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Some Properties of Bessel Elliptic Kernel and Bessel Ultrahyperbolic Kernel

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Abstract : In this paper we introduce the distributional function families $S_\alpha(x)$ and $R^B_\alpha(u)$ named Bessel Elliptic kernel and Bessel Ultrahyperbolic kernel, where α is a complex number. In particular when $\alpha = 2k, S_{2k}(x)$ and $R^B_{2k}(u), k = 1, 2, \ldots$ appear in [1]. We found some relations between the operator Δ^k_B and Δ^k and between $R^B_\alpha(u)$ and $R^H_\alpha(u)$, where $R^H_\alpha(u)$ is Marcel Riesz Ultrahyperbolic kernel, Δ^k_B is the Laplace-Bessel operator and Δ is the Laplacian operator.

We give a sense to the properties $\Delta^k_B\{S_\alpha\}, S_\alpha * S_\beta, \Delta^k_B\{M_\alpha\}, \Delta^k_B\{N_{2k}\}, \Box^k_B\{R^B_\alpha\}$ and $\Box^k_B\{R^B_{2k}\}$, where M_α is defined by (1.17), N_{2k} by (1.49) and $\Box^k_B$ by (2.21).

Finally, we study $R^B_\alpha(u)$ at $\alpha = 0$, taking into account the signature p, q of $u(x_1, x_2, \ldots, x_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ and we obtain the equation $\Box^k_B\{E_{2k}\} = \delta$ which means that E_{2k} is elemental solution of the operator $\Box^k_B$, where E_{2k} is defined by (2.21).

Keywords : Distribution theory, Bessel elliptic kernel, Bessel ultrahyperbolic kernel.
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1 Introduction

Definition 1.1 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $S_\alpha(x)$ be the distributional function family defined by

$$S_\alpha = C_v a_\alpha |x|^{\alpha - 2|v| - n} \quad \text{if } \alpha \neq 2|v| - 2j,\ j = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$C_v = \frac{1}{\prod_{i=1}^n 2^{vi - \frac{1}{2}} \Gamma(vi + \frac{1}{2})} \tag{1.2}$$

$$|v| = v_1 + v_2 + \ldots + v_n \tag{1.3}$$

and

$$a_\alpha = \frac{2^{n+2|\nu| - 2\alpha} \Gamma(\frac{n-\alpha}{2} + |\nu|)}{\Gamma(\frac{\alpha}{2})}. \tag{1.4}$$

Letting $\alpha = 2k$ in (1.1) we obtain

$$S_{2k} = C_v.a_{2k} \left|x\right|^{2k-2|\nu|-n}. \tag{1.5}$$

The formula of distributional function family S_{2k} (see(1.5)) appear in [1].

On the other hand, from ([1], p. 377) the following formula is valid

$$F_B\left\{\left|x\right|^{-\alpha}\right\} = a_\alpha \left|y\right|^{\alpha-n-2|\nu|} \tag{1.6}$$

([1], p. 377) where F_B is the Fourier-Bessel Transform defined by

$$(F_B f)(x) = C_v \int_{R_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i) y_i^{2\nu i}\right) dy \tag{1.7}$$

([1], p.377),

$$F_B^{-1}\left\{f\right\}(y) = F_B\left\{f\right\}(-y) \tag{1.8}$$

([1], p. 377), $J_{v_i-\frac{1}{2}}(x_i, y_i)$ is normalized Bessel function which is the eigenfunction of the Bessel differential operator ([2] and [3]).

From (1.6) and using (1.8) we have,

$$F_B^{-1}\left\{\left|y\right|^{-\beta}\right\}(x) = C_v a_\beta \left|x\right|^{\beta-n-2|\nu|}. \tag{1.9}$$

Putting $\beta = k + m$, $k, m = 1, 2, \ldots$, in (1.10) and using (1.5) we have

$$F_B^{-1}\left\{\left|y\right|^{-k-m}\right\} = S_{k+m}(x). \tag{1.10}$$

The formula (1.10) appear in ([1], p. 380).

From (1.9) and using (1.8) the Fourier Bessel transform of S_α is given by the formula

$$F_B\left\{S_\alpha(x)\right\}(y) = \left|y\right|^{-\alpha} \tag{1.11}$$

Using that $S_\alpha(x)$ is a tempered distribution for $\alpha - 2\left|\nu\right| - n \neq -n - n - 2, \ldots$, then the convolution product of $S_\alpha * S_\beta$ exists and is a tempered distributions for $\alpha, \beta \neq 2\left|\nu\right|, 2\left|\nu\right| - 2, 2\left|\nu\right| - 4, \ldots$.

Now using the properties $F_B\left\{f * g\right\}(x) = F_B f(x).F_B f(x)$. ([1], p.377) then the following formula is valid

$$\begin{aligned} F_B\left\{S_\alpha * S_\beta\right\} &= F_B\left\{S_\alpha\right\}.F_B\left\{S_\beta\right\} \\ &= \left|y\right|^{-\alpha}.\left|y\right|^{-\beta} = \left|y\right|^{-(\alpha+\beta)} \end{aligned} \tag{1.12}$$

if $\alpha, \beta \neq 2\left|\nu\right|, 2\left|\nu\right| - 2, 2\left|\nu\right| - 4, \ldots$.

From (1.12) and using (1.9) we have the following property :

$$S_\alpha * S_\beta = F_B^{-1}\left\{\left|y\right|^{-(\alpha+\beta)}\right\} = C_v a_{\alpha+\beta} \left|x\right|^{\alpha+\beta-n-2|\nu|} = S_{\alpha+\beta}. \tag{1.13}$$

Letting $\alpha = 2k, k = 1, 2, \ldots,$ and $\beta = 2m, m = 1, 2, \ldots$ in (1.13) we have

$$S_{2k} * S_{2m} = S_{2(k+m)}, \tag{1.14}$$

the formula (1.14) appears in ([1], p. 380).

We observe from (1.11) and using the property

$$F_B \{\delta(x)\} = 1 \tag{1.15}$$

where δ is the distribution of Dirac´s delta, we have,

$$S_0 = F_0^{-1} \{1\} = \delta \tag{1.16}$$

On the other hand, from ([1], p. 379, Lemma 3),

$$u(x) = (-1)^k S_{2k} \tag{1.17}$$

is elemental solution of Laplace-Bessel operator iterated $k-$ times defined by

$$\Delta_B^k = (B_{x_1} + \cdots + B_{x_n})^k = \left(\sum_{i=1}^{n} B_{x_i} \right)^k, \tag{1.18}$$

i.e.,

$$\Delta_B^k \left\{ (-1)^k S_{2k} \right\} = \delta \tag{1.19}$$

where

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\nu_i}{\partial x_i} . \frac{\partial}{\partial x_i} \tag{1.20}$$

$$2\nu_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, x_i > 0 \tag{1.21}$$

Lemma 1.2 *Let Δ_B^k the Laplace-Bessel operator iterated k- times defined by(1.18) then following formula is valid.*

$$\Delta_B^k \{S_\alpha(x)\} = (-1)^n S_{\alpha - 2k}(x). \tag{1.22}$$

Proof. From (1.1) and (1.12) and using the property

$$F_B \left\{ \left(\Delta_B^k \right) f \right\}(x) = (-1)^k |y|^{2k} F_B f \tag{1.23}$$

([1], p. 384) and (c.f. [4]). we have

$$\begin{aligned} F_B \left\{ \left(\Delta_B^k S_\alpha \right) \right\} &= (-1)^k |y|^{2k} F_B \{S_\alpha\} \\ &= (-1)^k |y|^{2k} |y|^{-\alpha} \\ &= (-1)^k |y|^{-(\alpha - 2k)} \end{aligned} \tag{1.24}$$

From (1.24)and using (1.9) we obtain

$$\begin{aligned}\Delta_B^k\left\{S_\alpha\right\} &= F_B^{-1}\left\{(-1)^k\left|y\right|^{-(\alpha-2k)}\right\}\\ &= (-1)^k C_v a_{\alpha-2k}(x)^{\alpha-2k-2|\nu|-n}\\ &= (-1)^k S_{\alpha-2k} \qquad (1.25)\end{aligned}$$

that complete the proof. □

In particular letting $\alpha = 2k$ in (1.22) and using (1.16) we obtain

$$\Delta_B^k\left\{(-1)^k S_{2k}(x)\right\} = \delta, \qquad (1.26)$$

which means that $(-1)^K S_{2k}(x)$ is elemental solution of Laplace-Bessel operator iterated k times.

Definition 1.3 Let be the function defined by

$$M_\alpha = b_\alpha\left|x\right|^{\alpha-n-2|\nu|} \qquad (1.27)$$

where

$$b_\alpha = \frac{1}{\Gamma(\frac{\alpha-n-2|\nu|}{2}+1)2^\alpha\Gamma(\frac{\alpha}{2})}. \qquad (1.28)$$

Lemma 1.4 *Let Δ_B be the Laplace-Bessel operator defined by (1.18) and Δ the Laplace operator defined by*

$$\Delta = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i}. \qquad (1.29)$$

Then the following formula is valid

$$\Delta_B^k\left\{\delta(x)\right\} = \frac{(-1)^k\Gamma(|\nu|)}{\Gamma(|\nu|-k)k!}\Delta^k\left\{\delta(x)\right\}. \qquad (1.30)$$

Proof. By direct calculus we have the following formulae

$$r^\lambda = \frac{\Gamma(\frac{\lambda}{2}+|\nu|+\frac{n}{2})\Gamma(-\frac{\lambda}{2}-k)}{2^{2k}(-1)^k\Gamma(-\frac{\lambda}{2})\Gamma(\frac{\lambda}{2}+|\nu|+\frac{n}{2}+k)}\Delta_B^k\left\{r^{\lambda+2k}\right\} \qquad (1.31)$$

where $r = \left|x\right|$.

Putting $\lambda = \alpha - 2\left|\nu\right|$ in (1.31) we have

$$r^{\alpha-2|\nu|} = \frac{\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{-\alpha}{2}+|\nu|-k)}{2^{2k}(-1)^k\Gamma(-\frac{\alpha}{2}+|\nu|)\Gamma(\frac{n+\alpha}{2}+k)}\Delta_B^K\left\{r^{\alpha-2|\nu|+2k}\right\} \qquad (1.32)$$

Now considering the following formula:

$$\operatorname*{Res}_{\lambda=-n-2s, s=0,1,2,....} r^{\lambda} = \frac{\Omega_n \Delta^s \{\delta\} \Gamma(\frac{n}{2})}{2^{2s} s! \Gamma(\frac{n}{2}+s)} \tag{1.33}$$

([5], p.792, formula (2.1)) and

$$\begin{aligned}\operatorname*{Res}_{\beta=-n} < \Delta_B^k |x|^B, \varphi > &= \operatorname*{Res}_{\beta=-n} < |x|^{\beta}, \Delta_B^k \varphi > \\ &= \Omega_n < \delta, \Delta_B^k \varphi > = \Omega_n \Delta_B^k \{\varphi(0)\} \\ &= \Omega_n < \Delta_B^k \delta, \varphi > \end{aligned} \tag{1.34}$$

for all $\varphi \in D$(test space), we have

$$\begin{aligned}\operatorname*{Res}_{\alpha-2|\nu|=-n-2k} < r^{\alpha-2|\nu|}, \varphi > &= \frac{\Gamma(|\nu|-k)\Gamma(\frac{n}{2})}{2^{2k}(-1)^k \Gamma(\frac{n}{2}+k)} . \operatorname*{Res}_{\alpha-2|\nu|=-n-2k} < \Delta_B^k r^{\alpha-2|\nu|+2k}, \varphi > \\ &= \frac{\Gamma(|\nu|-k)\Gamma(\frac{n}{2})\Omega_n}{2^{2k}(-1)^k\Gamma(\frac{n}{2}+k)\Gamma(|\nu|)} \operatorname{Res} < \Delta_B^k \delta, \varphi > .\end{aligned} \tag{1.35}$$

In consequence we have

$$\frac{\Omega_n \Delta^k \delta \Gamma(\frac{n}{2})}{2^{2k} k! \Gamma(\frac{n}{2}+k)} = \frac{\Omega_n \Gamma(|\nu|-k)\Gamma(\frac{n}{2})}{2^{2k} k! \Gamma(\frac{n}{2}+k)\Gamma(|\nu|)} \Delta_B^k \delta, \tag{1.36}$$

from (1.36) we obtain the formula (1.30). □

In particular putting $k=1$ in (1.30) we obtain

$$\begin{aligned}\Delta_B \delta(x) &= -\frac{\Gamma(|\nu|)}{\Gamma(|\nu|-1)} \Delta\delta(x) \\ &= -(|\nu|-1)\Delta\delta(x) \\ &= (1-|\nu|)\Delta\delta(x) \\ &= \left(1 - \sum_{i=1}^{n} \nu_i\right) \Delta\delta(x) \\ &= \Delta\delta(x) - \left(\sum_{i=1}^{n} \nu_i\right) \Delta\delta(x).\end{aligned} \tag{1.37}$$

In fact, from (1.18) and (1.20) we have

$$\Delta_B \delta(x) = \sum \frac{\partial^2}{\partial x_i} \delta + 2 \sum \frac{\nu_i}{x_i} \frac{\partial}{\partial x_i} \delta = \Delta\delta + 2 \sum \frac{\nu_i}{x_i} \frac{\partial}{\partial x_i} \delta. \tag{1.38}$$

Now using that

$$\delta(x) = R_0(x) = \lim_{\alpha \to 0} R_\alpha(x) \tag{1.39}$$

and

$$R_{-2j}(x) = (-1)^j \Delta^j \delta, j = 0, 1, \ldots, \tag{1.40}$$

where

$$R_\alpha(x) = \frac{r^{\alpha-n}}{D_n(\alpha)} \tag{1.41}$$

elliptic kernel of Marcel Riesz [6] and

$$D_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}, \tag{1.42}$$

by direct computer we have

$$\frac{1}{x_i}\frac{\partial}{\partial x_i}\delta = \frac{1}{2}R_{-2}(x) = -\frac{1}{2}\Delta\delta \tag{1.43}$$

where Δ is defined by (1.29). From (1.38) and (1.43) we obtain the property (1.37).

Lemma 1.5 *Let M_a be the distributional family defined by (1.27), then following formula is valid*

$$\Delta_B^k M_\alpha = M_{\alpha-2k} \tag{1.44}$$

where $k = 0, 1, 2, \ldots$.

Proof. From (1.27), (1.28) and considering (1.32) we have

$$\begin{aligned} \Delta_B^k M_\alpha &= b_\alpha \Delta_B^k |x|^{\alpha-n-2|\nu|} \\ &= b_\alpha \frac{2^{2k}(-1)^k \Gamma(\frac{n-\alpha}{2}+k+|\nu|)\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}-k)\Gamma(\frac{n-\alpha}{2}+|\nu|)} |x|^{\alpha-n-2|\nu|-2k}. \end{aligned} \tag{1.45}$$

Using the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi} \tag{1.46}$$

we have

$$\begin{aligned} \Gamma(\frac{n-\alpha}{2}+k+|\nu|) &= \frac{\pi}{\sin(\frac{n-\alpha}{2}+k+|\nu|)\Gamma(1-(\frac{n-\alpha}{2}+k+|\nu|))} \\ &= \frac{\pi}{\sin(\frac{n-\alpha}{2}+|\nu|)\pi(-1)^k\Gamma(\frac{n-\alpha}{2}-k-|\nu|+1)} \\ &= \frac{\Gamma(\frac{n-\alpha}{2}+|\nu|)\Gamma(1-(\frac{n-\alpha}{2}+k+|\nu|))}{(-1)^k\Gamma(1+\frac{\alpha-n}{2}-k-|\nu|)}. \end{aligned} \tag{1.47}$$

From (1.46) and (1.47) we have

$$\begin{aligned} \Delta_B^k M_\alpha &= \frac{1}{2^\alpha\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n-2|\nu|}{2}+1)} \frac{2^{2k}(-1)^k\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}-k)\Gamma(\frac{n-\alpha}{2}+|\nu|)} \frac{\Gamma(\frac{n-\alpha}{2}+|\nu|)\Gamma(1+(\frac{n-\alpha}{2}+|\nu|)}{(-1)^k\Gamma(1+(\frac{n-\alpha}{2}-k-|\nu|)} \\ &= \frac{1}{2^{\alpha-2k}\Gamma(\frac{\alpha}{2}-k)\Gamma(1+\frac{\alpha-n}{2}-k-|\nu|)} \\ &= M_{\alpha-2k}. \end{aligned} \tag{1.48}$$

□

Lemma 1.6 *Let N_{2k} be the distribution family defined by*

$$N_{2k}(x) = \frac{\Gamma(|\nu|)\,|x|^{2k-n}}{\Gamma(k+|\nu|)2^{2k}\Gamma(k-\frac{n}{2}+1)(-1)^{\frac{n}{2}+1}\pi^{\frac{n}{2}}} \tag{1.49}$$

then $N_{2k}(x)$ is elemental solution of operator Δ_B^k,where Δ_B^k is defined by (1.18).

Proof. From (1.27) we have,

$$M_{2|\nu|} = \lim_{\alpha\to 2|\nu|} M_\alpha = \frac{1}{2^{2|\nu|}\Gamma(|\nu|)} \lim_{\alpha\to 2|\nu|} \frac{x^{\alpha-n-2\alpha\to 2|\nu|}}{\Gamma(\frac{\alpha-n-2|\nu|}{2}+1)}. \tag{1.50}$$

Making $\beta = \alpha - n - 2\,|\nu|$, using the formula (1.33) and the following formula

$$\operatorname*{Res}_{z=-j,j=0,1,2,\ldots} \Gamma(z) = \frac{(-1)^j}{j!} \tag{1.51}$$

where $\Gamma(z)$ is defined by (3.2) we have

$$\begin{aligned}
\lim_{\beta\to -n} \frac{r^\beta}{\Gamma(\frac{\beta}{2}+1)} &= \lim_{\beta\to -n} \frac{(\beta+n)r^\beta}{\beta+n\Gamma(\frac{\beta}{2}+1)} \\
&= \frac{\operatorname*{Res}_{\beta=-n} r^\beta}{\lim_{\beta\to -n}(\beta+n)\frac{\beta}{2}\Gamma(\frac{\beta}{2})} \\
&= \frac{\Omega_n\delta(x)(\frac{n}{2}-1)!}{2(-1)^{\frac{n}{2}-1}}.
\end{aligned} \tag{1.52}$$

From (1.50) and using (1.52) we have

$$M_{2|\nu|} = \frac{(-1)^{\frac{n}{2}+1}\pi^{\frac{n}{2}}\delta(x)}{2^{2|\nu|}\Gamma(|\nu|)}. \tag{1.53}$$

Letting $\alpha = 2\,|\nu| + 2k$ in (1.44) and using (1.53) we have

$$\Delta_B^k M_{2|\nu|+2k} = M_{2|\nu|} = \frac{(-1)^{\frac{n}{2}+1}\pi^{\frac{n}{2}}\delta(x)}{2^{2|\nu|}\Gamma(|\nu|)}. \tag{1.54}$$

Now taking into account (1.27) and (1.28) we have

$$\Delta_B^k\left\{N_{2k}(x)\right\} = \Delta_B^k\left\{M_{2|\nu|+2k}\cdot\frac{2^{2|\nu|}\Gamma(|\nu|)}{(-1)^{\frac{n}{2}+1}\pi^{\frac{n}{2}}}\right\} = \delta(x). \tag{1.55}$$

The formula(1.55) means that $N_{2k}(x)$ is elemental solution of the Laplace-Bessel operator iterated $k-$ times Δ_B^k defined by(1.18). □

2 The Bessel ultrahyperbolic KernelR_α^B

Definition 2.1 Let $x = (x_1, \ldots, x_n)$ be a point of R^n and write

$$u = u(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

where $p + q = n$ is the dimension of the space.Denote by Γ_+ the interior of the forward cone defined by

$$\Gamma_+ = \{x \in \mathbb{R}^n : u > 0, x_i > 0, i = 1, 2, \ldots, p\}, \tag{2.2}$$

$\overline{\Gamma}_+$designate its closure. Similarly we define

$$\Gamma_- = \{x \in \mathbb{R}^n : u > 0, x_i < 0, i = 1, 2, \ldots, p\} \tag{2.3}$$

and $\overline{\Gamma}_-$designate its closure. For any complex number α, we define the distributional function family $R_\alpha^B(u)$ by the following form

$$R_\alpha^B(u) = \begin{cases} \frac{u^{\frac{\alpha-2|\nu|-n}{2}}}{K_n^{|\nu|}(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.4}$$

where

$$K_n^{|\nu|}(\alpha) = \frac{\pi^{\frac{n-1+2|\nu|}{2}} \Gamma(\frac{\alpha-n-2|\nu|+2}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{\alpha-p-2|\nu|+2}{2}) \Gamma(\frac{p-\alpha}{2})} \tag{2.5}$$

and $|\nu|$ is defined by (1.3).

The distributional function family $R_\alpha^B(u)$ is an ordinary function if Re$(\alpha - 2|\nu|) \geq n$ and is a distribution of $\alpha - 2|\nu|$ if $Re(\alpha - 2|\nu|) < n$.

Let supp $R_\alpha^B(u)$ be denote the support of $R_\alpha^B(u)$. Suppose supp $R_\alpha^B(u) \subset \overline{\Gamma}_+$
We shall call $R_\alpha^B(u)$ the Bessel Ultrahyperbolic Kernel. Letting $\alpha = 2k$ in (2.4) and (2.5) we obtain

$$R_{2k}^B(u) = \frac{u^{\frac{2k-n-2|\nu|}{2}}}{K_n(2k)} \tag{2.6}$$

where

$$K_n(2k) = \frac{\pi^{\frac{n+2|\nu|-1}{2}} \Gamma(\frac{2+2k-n-2|\nu|}{2}) \Gamma(\frac{1-2k}{2}) \Gamma(2k)}{\Gamma(\frac{2+2k-p-2|\nu|}{2}) \Gamma(\frac{p-2k}{2})}. \tag{2.7}$$

By putting $|\nu| = 0$ in (2.4) and (2.5) the formulae (2.4) and (2.5) reduces to

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+ \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.8}$$

and

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{\alpha-n}{2}+1)\Gamma(\frac{n-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{\alpha-p}{2}+1)\Gamma(\frac{n-\alpha}{2})}. \tag{2.9}$$

$R^H_\alpha(u)$ is precisely the Marcel Ultrahyperbolic Kernel, $R^H_\alpha(u)$ was introduced by Y. Nozaki [7].

$R^H_\alpha(u)$ has the following properties :

$$R^H_0(u) = \delta(x) \tag{2.10}$$

$$R^H_{-2k}(u) = \Box^k\left\{\delta(x)\right\} \tag{2.11}$$

$$\Box^k R^H_\beta(u) = R^H_{\beta-2k} \tag{2.12}$$

and

$$\Box^k R^H_{2k} = R^H_0 = \delta(x) \tag{2.13}$$

where

$$\Box^k = \left\{\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right\}^k \tag{2.14}$$

is ultrahyperbolic operator.

We observe that if $|\nu| \neq 0$ the following relationship is true

$$R^B_\alpha(u) = h_{\alpha,p,|\nu|}R^H_{\alpha-2|\nu|}(u) \tag{2.15}$$

where

$$h_{\alpha,p,|\nu|} = \frac{\Gamma(\frac{p-\alpha}{2}+|\nu|)\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{1-\alpha}{2}+|\nu|)\Gamma(\alpha-|\nu|)\Gamma(\frac{p-\alpha}{2})}. \tag{2.16}$$

If $|\nu| = l, l = 1,2,3,\ldots$, and using the formula

$$\frac{\Gamma(z)}{\Gamma(z-l)} = \frac{(-1)^l\Gamma(-z+l+1)}{\Gamma(1-z)} \tag{2.17}$$

$h_{\alpha,p,|\nu|}$ can be rewritten in the following form

$$h_{\alpha,p,|\nu|} = \frac{\Gamma(\frac{p-\alpha}{2}+|\nu|)\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{1-\alpha}{\alpha}+|\nu|)}.\left[\frac{(-1)^{|\nu|}\Gamma(-\alpha-|\nu|+1)}{\Gamma(1-\alpha)}\right]\frac{1}{\Gamma(\frac{p-\alpha}{2})}. \tag{2.18}$$

On the other hand, letting $\alpha = -2k$ in (2.15), we have the following property

$$R^B_{-2k}(u) = h_{-2k,p,|\nu|}R^H_{-2k-2|\nu|}(u). \tag{2.19}$$

From (2.15) and considering (2.11) we have

$$R^B_{-2k}(u) = h_{-2k,p,|\nu|}\Box^{k+|\nu|}\left\{\delta(x)\right\} \tag{2.20}$$

if |v| is positive integer(or if $|\nu| = 1, 2, 3, \ldots$), where $\Box^k$ is defined by (2.14) and

$$h_{-2k,p,|\nu|} = \frac{\Gamma(\frac{p}{2}+k+|\nu|)\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{1}{2}+k+|\mathrm{v}|)} \frac{(-1)^{|\nu|}\Gamma(2k+|\nu|+1)}{\Gamma(1+2k)} \frac{1}{\Gamma(\frac{p}{2}+k)}. \quad (2.21)$$

Lemma 2.2 *Let $R^B_\alpha(u)$ be the function defined by (2.4) and let $\Box^K_B$ be the Bessel ultrahyperbolic operator iterated $k-$times defined by*

$$\Box^k_B = (Bx_1 + \cdots + Bx_p - Bx_{p+1} - \cdots - Bx_{p+q})^k \quad (2.22)$$

([1], p. 379), where $p+q=n$ dimension of the space

$$Bx_i = \frac{\partial^2}{\partial x_i^2} + 2\frac{\nu_i}{x_i}\frac{\partial}{\partial x_i} \quad (2.23)$$

([1],p. 375) and $2v_i$ is defined by (1.21), then the following formula is valid

$$\Box^k_B \left\{R^B_\alpha(u)\right\} = d_{\alpha,p,|\nu|,k} R^B_{\alpha-2k}(u) \quad (2.24)$$

where

$$d_{\alpha,p,|\nu|,k} = \frac{(\frac{\alpha-p}{2}-|\nu|)(\frac{\alpha-p}{2}-|\nu|-1)\ldots(\frac{\alpha-p}{2}-|\nu|-(k-1))}{(\frac{\alpha-p}{2})(\frac{\alpha-p}{2}-1)\ldots(\frac{\alpha-p}{2}-(k-1))}. \quad (2.25)$$

Proof. From (2.4)and using (2.22) and (2.25) we have

$$\begin{aligned}\Box_B\left\{R^B_\alpha(u)\right\} &= \Box_B\left\{\frac{u^{\frac{\alpha-n}{2}-|\nu|}}{K^{|\nu|}_n(\alpha)}\right\}\\ &= \frac{1}{K^{|\nu|}_n(\alpha)}\left\{\Box + D_p - D_q\right\} u^{\frac{\alpha-n}{2}-|\nu|}\\ &= \frac{1}{K^{|\nu|}_n(\alpha)}\left\{\Box u^{\frac{\alpha-n}{2}-|\nu|} + D_p u^{\frac{\alpha-n}{2}-|\nu|} - D_q u^{\frac{\alpha-n}{2}-|\nu|}\right\}, \end{aligned} \quad (2.26)$$

where

$$\Box = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}, \quad (2.27)$$

$$D_p = \sum_{i=1}^{p} 2\nu_i \frac{\partial^2}{\partial x_i^2} \quad (2.28)$$

and

$$D_q = \sum_{i=p+1}^{p+q} 2\nu_i \frac{\partial^2}{\partial x_i^2}\ . \quad (2.29)$$

Using the propertie

$$\Box P^{\lambda+1} = 2(\lambda+1)(2\lambda+n)P^\lambda$$

([8], p. 258, formula 825), where $P = u = u(x)$ is defined by (2.1), we have

$$\square\left\{u^{\frac{\alpha-n}{2}-|\nu|}\right\} = 2(\frac{\alpha-n}{2}-|\nu|).2(\frac{\alpha-n}{2}-|\nu|-1+\frac{n}{2})u^{\frac{\alpha-n}{2}-|\nu|-1} \tag{2.30}$$

By direct calculus we have,

$$D_p u^{\frac{\alpha-n}{2}-|\nu|} = 2^2(\nu_1+\nu_2+\cdots+\nu_p)(\frac{\alpha-n}{2}-|\nu|)u^{\frac{\alpha-n}{2}-|\nu|-1} \tag{2.31}$$

and

$$D_q u^{\frac{\alpha-n}{2}-|\nu|} = -2^2(\nu_{p+1}+\nu_{p+2}+\cdots+\nu_{p+q})(\frac{\alpha-n}{2}-|\nu|)u^{\frac{\alpha-n}{2}-|\nu|-1}. \tag{2.32}$$

From (2.26) and using (2.30), (2.31) and (2.32) we have

$$\square_B\left\{R^B_\alpha(u)\right\} = \frac{1}{K^{|\nu|}_n(\alpha)}2^2(\frac{\alpha-n}{2}-|\nu|)(\frac{\alpha}{2}-1)u^{\frac{\alpha-n}{2}-|\nu|-1}. \tag{2.33}$$

On the other hand, from (2.5)and using the formula

$$\Gamma(z+l) = z(z+1)\ldots(z+l-1)\Gamma(z), l = 1,2,\ldots, \tag{2.34}$$

we have

$$K^{|\nu|}_n(\alpha-2) = K^{|\nu|}_n(\alpha)\frac{(\frac{\alpha-p-2|\nu|)}{2})}{(\frac{\alpha-n-2|\nu|)}{2})(\frac{\alpha-p}{2})2^2(\frac{\alpha}{2}-1)}. \tag{2.35}$$

Therefore, from (2.33) and (2.34) we obtain

$$\square_B\left\{R^B_\alpha(u)\right\} = \frac{(\frac{\alpha-p-2|\nu|)}{2})}{(\frac{\alpha-p}{2})}R^B_{\alpha-2}(u). \tag{2.36}$$

From (2.36) we have

$$\square^2_B\left\{R^B_\alpha(u)\right\} = \square_B\left\{\square_B R^B_\alpha(u)\right\} = \frac{(\frac{\alpha-p-2|\nu|)}{2})((\frac{\alpha-p}{2})-|\nu|-1)}{(\frac{\alpha-p}{2})(\frac{\alpha-p}{2}-1)}R^B_{\alpha-4}(u), \tag{2.37}$$

by iteration, we obtain

$$\square^k_B\left\{R^B_\alpha(u)\right\} = \frac{(\frac{\alpha-p}{2}-|\nu|)((\frac{\alpha-p}{2})-|\nu|-1)\ldots((\frac{\alpha-p}{2})-|\nu|-(k-1))}{(\frac{\alpha-p}{2})(\frac{\alpha-p}{2}-1)\ldots((\frac{\alpha-p}{2}-1)-(k-1))}R^B_{\alpha-2k}(u) \tag{2.38}$$

for $k = 1,2,3,\ldots$.

From (2.38) we obtain the formula (2.24). □

We observe that using the formula

$$(-1)^n z(z+1)\ldots(z-(s-1)) = \frac{(-1)^s\Gamma(z+1)}{\Gamma(z-s+1)} = \frac{\Gamma(-z+s)}{\Gamma(-z)} \tag{2.39}$$

if $s = 1, 2, \ldots, d_{\alpha,p,|\nu|,k}$ defined by (2.25), can be rewritten in the following form

$$d_{\alpha,p,|\nu|,k} = \frac{\Gamma(\frac{\alpha-p}{2} - |\nu| + 1)}{\Gamma(\frac{\alpha-p}{2} - |\nu| + 1 - k)} . \frac{\Gamma(\frac{\alpha-p}{2} - k + 1)}{\Gamma(\frac{\alpha-p}{2} + 1)} \tag{2.40}$$

and

$$d_{\alpha,p,|\nu|,k} = \frac{(-1)^k \Gamma(-(\frac{\alpha-p}{2} - |\nu|) + k)}{\Gamma(-(\frac{\alpha-p}{2} - |\nu|))} . \frac{\Gamma(-(\frac{\alpha-p}{2}))}{\Gamma(-(\frac{\alpha-p}{2}) + k)} \tag{2.41}$$

In the next section we are going to $R^B_\alpha(u)$ for the case $\alpha = 0$.

3 R_0

From (2.4) and (2.5) we have

$$\begin{aligned} R^B_0(u) &= \lim_{\alpha\to 0} R^B_\alpha(u) \\ &= \lim_{\alpha\to 0} \frac{\Gamma(\frac{\alpha-p-2|\nu|+2}{2})\Gamma(\frac{p-\alpha}{2})}{2^{-1}\pi^{\frac{n+2|\nu|}{2}}\Gamma(\frac{\alpha-n-2|\nu|+2}{2})} . \lim_{\alpha\to 0} \frac{u^{\frac{\alpha-2|\nu|-n}{2}}}{\Gamma(\frac{\alpha}{2})} \end{aligned} \tag{3.1}$$

Now we consider three cases:
Case 1 : q even, p odd (n odd)

In this case considering that $P^\lambda = u^\lambda$ has simple poles at $\lambda = -k, k = 1, 2, \ldots$ and $\lambda = -\frac{n}{2} - l, l = 0, 1, 2, \ldots$([8], page...) then $u^{\frac{\alpha-n}{2}-|\nu|}$ has simple poles at $\alpha = 0$ when $|\nu| = s = 0, 1, 2, \ldots$.

Now taking into account that

$$\begin{aligned} \operatorname*{Res}_{\lambda=-\frac{n}{2}-s, s=0,1,2,\ldots} P^\lambda_+ &= \lim_{\lambda\to -\frac{n}{2}-s} (\lambda + \frac{n}{2} + s) P^\lambda_+ \\ &= \lim_{\gamma\to 0} \gamma P_+^{\gamma-\frac{n}{2}-s} \\ &= \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{2^{2s}s!\Gamma(\frac{n}{2}+s)} \square^s \{\delta(x)\} \end{aligned} \tag{3.2}$$

if q is even and p odd ([8], p. 260, formula 30) and using the formula (1.51) where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \tag{3.3}$$

we have

$$\lim_{\alpha\to 0} \frac{u^{\frac{\alpha-n}{2}-|\nu|}}{\Gamma(\frac{\alpha}{2})} = \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{22^{2s}s!\Gamma(\frac{n}{2}+s)} \square^s \{\delta(x)\} \tag{3.4}$$

if $|\nu| = s = 0, 1, 2, \ldots$.

From (3.1) and (3.4), we obtain the following formula

$$R_0^B(u) = \frac{\Gamma(\frac{p}{2})(-1)^s \pi^{\frac{n}{2}}}{\pi^{\frac{n}{2}+s} s! \Gamma(\frac{p}{2}+s)} \Box^s \{\delta(x)\} \tag{3.5}$$

when $|\nu| = s = 0, 1, 2, \ldots$.

The properties (3.5) generalized the property (2.10). In fact, letting $s = |\nu| = 0$ in (2.4) and considering (2.5), (2.8) and (2.9) we have

$$R_\alpha^B(u) = R_\alpha^H(u) \tag{3.6}$$

if $|\nu| = 0$. Now from (2.22), (2.23), (2.14) and using the property

$$\Box_B^k = \Box^k \tag{3.7}$$

if $|\nu| = 0$, we have

$$R_0^H(u) = R_0^B(u) = \delta(x) \tag{3.8}$$

if $|\nu| = 0$.

On the other hand if $|\nu| \neq 0, 1, 2, \ldots$ but if $|\nu| = -\frac{n}{2} + l, l = 1, 2, \ldots$ with n odd, $u^{\frac{\alpha-n}{2}-|\nu|}$ at $\alpha = 0$ has simple poles, therefore

$$\begin{aligned}
\lim_{\alpha\to 0} \frac{u^{\frac{\alpha-n}{2}-|\nu|}}{\Gamma(\frac{\alpha}{2})} &= \lim_{\alpha\to 0} \frac{u^{\frac{\alpha-n}{2}-(-\frac{n}{2}+l)}}{\Gamma(\frac{\alpha}{2})} \\
&= \lim_{\lambda\to -l} \frac{u^\lambda}{\Gamma(\lambda+l)} \\
&= \lim_{\lambda\to -l} \frac{(\lambda+l)u^\lambda}{(\lambda+l)\Gamma(\lambda+l)} \\
&= \frac{1}{(-1)^l l!} \frac{\operatorname*{Res}_{\lambda=-l,l=1,2,\ldots} u^\lambda}{\operatorname*{Res}_{\lambda=-l,l=1,2,\ldots} \Gamma(\lambda)}.
\end{aligned} \tag{3.9}$$

On the other hand, we know that the following formula is true

$$\operatorname*{Res}_{\lambda=-l,l=1,2,\ldots} u^\lambda = \frac{(-1)^{l-1}}{(l-1)!} \delta^{(l-1)}(u) \tag{3.10}$$

if p is odd and q is even ([8], p. 256). From (3.1) and considering (**??**) and (3.10) we obtain

$$R_0^H(u) = \begin{cases} \frac{(-1)^{l-1}(-1)^{\frac{q}{2}}\Gamma(\frac{p}{2})}{(l-1)!\Gamma(l-\frac{q}{2})\pi^l 2^{-1}} \delta^{(l-1)}(u), & \text{if } l > \frac{q}{2} \\ 0, & \text{if } l \le \frac{q}{2} \end{cases} \tag{3.11}$$

Summary, for the case q even and p odd we have

$$R_0^H(u) = \frac{\Gamma(\frac{p}{2})(-1)^s}{\Gamma(\frac{p}{2}+s)\pi^s 2^{2s} s!} \Box^s \{\delta(x)\} \tag{3.12}$$

if $s=|\nu|, |\nu|=0,1,2,\ldots,|\nu|=-\frac{n}{2}+l.l=1,2,\ldots$ and

$$R_0^H(u)=\begin{cases}\frac{(-1)^{l-1}(-1)^{\frac{q}{2}}\Gamma(\frac{p}{2})}{(l-1)!\Gamma(l-\frac{q}{2})\pi^l 2^{-1}}\delta^{(l-1)}(u), & \text{if } l>\frac{q}{2}, |\nu|=-\frac{n}{2}+l, l=1,2,\ldots;\\ 0, & \text{if } l\leq\frac{q}{2}, |\nu|=-\frac{n}{2}+l.l=1,2,\ldots;\\ 0, & \text{if } |\nu|\neq 0,1,2,\ldots \text{ and } |\nu|\neq -\frac{n}{2}+l.l=1,2,\ldots.\end{cases} \tag{3.13}$$

Case 2 : q even and p even (n even)
To study this case we need the following formula

$$\underset{\lambda=-\frac{n}{2}-s,s=0,1,2,\ldots}{\text{Res}} P_+^\lambda = \lim_{\lambda\to-\frac{n}{2}-s}(\lambda+\frac{n}{2}+s)P_+^\lambda$$
$$=\lim_{\gamma\to 0}\gamma P_+^{\gamma-\frac{n}{2}-s}$$
$$=\frac{(-1)(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{2^{2s}s!\Gamma(\frac{n}{2}+s)}\Box^{s-\frac{n}{2}+1}\{\delta(x)\} \text{ if } s\geq\frac{n}{2}-1, \tag{3.14}$$

q even and p even [9]. From (3.1) and using (3.14) we have

$$R_0^B(u)=\frac{(-1)\Gamma(\frac{p}{2})(-1)^s}{\Gamma(\frac{p}{2}+s)\pi^s 2^{2s}s!}\Box^{s-\frac{n}{2}+1}\{\delta(x)\} \tag{3.15}$$

if $s\geq\frac{n}{2}-1$ and $|\nu|=s, s=0,1,2,\ldots$ In particular if $s=|\nu|=\frac{n}{2}-1$, from (3.14) we have

$$R_0^B(u)=\frac{\Gamma(\frac{p}{2})(-1)^{\frac{n}{2}}}{\Gamma(\frac{p}{2}+\frac{n}{2}-1)\pi^{\frac{n}{2}-1}2^{n-2}(\frac{n}{2}-1)!}\delta(x) \tag{3.16}$$

Case 3 : p odd and q odd (n even)
To study this case we need the following formula

$$\underset{\lambda=-\frac{n}{2}-s,s=0,1,2,\ldots}{\text{Res}} P_+^\lambda = \lim_{\lambda\to-\frac{n}{2}-s}(\lambda+\frac{n}{2}+s)P_+^\lambda$$
$$=\lim_{\gamma\to 0}\gamma P_+^{\gamma-\frac{n}{2}-s}$$
$$=\frac{(-1)(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2s}s!\Gamma(\frac{n}{2}+s)}\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\Box^{s-\frac{n}{2}+1}\{\delta(x)\}, \tag{3.17}$$

if $s\geq\frac{n}{2}-1$ ([9]), where

$$\psi(x)=\frac{\Gamma^{`}(x)}{\Gamma(x)}, \tag{3.18}$$

$$\psi(k)=-C+1+\frac{1}{2}+\cdots+\frac{1}{k-1}, k=2,3,\ldots \tag{3.19}$$

$$\psi(k+\frac{1}{2}) = -C - 2\ln 2 + 2\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}\right), k=1,2,\dots \tag{3.20}$$

and C is Euler´s constant.

Now using from (2.4), using (1.46), 2.5) and (3.1) we have

$$R_0^H(u) = \frac{1}{\pi^{\frac{1}{2}}}\frac{\Gamma(1-\frac{p}{2}-s)\Gamma(\frac{p}{2})(-1)(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{\Gamma(1-\frac{n}{2}+s)\Gamma(\frac{p}{2}+s)2^{2s}s!}$$
$$\cdot\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\Box^{s-\frac{n}{2}+1}\{\delta(x)\} \tag{3.21}$$

if $1-\frac{n}{2}+s>0$ and $|\nu| = s = 0,1,2\dots.$

We observe that if $1-\frac{n}{2}+s\leq 0$ then

$$\frac{1}{\Gamma(1-\frac{n}{2}+s)} = 0, \tag{3.22}$$

therefore from (3.20) and using (3.21) we have

$$R_0^H(u) = 0 \tag{3.23}$$

if $s<\frac{n}{2}-1$.

We are goin to study the special case when $|\nu| = \frac{n}{2}-1$.

In order to do it we observe from ([8], p. 269) that $u = P_+^\lambda$ has poles of order 2 when p and q are odd and the following formula are true

$$A_{-2}^{(k)} = \lim_{\lambda\to-\frac{n}{2}-k}(\lambda+\frac{n}{2}+k)^2P_+^\lambda$$
$$= \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2k}k!\Gamma(\frac{n}{2}+k)\Gamma(n-1)}L^k\{\delta(x)\} \tag{3.24}$$

where $L^k = \Box^k$ is defined by(2.14). In particular from (3.23) we have

$$A_{-2}^{(\frac{n}{2}-1)} = \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{n-2}(\frac{n}{2}-1)!\Gamma(n-1)}L^{(\frac{n}{2}-1)}\{\delta(x)\}. \tag{3.25}$$

On the other hand, using the formulae (1.46) and (2.17) we have

$$\lim_{\alpha\to 0}\Gamma(2+\frac{\alpha}{2}-\frac{q}{2}-p) = \lim_{\alpha\to 0}\frac{\Gamma(2+\frac{\alpha}{2}-\frac{q}{2})\Gamma(1-(2+\frac{\alpha}{2}-\frac{q}{2}))}{(-1)^p\Gamma(1-(2+\frac{\alpha}{2}-\frac{q}{2})+p)}$$
$$= \frac{\pi}{\sin(2+\frac{\alpha}{2}-\frac{q}{2})\pi(-1)^p\Gamma(-2-\frac{\alpha}{2}+\frac{q}{2}+p+1)}$$
$$= \frac{1}{(-1)(-1)^{\frac{q-1}{2}}(-1)^p\Gamma(\frac{q}{2}+p-1)}. \tag{3.26}$$

Now from (2.4), (2.5) and using the formula

$$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos(z\pi)} \tag{3.27}$$

for the case $|\nu| = \frac{n}{2} - 1$, we have

$$\begin{aligned} R_0^B(u) &= \lim_{\alpha\to 0} R_\alpha^B(u) \\ &= \lim_{\alpha\to 0} \frac{\Gamma(\frac{\alpha-p-2|\nu|+2}{2})\Gamma(\frac{p-\alpha}{2})}{2^{-1}\pi^{\frac{n+2|\nu|}{2}}\Gamma(\frac{\alpha-n-2|\nu|+2}{2})} \cdot \frac{\cos\frac{\alpha}{2}\pi}{2^{\alpha-1}} \\ &\cdot \lim_{\alpha\to 0} \frac{u^{\frac{\alpha-2|\nu|-n}{2}}}{\Gamma(\frac{\alpha}{2})} \\ &= \lim_{\alpha\to 0} \frac{\Gamma(2+\frac{\alpha}{2}-\frac{q}{2}-p))\Gamma(\frac{p-\alpha}{2})}{2^{-1}\pi^{\frac{n-1}{2}}\pi^{\frac{n}{2}-1}\pi^{-\frac{1}{2}}\pi\Gamma(\frac{\alpha}{2})} \cdot \lim_{\alpha\to 0} \frac{u^{\frac{\alpha--n}{2}-\frac{n}{2}+1}}{\Gamma(\frac{\alpha}{2})}. \end{aligned} \tag{3.28}$$

From (3.28) and using (3.26) we have

$$\begin{aligned} R_0^B(u) = &\frac{1}{(-1)(-1)^{\frac{q-1}{2}}(-1)^p\Gamma(\frac{q}{2}+p-1)} \frac{\Gamma(\frac{p}{2})}{2^{-1}\pi^{\frac{n-1}{2}}\pi^{\frac{n}{2}-1}\pi^{-\frac{n}{2}}\pi^{-\frac{1}{2}}\pi} \\ &\cdot \lim_{\alpha\to 0} \frac{1}{\Gamma(\frac{\alpha}{2})}.\frac{u^{\frac{\alpha-n}{2}-\frac{n}{2}+1}}{\Gamma(\frac{\alpha}{2})}. \end{aligned} \tag{3.29}$$

Letting $\lambda = \frac{\alpha-n}{2} - \frac{n}{2} + 1$, we have

$$\begin{aligned} \lim_{\alpha\to 0} \frac{1}{\Gamma(\frac{\alpha}{2})}.\frac{u^{\frac{\alpha-n}{2}-\frac{n}{2}+1}}{\Gamma(\frac{\alpha}{2})} &= \lim_{\lambda\to -\frac{n}{2}-\frac{n}{2}+1} \frac{1}{\Gamma(\lambda+n-1)}.\frac{1}{\Gamma(\lambda+n-1)}.u^\lambda \\ &= \lim_{\lambda\to -\frac{n}{2}-(\frac{n}{2}-1)} \frac{(\lambda+n-1)^2 u^\lambda}{[(\lambda+n-1)\Gamma(\lambda+n-1)]\,[(\lambda+n-1)\Gamma(\lambda+n-1)]} \\ &= \frac{\lim_{\lambda\to -\frac{n}{2}-(\frac{n}{2}-1)}(\lambda+n-1)^2 u^\lambda}{\lim_{\lambda\to -\frac{n}{2}-(\frac{n}{2}-1)}[(\lambda+n-1)\Gamma(\lambda+n-1)]} \\ &\cdot \frac{1}{\lim_{\lambda\to -\frac{n}{2}-(\frac{n}{2}-1)}[(\lambda+n-1)\Gamma(\lambda+n-1)]} \\ &= \frac{A_{-2}^{(\frac{n}{2}-1)}}{\lim_{z\to 0} z\Gamma(z)\lim_{z\to 0} z\Gamma(z)} \\ &= \frac{A_{-2}^{(\frac{n}{2}-1)}}{\operatorname*{Res}_{z=0}\Gamma(z).\operatorname*{Res}_{z=0}\Gamma(z)} \\ &= \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{n-2}(\frac{n}{2}-1)!\Gamma(n-1)} \square^{(\frac{n}{2}-1)}\{\delta(x)\}. \end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we obtain the following formula

$$\begin{aligned} R_0^B(u) &= \frac{1}{(-1)^{\frac{q-1}{2}}\pi^{n-1}2^{-1}\Gamma(\frac{q}{2}+p-1)} \cdot \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}\Gamma(\frac{p}{2})}{2^{n-2}(\frac{n}{2}-1)!\Gamma(n-1)} \Box^{(\frac{n}{2}-1)}\{\delta(x)\} \\ &= \frac{\Gamma(\frac{p}{2})}{\pi^{\frac{n}{2}}(\frac{n}{2}-1)!\Gamma(n-1)2^{n-3}\Gamma(\frac{q}{2}+p-1)} \Box^{(\frac{n}{2}-1)}\{\delta(x)\} \end{aligned} \tag{3.31}$$

if $|\nu| = \frac{n}{2} - 1$.

From (3.21), (3.23) and (3.31), we have the following formula

$$R_0^B(u) = \begin{cases} \dfrac{\Gamma(1-\frac{p}{2}-s)\Gamma(\frac{p}{2})(-1)(-1)^{\frac{q+1}{2}}\pi^{\frac{n-1}{2}-1}}{\Gamma(1-\frac{n}{2}+s)\Gamma(\frac{p}{2}+s)2^{2s}s!} \cdot \\ \left[\psi(\dfrac{p}{2}) - \psi(\dfrac{n}{2})\right] \Box^{s-\frac{n}{2}+1}\{\delta(x)\}, & \text{if } s > \frac{n}{2}-1; \\ 0, & \text{if } s < \frac{n}{2}-1; \\ \dfrac{\Gamma(\frac{p}{2})}{\pi^{\frac{n}{2}}(\frac{n}{2}-1)!\Gamma(n-1)2^{n-3}\Gamma(\frac{q}{2}+p-1)} \Box^{\frac{n}{2}-1}\{\delta(x)\}, & \text{if } s = |\nu| = \frac{n}{2}-1. \end{cases} \tag{3.32}$$

In the next section we are going to study the property(2.24) at $\alpha = 2k, k = 1, 2, \ldots$.

4 $\Box_B^k \left\{R_{2k}^B\right\}$

Putting $\alpha = 2k, k = 1, 2, \ldots$ in (2.23), we have

$$\Box_B^k \left\{R_{2k}^B\right\} = d_{2k,p,|\nu|,k}\; R_0^B. \tag{4.1}$$

Now considering the case 1, case 2 and case 3 of the above section we have the following results :

Case 1 : q even, p odd (n odd)

From (4.1) and considering (3.12) and (3.13) we have

$$\Box_B^k\left\{R_{2k}^B\right\} \begin{cases} \dfrac{(-1)^{k+s}\Gamma(\frac{p}{2}-k)}{\pi^s\Gamma(\frac{p}{2}-k+s)2^{2s}s!} \Box_B^s\{\delta(x)\}, & \text{if } s = |\nu|, |\nu| = 0, 1, 2, \ldots; \\ \dfrac{(-1)^{l-1}(-1)^k(-1)^{\frac{q}{2}}\Gamma(\frac{p}{2}+s)}{(l-1)!\Gamma(l-\frac{q}{2})\pi^l 2^{-1}\Gamma(\frac{p}{2}-k+s)} \delta^{(l-1)}(u), & \text{if } |\nu| = -\frac{n}{2}+l \text{ and } l > \frac{q}{2}; \\ 0, & \text{if } |\nu| = -\frac{n}{2}+l \text{ and } l \le \frac{q}{2}; \\ 0, & \text{if } |\nu| \neq 0, 1, 2, \ldots. \end{cases} \tag{4.2}$$

Case 2 : q even, p even (n even)

From (3.32) and considering (3.14) we have

$$\square_B^k\left\{R_{2k}^B\right\}=\begin{cases}(-1)^{k+1}\left\{\delta(x)\right\}, & \text{if } s=|\nu|=0, k=1,2,\ldots \text{ and } n=2;\\ \dfrac{(-1)^{k+s+1}(-1)^s\Gamma(\frac{p}{2}-k)}{\Gamma(\frac{p}{2}-k+s)((\pi^s 2^{2s}s!}\square^{s-\frac{n}{2}+1}\left\{\delta(x)\right\}, & \text{if } s\geq\frac{n}{2}-1, s=1,2,\ldots.\end{cases} \tag{4.3}$$

In particular letting $s=|\nu|=\frac{n}{2}-1$ in (4.2) we have

$$\square_B^k\left\{R_{2k}^B\right\}=\frac{(-1)^{k+\frac{n}{2}}\Gamma(\frac{p}{2}-k)}{\Gamma(\frac{p}{2}+\frac{n}{2}-k-1)\pi^{\frac{n}{2}-1}2^{n-2}(\frac{n}{2}-1)!}\left\{\delta(x)\right\} \tag{4.4}$$

Now considering E_{2k} defined the following form

$$E_{2k}=(\frac{p}{2}-k)(\frac{p}{2}-k+1)\ldots(\frac{p}{2}-k+\frac{n}{2}-2)\frac{\pi^{\frac{n}{2}-1}2^{n-2}(\frac{n}{2}-1)!}{(-1)^{k+\frac{n}{2}}}R_{2k}^B, \tag{4.5}$$

from (4.4) and using (2.34) we have the following property

$$\square_B^k\left\{E_{2k}\right\}=\delta(x). \tag{4.6}$$

The property (4.6) we means that E_{2k} is elemental solution of the Bessel ultrahyperbolic operator iterated k-times.

On the other hand for $|\nu|=\frac{n}{2}-1$,

$$\begin{aligned}
&\lim_{\alpha\to 2k}\frac{\Gamma(\frac{\alpha-n-2|\nu|+2}{2})}{\Gamma(\frac{\alpha-p-2|\nu|+2}{2})}\\
&=\lim_{\alpha\to 2k}\frac{\Gamma(\frac{\alpha-p}{2}+1)\Gamma(\frac{p-\alpha}{2})}{(-1)^{\frac{n}{2}-1}\Gamma(-\frac{\alpha}{2}+\frac{p}{2}-1+\frac{n}{2})}\cdot\frac{(-1)^{\frac{q}{2}-1}\Gamma(-\frac{\alpha}{2}+\frac{p}{2}+\frac{q}{2}-1)}{\Gamma(\frac{\alpha-p}{2}+1)\Gamma(\frac{p-\alpha}{2})}\\
&=\lim_{\alpha\to 2k}\frac{(-1)^{\frac{q}{2}-\frac{n}{2}}(-1)\left[(-\frac{\alpha}{2})(-\frac{\alpha}{2}+1)\cdots(-\frac{\alpha}{2}+\frac{n}{2}-2)\right]}{\left[(-\frac{\alpha}{2})(-\frac{\alpha}{2}+1)\cdots(-\frac{\alpha}{2}+\frac{p}{2}+\frac{n}{2}+2)\right]}\\
&=\frac{(-1)^{-p}k(k-1)(k-2)\cdots(k-\frac{n}{2}-\frac{p}{2})(k-\frac{n}{2}-\frac{p}{2}+1)}{k(k-1)(k-2)\cdots(k-\frac{n}{2}-\frac{p}{2})(k-\frac{n}{2}-\frac{p}{2}+1)(k-\frac{n}{2}-\frac{p}{2}+2)}\\
&\quad\cdot\frac{1}{(k-\frac{n}{2}-\frac{p}{2}+2)(k-\frac{n}{2}-\frac{p}{2}+3)\cdots(k-\frac{n}{2}-\frac{p}{2}+(\frac{p}{2}+2))}\\
&=\left(k-\frac{n}{2}-\frac{p}{2}+3\right)\cdots\left(k-\frac{n}{2}-\frac{p}{2}+(\frac{p}{2}+2)\right).
\end{aligned} \tag{4.7}$$

From (2.4), (2.5) and using (2.17) and (4.7) we have

$$\begin{aligned}
R^B_{2k}(u) &= \lim_{\alpha\to 2k} R^B_\alpha(u) \\
&= \lim_{\alpha\to 2k} \frac{u^{\frac{\alpha-2|\nu|-n}{2}}}{K_n^{|\nu|}(\alpha)} \\
&= \frac{\Gamma(\frac{p}{2}-k)}{\pi^{n-\frac{3}{2}}\Gamma(\frac{1}{2}-k)\Gamma(2k)}.u^{k-n+1} \\
&= \lim_{\alpha\to 2k} \frac{\Gamma(\frac{\alpha-n-2|\nu|+2}{2})}{\Gamma(\frac{\alpha-p-2|\nu|+2}{2})} \\
&= \frac{\Gamma(\frac{p}{2}-k)}{\pi^{n-\frac{3}{2}}\Gamma(\frac{1}{2}-k)\Gamma(2k)}.u^{k-n+1}\left(k-\frac{n}{2}-\frac{p}{2}+3\right)\cdots\left(k-\frac{n}{2}-\frac{p}{2}+(\frac{p}{2}+2)\right)
\end{aligned} \tag{4.8}$$

Now considering that u^λ has simple poles at $\lambda = -\frac{n}{2}-l, l = 0,1,2,\ldots$ when p and q are both even ([8], p. 268) and the function $\Gamma(z)$ has simple poles at $z = 0,-1,-2,\ldots$, from (4.5) and considering (4.8) E_{2k} can be rewritten in the following form

$$\begin{aligned}
E_{2k} &= \frac{\Gamma(\frac{p}{2}+\frac{n}{2}-k-1)}{\Gamma(\frac{p}{2}-k)}\frac{\pi^{\frac{n}{2}-1}2^{n-2}(\frac{n}{2}-1)!}{(-1)^{k+\frac{n}{2}}}R^B_{2k} \\
&= \frac{\Gamma(\frac{p}{2}+\frac{n}{2}-k-1)}{\Gamma(\frac{p}{2}-k)}\cdot\frac{\pi^{\frac{n}{2}-1}2^{n-2}(\frac{n}{2}-1)!}{(-1)^{k+\frac{n}{2}}} \\
&\quad\cdot\frac{\Gamma(\frac{p}{2}-k)}{\pi^{n-\frac{3}{2}}\Gamma(\frac{1}{2}-k)\Gamma(2k)}\cdot u^{k-n+1}\left(k-\frac{n}{2}-\frac{p}{2}+3\right)\cdots\left(k-\frac{n}{2}-\frac{p}{2}+(\frac{p}{2}+2)\right) \\
&= \frac{2^{n-2}(\frac{n}{2}-1)!\Gamma(\frac{p}{2}+\frac{n}{2}-k-1)}{(-1)^{k+\frac{n}{2}}\pi^{\frac{n-1}{2}}\Gamma(\frac{1}{2}-k)\Gamma(2k)}\left(k-\frac{n}{2}-\frac{p}{2}+3\right)\cdots\left(k-\frac{n}{2}-\frac{p}{2}+(\frac{p}{2}+2)\right)u^{k-n+1}
\end{aligned} \tag{4.9}$$

under conditions $\frac{n}{2}-1 < k < \frac{p}{2}+\frac{n}{2}-2$.

Case 3 : q odd and p odd (n even)

From (3.29) and considering (2.40) and (3.32) we obtain the following formulae :

$$\begin{aligned}
\Box_B^k\left\{R^B_{2k}\right\} &= \frac{\Gamma(1-\frac{p}{2}-s)\Gamma(\frac{p}{2})(-1)(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}\Gamma(k-\frac{p}{2}-s+1)}{2^{-1}2^{2s}\Gamma(1-\frac{n}{2}+s)\Gamma(\frac{p}{2}+s)2^{2s}s!\Gamma(-\frac{p}{2}-s+1)} \\
&\quad\cdot\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\Box^{s-\frac{n}{2}+1}\left\{\delta(x)\right\} \\
&\quad\cdot\frac{\Gamma(1-\frac{p}{2})}{\Gamma(1-\frac{p}{2}+k)}\left[\psi(\frac{p}{2})-\psi(\frac{n}{2})\right]\Box^{s-\frac{n}{2}+1}\left\{\delta(x)\right\},
\end{aligned} \tag{4.10}$$

if $|\nu| = s > \frac{n}{2} - 1$,

$$\square_B^k \left\{ R_{2k}^B \right\} = 0, \tag{4.11}$$

if $s < \frac{n}{2} - 1$, and

$$\begin{aligned} \square_B^k \left\{ R_{2k}^B \right\} = & \frac{(-1)^{\frac{p+1}{2}}}{\pi^{\frac{n}{2}-1}(\frac{n}{2}-1)!\Gamma(n-1)2^{n-3}\Gamma(\frac{q}{2}+p-1)\Gamma(1-\frac{p}{2}+k)} \\ & \cdot \frac{\Gamma(k-p-\frac{q}{2}+2)}{\Gamma(-p-\frac{q}{2}+2)} \square^{\frac{n}{2}-1} \left\{ \delta(x) \right\} \end{aligned} \tag{4.12}$$

if $s = |\nu| = \frac{n}{2} - 1$.

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