y \in X$ such that $\lim_{n \rightarrow \infty} gx_n = x$ and $\lim_{n \rightarrow \infty} gy_n = y$. That is from Proposition 2.4, we have

$$\lim_{n \rightarrow \infty} G(gx_n, gx_n, x) = \lim_{n \rightarrow \infty} G(gx_n, x, x) = 0, \tag{3.9}$$

$$\lim_{n \rightarrow \infty} G(gy_n, gy_n, y) = \lim_{n \rightarrow \infty} G(gy_n, y, y) = 0. \tag{3.10}$$

From (3.9), (3.10), continuity of g and proposition 2.7, we get

$$\lim_{n \rightarrow \infty} G(g(gx_n), g(gx_n), gx) = \lim_{n \rightarrow \infty} G(g(gx_n), gx, gx) = 0, \tag{3.11}$$

$$\lim_{n \rightarrow \infty} G(g(gy_n), g(gy_n), gy) = \lim_{n \rightarrow \infty} G(g(gy_n), gy, gy) = 0. \tag{3.12}$$

From (3.2) and commutativity of F and g ,

$$g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n), \tag{3.13}$$

$$g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n). \tag{3.14}$$

We now show that $F(x, y) = gx$ and $F(y, x) = gy$.

Taking the limit as $n \rightarrow +\infty$ in (3.13) and (3.14), by (3.11), (3.12), continuity of F , we get

$$g(x) = g(\lim_{n \rightarrow \infty} gx_{n+1}) = \lim_{n \rightarrow \infty} g(gx_{n+1}) = \lim_{n \rightarrow \infty} g(F(x_n, y_n)) = \lim_{n \rightarrow \infty} F(gx_n, gy_n) = F(x, y)$$

and

$$g(y) = g(\lim_{n \rightarrow \infty} gy_{n+1}) = \lim_{n \rightarrow \infty} g(gy_{n+1}) = \lim_{n \rightarrow \infty} g(F(y_n, x_n)) = \lim_{n \rightarrow \infty} F(gy_n, gx_n) = F(y, x).$$

Thus we prove that $F(x, y) = gx$ and $F(y, x) = gy$.

In the next theorem, we omit the continuity hypothesis of F .

Theorem 3.2. Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space such that (X, G, \leq) is regular. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be mapping having the mixed g -monotone property on X . Assume that there exists $\psi \in \Psi$ and $\beta : X^2 \times X^2 \times X^2 \rightarrow [0, \infty)$, such that for all $x, y, z, u, v, w \in X$, the following holds

$$\begin{aligned} \beta((gx, gu), (gy, gv), (gz, gw))[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ \leq \psi(G(gx, gy, gz) + G(gu, gv, gw)) \end{aligned}$$

for all $gx \geq gy \geq gz$ and $gu \leq gv \leq gw$.

Suppose also that

- (i) F is (β, g) -admissible.
- (ii) For any two sequences (x_n) and (y_n) in X with $\beta((x_{n+1}, y_{n+1}), (x_n, y_n), (x_n, y_n)) \geq 1$ for all n , if there is $x, y \in X$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\beta((x, y), (x_n, y_n), (x_n, y_n)) \geq 1$ for all n .
- (iii) $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and $(g(X), G)$ is complete.
- (iv) there exist $x_0, y_0 \in X$ such that

$$\beta((F(x_0, y_0), F(y_0, x_0)), (gx_0, gy_0), (gx_0, gy_0)) \geq 1.$$

If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is F has a coupled coincidence point.

Proof Proceeding exactly as in Theorem 3.1, we have that $\beta((gx_n, gy_n), (gx_n, gy_n), (gx_{n-1}, gy_{n-1})) \geq 1$ and $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete G -metric space $(g(X), G)$. Then, there exists $x, y \in X$ such that $\{gx_n\} \rightarrow gx$ and $\{gy_n\} \rightarrow gy$, by the assumption (ii), we have $\beta((gx, gy), (gx_n, gy_n), (gx_n, gy_n)) \geq 1$ for all $n \geq 1$

. Since $\{gx_n\}$ is non-decreasing and $\{gy_n\}$ is non-increasing, by assumption (iii), (X, G, \leq) is regular, we have $gx_n \leq gx$ and $gy_n \geq gy$, for all n . Now by (3.1), the rectangle inequality and $\psi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}
& G(F(x, y), gx, gx) + G(F(y, x), gy, gy) \\
& \leq G(F(x, y), gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx, gx) \\
& \quad + G(F(y, x), gy_{n+1}, gy_{n+1}) + G(gy_{n+1}, gy, gy) \\
& = G(F(x, y), F(x_n, y_n), F(x_n, y_n)) + G(gx_{n+1}, gx, gx) \\
& \quad + G(F(y, x), F(y_n, x_n), F(y_n, x_n)) + G(gy_{n+1}, gy, gy) \\
& \leq \beta((gx, gy), (gx_n, gy_n), (gx_n, gy_n)) [G(F(x, y), F(x_n, y_n), F(x_n, y_n)) + G(F(y, x), F(y_n, x_n), F(y_n, x_n))] \\
& \quad + G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy) \\
& \leq \psi(G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)) \\
& \quad + G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy) \\
& < G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n) \\
& \quad + G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy).
\end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, we obtain

$$G(F(x, y), gx, gx) + G(F(y, x), gy, gy) = 0.$$

Which implies that $gx = F(x, y)$, and $gy = F(y, x)$. Thus we prove that (x, y) is a coupled coincidence point of F and g .

The following example is valid for Theorem 3.1.

Example 3.3. Let $X = \mathbb{R}$. Define $G : X \times X \times X \rightarrow [0, +\infty)$ by $G(x, y, z) = |x - y| + |x - z| + |y - z|$ and $F : X \times X \rightarrow X$ be defined by

$$F(x, y) = \frac{x - 2y}{4}, \quad (x, y) \in X^2,$$

and $g : X \rightarrow X$ by $g(x) = \frac{3x}{2}$, clearly, F has a mixed g -monotone property. Consider the mapping $\beta : X^2 \times X^2 \times X^2 \rightarrow (0, +\infty]$ such that

$$\beta((x, u), (y, v), (z, w)) = \begin{cases} 1 & \text{if } x \geq y \geq z \text{ and } u \leq v \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we claim that F satisfies condition (3.1). If $\beta((x, u), (y, v), (z, w)) = 0$, then the result is straightforward. Let $x, u, y, v, z, w \in X$. Without loss of generality, assume that $gx \geq gy \geq gz$ and $gu \leq gv \leq gw$, we have $\beta((gx, gu), (gy, gv), (gz, gw)) =$

1. Then we have

$$\begin{aligned}
 & \beta((gx, gu), (gy, gv), (gz, gw))[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
 &= [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
 &= \left| \frac{x-2u}{4} - \frac{y-2v}{4} \right| + \left| \frac{x-2u}{4} - \frac{z-2w}{4} \right| + \left| \frac{y-2v}{4} - \frac{z-2w}{4} \right| \\
 &\quad + \left| \frac{u-2x}{4} - \frac{v-2y}{4} \right| + \left| \frac{u-2x}{4} - \frac{w-2z}{4} \right| + \left| \frac{v-2y}{4} - \frac{w-2z}{4} \right| \\
 &\leq 3 \left| \frac{x-y}{4} \right| + 3 \left| \frac{x-z}{4} \right| + 3 \left| \frac{y-z}{4} \right| + 3 \left| \frac{u-v}{4} \right| + 3 \left| \frac{u-w}{4} \right| + 3 \left| \frac{v-w}{4} \right| \\
 &= \frac{3}{4}(|x-y| + |x-z| + |y-z|) + \frac{3}{4}(|u-v| + |u-w| + |v-w|).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 G(gx, gy, gz) + G(gu, gv, gw) &= G\left(\frac{3x}{2}, \frac{3y}{2}, \frac{3z}{2}\right) + G\left(\frac{3u}{2}, \frac{3v}{2}, \frac{3w}{2}\right) \\
 &= \frac{3}{2}(|x-y| + |x-z| + |y-z|) + \frac{3}{2}(|u-v| + |u-w| + |v-w|)
 \end{aligned}$$

Now, choose $\psi \in \Psi$ such that $\psi(t) = t/2$, then

$$\begin{aligned}
 & \beta((gx, gu), (gy, gv), (gz, gw))[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
 &\leq \psi(G(gx, gy, gz) + G(gu, gv, gw))
 \end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold, we know that F has a coupled coincidence point $(0, 0)$.

Example 3.4. Let $X = [-1, 1]$. Define $G : X \times X \times X \rightarrow [0, +\infty)$ by $G(x, y, z) = |x-y| + |x-z| + |y-z|$ and $F : X \times X \rightarrow X$ be defined by

$$F(x, y) = \frac{x^3 - 2y^3}{4}, \quad (x, y) \in X^2,$$

and $g : X \rightarrow X$ by $g(x) = \frac{9x}{2}$, clearly, F has a mixed g -monotone property. Consider the mapping $\beta : X^2 \times X^2 \times X^2 \rightarrow (0, +\infty]$ such that

$$\beta((x, u), (y, v), (z, w)) = \begin{cases} 1 & \text{if } x \geq y \geq z \text{ and } u \leq v \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we claim that F satisfies condition (3.1). If $\beta((x, u), (y, v), (z, w)) = 0$, then the result is straightforward. Let $x, u, y, v, z, w \in X$. Without loss of generality, assume that $gx \geq gy \geq gz$ and $gu \leq gv \leq gw$, we have $\beta((gx, gu), (gy, gv), (gz, gw)) =$

1. Then we have

$$\begin{aligned}
& \beta((gx, gu), (gy, gv), (gz, gw))[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
&= [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
&= \left| \frac{x^3 - 2u^3}{4} - \frac{y^3 - 2v^3}{4} \right| + \left| \frac{x^3 - 2u^3}{4} - \frac{z^3 - 2w^3}{4} \right| + \left| \frac{y^3 - 2v^3}{4} - \frac{z^3 - 2w^3}{4} \right| \\
&\quad + \left| \frac{u^3 - 2x^3}{4} - \frac{v^3 - 2y^3}{4} \right| + \left| \frac{u^3 - 2x^3}{4} - \frac{w^3 - 2z^3}{4} \right| + \left| \frac{v^3 - 2y^3}{4} - \frac{w^3 - 2z^3}{4} \right| \\
&= \frac{3}{4}(|x^3 - y^3| + |x^3 - z^3| + |y^3 - z^3|) + \frac{3}{4}(|u^3 - v^3| + |u^3 - w^3| + |v^3 - w^3|) \\
&\leq \frac{9}{4}(|x - y| + |x - z| + |y - z|) + \frac{9}{4}(|u - v| + |u - w| + |v - w|)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
G(gx, gy, gz) + G(gu, gv, gw) &= G\left(\frac{9x}{2}, \frac{9y}{2}, \frac{9z}{2}\right) + G\left(\frac{9u}{2}, \frac{9v}{2}, \frac{9w}{2}\right) \\
&= \frac{9}{2}(|x - y| + |x - z| + |y - z|) + \frac{9}{2}(|u - v| + |u - w| + |v - w|)
\end{aligned}$$

Now, choose $\psi \in \Psi$ such that $\psi(t) = t/2$, then

$$\begin{aligned}
& \beta((gx, gu), (gy, gv), (gz, gw))[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\
&\leq \psi(G(gx, gy, gz) + G(gu, gv, gw))
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 hold. Notice that F has a coupled coincidence point $(0, 0)$.

Next, we give a sufficient condition for the uniqueness of the coupled coincidence point in Theorem 3.1 .

Theorem 3.5. *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that*

- (i) $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$,
- (ii) $\beta((gu, gv), (gx, gy), (gx, gy)) \geq 1$ and $\beta((gu, gv), (gx^*, gy^*), (gx^*, gy^*)) \geq 1$.

Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Proof From Theorem 3.1, the set of coupled coincidence point is non-empty. Suppose (x, y) and (x^*, y^*) are coupled coincidence point of F , that is

$$gx = F(x, y), gy = F(y, x), gx^* = F(x^*, y^*) \text{ and } gy^* = F(y^*, x^*).$$

We shall show that

$$gx^* = gx \text{ and } gy^* = gy. \quad (3.15)$$

By assumption there is $(u, v) \in X \times X$ such that

$$\beta((gu, gv), (gx, gy), (gx, gy)) \geq 1 \quad \text{and} \quad \beta((gu, gv), (gx^*, gy^*), (gx^*, gy^*)) \geq 1.$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$, such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_n = F(u_{n-1}, v_{n-1}) \quad \text{and} \quad gv_n = F(v_{n-1}, u_{n-1}) \quad \text{for all } n \geq 1.$$

Since F is (β, g) -admissible and $\beta((gu_0, gv_0), (gx, gy), (gx, gy)) \geq 1$, we have

$$\beta((F(u_0, v_0), F(v_0, u_0)), (F(x, y), F(y, x)), (F(x, y), F(y, x))) \geq 1.$$

That is $\beta((gu_1, gv_1), (gx, gy), (gx, gy)) \geq 1$.

From $\beta((gu_1, gv_1), (gx, gy), (gx, gy)) \geq 1$, if we use again the property of F is (β, g) -admissible, then it follow that

$$\beta((F(u_1, v_1), F(v_1, u_1)), (F(x, y), F(y, x)), (F(x, y), F(y, x))) \geq 1$$

and so

$$\beta((gu_2, gv_2), (gx, gy), (gx, gy)) \geq 1.$$

By repeating this process, we get

$$\beta((gu_n, gv_n), (gx, gy), (gx, gy)) \geq 1 \quad \text{for all } n \geq 1. \quad (3.16)$$

Since $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and F is mixed g -monotone mapping. It is easy to show that $gx \leq gu_n$ and $gy \geq gv_n$, for all $n \geq 1$. Thus from (3.1) and (3.16), we have

$$\begin{aligned} & G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy) \\ &= G(F(u_n, v_n), F(x, y), F(x, y)) + G(F(v_n, u_n), F(y, x), F(y, x)) \\ &\leq \beta((gu_n, gv_n), (gx, gy), (gx, gy))G(F(u_n, v_n), F(x, y), F(x, y)) + G(F(v_n, u_n), F(y, x), F(y, x)) \\ &\leq \psi(G(gu_n, gx, gx) + G(gv_n, gy, gy)). \end{aligned} \quad (3.17)$$

Thus from(3.17), we have

$$G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy) \leq \psi(G(gu_n, gx, gx) + G(gv_n, gy, gy)) \quad (3.18)$$

Since ψ is non-decreasing, from (3.18), we get

$$G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy) \leq \psi^n(G(gu_1, gx, gx) + G(gv_1, gy, gy)) \quad (3.19)$$

for each $n \geq 1$. Letting $n \rightarrow +\infty$ in (3.19) and using lemma 2.15. Which implies

$$\lim_{n \rightarrow \infty} G(gu_{n+1}, gx, gx) = \lim_{n \rightarrow \infty} G(gv_{n+1}, gy, gy) = 0. \quad (3.20)$$

Similarly, we obtain that

$$\lim_{n \rightarrow \infty} G(gu_{n+1}, gx^*, gx^*) = \lim_{n \rightarrow \infty} G(gv_{n+1}, gy^*, gy^*) = 0. \quad (3.21)$$

Hence, from (3.20), (3.21) and proposition 2.4, we get $gx^* = gx$ and $gy^* = gy$. Since $gx = F(x, y)$ and $gy = F(y, x)$, by commutativity of F and g , we have

$$g(gx) = g(F(x, y)) = F(gx, gy) \text{ and } g(gy) = g(F(y, x)) = F(gy, gx). \quad (3.22)$$

Denote $gx = z$ and $gy = w$. Then from (3.22)

$$gz = F(z, w) \text{ and } gw = F(w, z). \quad (3.23)$$

Therefore, (z, w) is a coupled coincidence fixed point of F and g . Then from (3.15) with $x^* = z$ and $y^* = w$. It follows $gz = gx$ and $gw = gy$, that is,

$$gz = z \text{ and } gw = w. \quad (3.24)$$

From (3.23) and (3.24), $z = gz = F(z, w)$ and $w = gw = F(w, z)$. Therefore, (z, w) is a coupled common fixed point of F and g .

To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (3.15) we have $p = gp = gz = z$ and $q = gq = gw = w$.

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